1.10. Comparison of Semigroups.

In perturbation theory one starts from a semigroup S and an operator P, which is "small" with respect to the generator H of S, and then constructs a perturbed semigroup S^P, with generator $H + P$, which is "close" to S. The notions of "smallness" of the perturbation and "closeness" of the semigroups are intimately related. In particular one can estimate from the identity

$$S_t - S_t^P = \int_0^t ds \frac{d}{ds} \left(S_{t-s}^P S_s\right)$$

$$= \int_0^t ds\, S_{t-s}^P P\, S_s$$

that

$$\left\| S_t - S_t^P \right\| = O(t) \ ,$$

as $t \to 0$, if P is bounded, or

$$\left\| \left(S_t - S_t^P\right) a \right\| = O(t)$$

for all $a \in D(H)$, as $t \to 0$, if P is relatively bounded with respect to H. Our aim is to prove converses to these statements.

We now begin with two semigroups satisfying the estimate (*), or (**), and attempt to prove that the corresponding generators differ by a bounded, or a relatively bounded, pertúrbation. The difficulty is that these converse statements are not valid for general C_0-semigroups. Nevertheless they are valid for C_0^*-semigroups, with some slight qualification, and hence for C_0-semigroups on reflexive Banach spaces. In general another phenomenon of intertwining of generators has to be taken into account. We will discuss this after describing the basic results on C_0^*-semigroups, and their corollaries.

THEOREM 1.10.1. *Let* S *and* T *be two* C_0^**-semigroups on the Banach space* $\mathcal{B}$ *with generators* H *and* K *, respectively. The following conditions are equivalent:*

1. $\|S_t - T_t\| = O(t)$ *as* $t \to 0+$ *,*

2. $D(H) = D(K)$ *and* $K = H + P$ *where* P *is a bounded operator from the norm closure* $\overline{D(H)}$ *of* $D(H)$ *to* $\mathcal{B}$ *.*

Proof. $1 \Rightarrow 2$. Condition 1 states that there are constants N , $\delta > 0$ such that

$$\|S_t - T_t\| \leq Nt$$

for $0 \leq t < \delta$. Now for $f \in D(H)$ consider the one-parameter family $f_t = (S_t - T_t)f/t \in \mathcal{B}$. One has $\|f_t\| \leq N\|f\|$ for $0 \leq t < \delta$. But the unit ball of $\mathcal{B}$ is compact in the weak*-topology, by the Alaoglu-Birkhoff theorem, and hence there exists a subnet f_{t_α} which is weak*-convergent, as $t_\alpha \to 0+$, to a limit g . Now if K_* and T_{*t} denote the adjoints of K and T_t , on $\mathcal{B}_*$, which exist by Lemma 1.5.1, one has

$$f(K_* a) = \lim_\alpha f\left(\left(I-T_{*t_\alpha}\right)a\right)/t_\alpha$$

$$= \lim_\alpha \left(\left(I-S_{t_\alpha}\right)f\right)(a)/t_\alpha + \lim_\alpha f_{t_\alpha}(a)$$

$$= (Hf)(a) + g(a)$$

for all $a \in D(K_*)$ and $f \in D(H)$. Since the right hand side is continuous in a, and since $D(K_*)$ is norm-dense in $\mathcal{B}_*$, one concludes that $f \in D(K)$ and hence $D(H) \subseteq D(K)$. But reversing the roles of S and T in this argument gives $D(K) \subseteq D(H)$ and hence $D(H) = D(K)$. Furthermore the foregoing identity gives

$$Kf = Hf + g .$$

But $\|g\| \le N\|f\|$ and hence $K - H$ extends by closure to a bounded operator P from $\overline{D(H)}$ to $\mathcal{B}$, with $\|P\| \le N$.

