INTERPOLATION IN ORLICZ AND SOBOLEV-ORLICZ SPACES

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§1. INTRODUCTION

The last two decades witnessed a rapid development of an umbrella-type theory of function spaces, in particular, of those frequently used in the theory of p.d.e.'s. The interpolation theory has become one of the powerful tools. A comprehensive monograph surveying results until the middle seventies is [T]. The classical interpolation theorems due to Riesz, Thorin, and Marcinkiewicz had become the basis for the complex and real interpolation method, resp. We shall restrict ourselves to the real method in virtue of the fact that only this method gives finer results in the framework of L_p-spaces.

Let X_1 and X_2 be B-spaces embedded into some Hausdorff topological space A. As usual let $\Delta(\bar{X}) = X_1 \cap X_2$ with the norm $\|x\|_\Delta = \max(\|x\|_{X_1}, \|x\|_{X_2})$ and $\Sigma(\bar{X}) = X_1 + X_2$ endowed with the norm $\|x\|_\Sigma = \inf\{\|x_1\|_{X_1} + \|x_2\|_{X_2}\ ;\ x = x_1 + x_2,\ x_i \in X_i,\ i = 1,2\}$. Any B-space X such that $\Delta(\bar{X}) \subset X \subset \Sigma(\bar{X})$ (with a continuous embedding) is said to be an *interpolation space* between X_1 and X_2 if for each linear operator $T : \Sigma(\bar{X}) \to \Sigma(\bar{X})$ such that $T = X_i \to X_i$ and $T|_{X_i}$ is bounded, $i = 1,2$, T maps X into X and is bounded here. *An interpolation method* is a mapping $\mathcal{T}$ from the family $C(A)$ of all couples $\bar{X} = (X_1, X_2)$ with properties as above into the set of all B-spaces embedded into A which satisfies the following condition: Whenever $\bar{X}$, $\bar{Y} \in C(A)$ and T is a linear operator such that T maps X_i continuously into Y_i, $i = 1,2$, then T maps $\mathcal{T}(\bar{X})$ continuously into $\mathcal{T}(\bar{Y})$. Let us note that various conditions may be added, expressing a dependance of $\|T\|_{\mathcal{T}(\bar{X})\to\mathcal{T}(\bar{Y})}$ upon $\|T\|_{X_i\to Y_i}$, $i = 1,2$; we shall meet them in due course.

The so-called real interpolation method goes back to Peetre, Gagliardo, and Lions. Peetre gave the following abstract construction: Let $\bar{X} \in C(A)$ and define, for $x \in \Sigma(\bar{X})$ and $t > 0$, the so-called *K-functional*

$$K(t,x) = \inf\{\|x_1\|_{X_1} + t\|x_2\|_{X_2} \ ; \ x = x_1 + x_2 \ , \ x_i \in X_i \ , \ i = 1,2\}$$

and, for each $\theta \in (0,1)$, $p \in \langle 1,\infty)$, the space

$$\bar{X}_{\theta,p} = \{x \in \Sigma(\bar{X}) \ ; \ \|x\|_{\theta,p} = \left(\int_0^\infty [t^{-\theta}K(t,x)]^p dt/t\right)^{1/p} < \infty\}$$

(with an evident change for $p = \infty$).

The space $\bar{X}_{\theta,p}$ is a B-space, it is an interpolation space between X_1 and X_2, and the mapping $\bar{X} \to \bar{X}_{\theta,p}$ is an interpolation method. Moreover,

$$\|x\|_{\theta,p} \leq \|x\|_{X_1}^{1-\theta} \ \|x\|_{X_2}^{\theta} \ , \ x \in \bar{X}_{\theta,p} \ , \ 0 < \theta < 1 \ , \ 1 \leq p \leq \infty \ .$$

It is very well known that this - the so-called K-method (together with several equivalent definitions) has been very successfully applied to L_p-spaces and to those which are based on them - i.e. Sobolev, Besov and Triebel-Lizorkin spaces (see e.g. [T]).