2 ⇒ 1. If $f \in D(H)$ then $S_s f \in D(H)$ and

$$\left(\left(S_t - T_t\right)f\right)(a) = \int_0^t ds\, \left(T_{t-s} P S_s f\right)(a) .$$

Therefore

$$\|S_t - T_t\| \le t\|P\| \sup\left\{\|T_{t-s}\|\ \|S_s\| \ ; \quad 0 \le s \le t\right\}$$

$$= O(t)$$

as $t \to 0+$ because $\|S_t\|, \|T_t\| \le M \exp\{\omega t\}$ for suitable $M, \omega \ge 0$. □

Note that in the proof of 1 ⇒ 2 one establishes that the perturbation P satisfies the estimate

$$\|P\| \le \sup_{t>0} \|S_t - T_t\|/t .$$

But in the proof of 2 ⇒ 1 one has the converse estimate

$$\sup_{t>0} \|S_t - T_t\|/t \le \|P\|M^2 .$$

Thus if S and T are contraction semigroups, or more generally if $\|S_t\|$, $\|T_t\| \le \exp\{\omega t\}$ for some $\omega \ge 0$, then

$$\|P\| = \sup_{t\ge0} \|S_t - T_t\|/t .$$

The magnitude of the perturbation is measured by the "distance" $\|S_t - T_t\|/t$ between the semigroups for small t .

The difficulty in interpreting Condition 2 of Theorem 1.10.1 as a perturbation result is that the perturbation $P = K - H$ is only defined on the weak*-dense domain $D(H)$. Although it is bounded as an operator from the norm closure of $D(H)$ to $\mathcal{B}$ it is not clear that it has a bounded extension from $\mathcal{B}$ to $\mathcal{B}$. This is the case, however, if $D(H)$ is norm dense. In particular this follows if $\mathcal{B}$ is reflexive because then the norm topology and weak*-topology coincide. Therefore Theorem 1.10.1 has the following corollary.

COROLLARY 1.10.2. *Let* S *and* T *be two* C_0*-semigroups on the reflexive Banach space* $\mathcal{B}$ *with generators* H *and* K *respectively. The following conditions are equivalent:*

1. $\|S_t - T_t\| = O(t)$, *as* $t \to 0+$,

2. *There is a bounded operator* P *on* $\mathcal{B}$ *such that* $K = H - P$.

Reflexivity of $\mathcal{B}$ means that $\mathcal{B}^* = \mathcal{B}_*$ and hence weak*-continuity is equivalent to weak, or strong continuity. Therefore C_0^*-semigroups are C_0-semigroups and this result follows from Theorem 1.10.1. But it is not generally true without reflexivity. Before giving a counterexample and discussing the new phenomenon which arises we will, however, describe the relative boundedness version of Theorem 1.10.1.

THEOREM 1.10.3. *Let* S *and* T *be two* C_0^**-semigroups on the Banach space* $\mathcal{B}$ *with generators* H *and* K *, respectively. The following conditions are equivalent:*

1. $\|(S_t-T_t)f\| = O(t)$ *as* $t \to 0+$ *, for all* $f \in D(H)$,

2. $$K \supseteq H + P$$

where $D(P) = D(H)$ *and*

$$\|Pf\| \le a\|f\| + b\|Hf\|$$

for all $f \in D(H)$ *and some* $a, b \ge 0$.

Proof. 1 ⇒ 2. First remark that $f \in D(H)$ if, and only if, $\|(I-S_t)f\| = O(t)$ at $t \to 0+$, by Exercise 1.5.2. But then since

$$\|(I-T_t)f\| \le \|(I-S_t)f\| + \|(S_t-T_t)f\| = O(t)$$

one must have $D(H) \subseteq D(K)$ and one can define P by $D(P) = D(H)$

and $P = K - H$.

Next note that H and K are both weak*-closed, and hence strongly closed, and consider the graph $G(H) = \{(f, Hf) ; f \in D(H)\}$ equipped with the norm

$$\|(f, Hf)\| = \|f\| + \|Hf\| .$$

The graph $G(H)$ is a closed subspace of $\mathcal{B} \times \mathcal{B}$ and the mapping $(f, Hf) \longmapsto Kf$ is a linear operator from $G(H)$ into $\mathcal{B}$. But this operator is closed, because if (f_n, Hf_n) converges in $G(H)$ and Kf_n converges in $\mathcal{B}$ then $\|f_n - f\| \to 0$, and $\|Kf_n - g\| \to 0$, for some $f, g \in \mathcal{B}$ and $g = Kf$ since K is closed. Therefore the mapping is bounded by the closed graph theorem, i.e., there is a constant $c > 0$ such that

$$\|Kf\| \le c\left(\|f\|+\|Hf\|\right) .$$

Consequently

$$\begin{aligned}\|Pf\| &= \|(K-H)f\| \\ &\le c\|f\| + (c+1)\|Hf\| .\end{aligned}$$

$2 \Rightarrow 1.$ If $f \in D(H)$ then $S_t f \in D(H) \subseteq D(K)$ and

$$\left(\left(S_t-T_t\right)f\right)(a) = \int_0^t ds \left(T_{t-s}PS_sf\right)(a)$$

for all $a \in \mathcal{B}_*$. Therefore

$$\begin{aligned}\left\|\left(S_t-T_t\right)f\right\| &\le t \sup_{0\le s\le t} \|T_{t-s}\| \left(a\|S_sf\|+b\|HS_sf\|\right) \\ &\le t \sup_{0\le s\le t} \|T_{t-s}\| \|S_s\| \left(a\|f\|+b\|Hf\|\right) .\end{aligned}$$

But $\|S_t\|$, $\|T_t\| \le M\exp\{\omega t\}$ for suitable $M, \omega \ge 0$ and hence

$$\|(S_t - T_t)f\| = O(t)$$

as $t \to 0+$ for each $f \in D(H)$. □

The analogues of Theorems 1.10.1 and 1.10.3 are not true for general C_0-semigroups because of another effect which is illustrated by the following example.

Example 1.10.4. Let $\mathcal{B} = C_0(\mathbb{R})$ be the continuous functions on the real line which vanish at infinity, equipped with the usual supremum norm, and let S denote the C_0-group of translations,

$$(S_t f)(x) = f(x-t)$$

for $f \in \mathcal{B}$ and $t \in \mathbb{R}$. Thus the generator H of S is the operator of differentiation with domain the differentiable functions $f \in C_0(\mathbb{R})$ whose derivatives f' are also in $C_0(\mathbb{R})$. Next let M be the operator of multiplication by a bounded function m which is non-differentiable at some points but is uniformly Hölder continuous in the sense that

$$|m(x)-m(y)| \le c|x-y|$$

for some c . Define W by $W = \exp\{iM\}$ and T by $T_t = WS_tW^{-1}$. Since

$$((S_t - T_t)f)(x) = (1-\exp\{i(m(x)-m(x-t))\})f(x-t)$$

one has the estimate

$$\|S_t - T_t\| \le c|t| .$$

But the generator K of T is given by $K = WHW^{-1}$ and $D(K) \neq D(H)$ because m is chosen to be non-differentiable. In fact one can choose m to be non-differentiable at a dense set of points and then one obtains the extreme case $D(H) \cap D(K) = \{0\}$.

Note that the same construction on $L^\infty(\mathbb{R})$ does not lead to a similar conclusion because the domain of the differentiation operator which generates translations is much larger and contains functions which are not continuously differentiable. □

The infinitesimal comparison of C_0-groups which are close together can be explained by a combination of a perturbation and a twist of the type occurring in Example 1.10.4. It is possible that this is also true for C_0-semigroups but the following proof does use the group property in an essential way. It also broadens the comparison criterion.

THEOREM 1.10.5. *Let* S *and* T *be two* C_0- *or* C_0^*-*groups on the Banach space* $\mathcal{B}$ *with generators* H *and* K, *respectively and let* $0 < \alpha \le 1$. *The following conditions are equivalent:*

1. $\|S_t - T_t\| = O(t^\alpha)$ *as* $t \to 0$,

2. *there exist bounded operators* P , W , *such that* W *has a bounded inverse,*

$$K = W(H+P)W^{-1}$$

and

$$\left\|S_t - WS_tW^{-1}\right\| = 0\left(t^{\alpha}\right) \quad as \quad t \to 0 .$$

Proof. $1 \Rightarrow 2$. Define

$$W = \frac{1}{r} \int_0^r ds\ T_sS_{-s}$$

where r is chosen sufficiently small that $\|I - W\| < 1$, and hence W has a bounded inverse. This is possible by Condition 1. Next introduce

$$U_t = W^{-1}T_tWS_{-t} .$$

One then has the identity

$$\left(I-U_h\right)/h = (rhW)^{-1} \int_0^h ds\ T_sS_{-s} - (rhW)^{-1} \int_r^{r+h} ds\ T_sS_{-s}$$

which implies the existence of the strong, or weak*-, limit

$$P = \lim_{h\to 0} \left(I-U_h\right)/h$$

and gives the identification

$$P = W^{-1}\left(I-T_rS_{-r}\right)/r .$$

Thus P is bounded. Next remark that

$$\left(I-T_t\right)Wa/t = W\left(I-S_t\right)a/t + W\left(I-U_t\right)S_ta/t .$$

But the right hand side converges for all $a \in D(H)$ in the limit

$t \to 0$. Hence $Wa \in D(K)$ and

$$KWa = W(H+P)a \ .$$

Similarly if $a \in D(K)$ then $W^{-1}a \in D(H)$ and

$$W^{-1}Ka = (H+P)W^{-1}a \ .$$

Thus $D(K) = WD(H)$ and

$$K = W(H+P)W^{-1} \ .$$

Finally one has

$$S_t - WS_tW^{-1} = \left(S_t - T_t\right) + \left(T_t - WS_tW^{-1}\right) \ .$$

But $t \mapsto WS_tW^{-1}$ is the group with generator

$$WHW^{-1} = K - WPW^{-1}$$

and hence

$$\left\|T_t - WS_tW^{-1}\right\| = O(t)$$

by perturbation theory, e.g., by Theorem 1.9.2. Thus

$$\left\|S_t - WS_tW^{-1}\right\| \le \left\|S_t - T_t\right\| + \left\|T_t - WS_tW^{-1}\right\|$$

$$= O\left(t^{\alpha}\right) + O(t) = O\left(t^{\alpha}\right) \ .$$

$2 \Rightarrow 1$. Define $Q = -WPW^{-1}$ then $H = W^{-1}(K+Q)W$ and WSW^{-1} is the group generated by $K + Q$. Thus

$$\|T_t - WS_tW^{-1}\| = O(t)$$

as $t \to 0$ by another application of perturbation theory. But

$$S_t - T_t = \left(S_t - WS_tW^{-1}\right) + \left(WS_tW^{-1} - T_t\right)$$

and hence

$$\|S_t - T_t\| \le \|S_t - WS_tW^{-1}\| + \|WS_tW^{-1} - T_t\|$$

$$= O\left(t^{\alpha}\right) + O(t) = O\left(t^{\alpha}\right) . \qquad \square$$

Exercises.

1.10.1. Prove that if S and T are two C_0-, or C_0^*-, semigroups with

$$\|S_t - T_t\| = o(t)$$

as $t \to 0+$ then $S = T$.

1.10.2. If S is a C_0- or C_0^*-semigroup prove that S is uniformly continuous if, and only if, there exist $\varepsilon, \delta > 0$ such that

$$\|I - S_t\| \le 1 - \varepsilon , \qquad 0 < t < \delta .$$

1.10.3. If S and T are two C_0- or C_0^*-groups with generators H and K prove that there exist $\varepsilon_1, \delta_1 > 0$ such that

$$\|S_t T_{-t} - I\| \le 1 - \varepsilon_1 , \qquad 0 < t < \delta_1 ,$$

if, and only if, there exist $\varepsilon_2, \delta_2 > 0$ and bounded operators P , W , such that W has a bounded inverse $K = W(H+P)W^{-1}$ and

$$\|S_t W S_{-t} W^{-1} - I\| \le 1 - \varepsilon_2 , \qquad 0 < t < \delta_2 .$$

Hint: Follow the proof of Theorem 1.10.5. Note that Exercise 1.10.2 follows by setting $T = I$.