Let us briefly survey the basic interpolation properties of them. It holds that $(L_{p_1},L_{p_2})_{\theta,q} = L_{pq}$ (the Marcinkiewicz space) for $0 < \theta < 1$, $1 \leq p_1,p_2,q \leq \infty$, $1/p = (1-\theta)/p_1 + \theta/p_2$, which, in particular gives that $(L_{p_1},L_{p_2})_{\theta,p} = L_p$, further, for Besov spaces, $(B^{s_1}_{pq_1},B^{s_2}_{pq_2})_{\theta,q} = B^s_{pq}$,

$$(B^{s}_{p_1q_1}, B^{s}_{p_2q_2})_{\theta,q} = B^{s}_{pq}\ , \quad (B^{s_1}_{p_1q_1}, B^{s_2}_{p_2q_2})_{\theta,q} = B^{s}_{pq} \quad \text{(if } p=q) \ ,$$

where $s = (1-\theta)s_1 + \theta s_2$, $1/p = (1-\theta)/p_1 + \theta/p_2$, $1/q = (1-\theta)/q_1 + \theta/q_2$, and this is valid for spaces on any domain in $\mathbb{R}^n$ permitting a bounded embedding into corresponding spaces on $\mathbb{R}^n$. Analogous results hold for Triebel-Lizorkin spaces F^s_{pq} which include Sobolev (potential) spaces as a particular case $(H^s_p = F^s_{p2})$. There are no restrictions on the number s and generalizations to cases $p \in (0,1)$ were given, too.

It is clear from this very brief and short survey that despite the fact that K-method works very well in abstract B-spaces it does not go "further" from L_p-type spaces than to Marcinkiewicz spaces. (Note that the restriction $p = q$ is not in fact substantial - it can be removed by considering generalized "Besov-Lorentz spaces" instead of standard Besov spaces B^s_{pq} which perhaps might be called here "Besov-Lebesgue spaces".) This is, roughly speaking, yet more restrictive in the case of the complex method (with the exception of the interpolation between $B^{s_0}_{p_0q_0}$ and $B^{s_1}_{p_1q_1}$ which does not go out of the scale of usual Besov spaces).

Another aspect of interpolation properties of function spaces and linear operators is devoted to the study of spaces with interpolation properties with respect to a given couple of B-spaces. Very general results in this direction for the couple (L_1, L_∞) can be found e.g. in [K-P-S]. A great deal of the most general results concerns interpolation properties of rearrangement invariant spaces, which include not only L_p-spaces but also Orlicz spaces.

Very soon, there was a need for some more general interpolation methods for Orlicz (and Sobolev-Orlicz) spaces, especially in connection with pioneering existence theorems for p.d.e.'s in Sobolev-Orlicz spaces (see [D], [Go]) and also with the development of the most general theories which enable the handling of function spaces from the point of view of several general abstract principles. First results involving Orlicz spaces dealt with interpolation of linear operators (see e.g. [S]) in (reflexive) Orlicz spaces and we shall not recall these in detail for they will be included as a special case in what follows.

§2. INTERPOLATION METHODS FOR MODULAR SPACES

2.1 Definition Let X be a (real) linear space. A mapping $m : X \to \langle 0,\infty \rangle$ will be called *the modular* (on X) if $m(x) = 0$ iff $x = 0$; $m(-x) = m(x)$, $x \in X$; $m(\lambda x+\eta y) \leqq \lambda m(x) + \mu m(y)$, $x,y \in X$, $\lambda,\eta \geqq 0$, $\lambda + \eta = 1$.

Let

$$X(m) = \{x \in X ; \lim_{\nu\to 0} m(\nu x) = 0\} .$$

The couple $(X(m),m)$ is called *the modular space* (m-space) generated by X and m .

For our purposes, we shall always start directly with $X(m)$ instead of X and we shall write simply X instead of $X(m)$.

In X there can be introduced the (Luxemburg) norm

$$\|x\| = \inf\{\lambda ; m(x/\lambda) \leqq 1\} .$$

A particularly important example of an m-space is the Sobolev-Orlicz space. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $\phi : \mathbb{R} \to \langle 0,\infty)$ be a *Young function* i.e. an even, convex function such that $\lim_{t\to 0} \phi(t)/t = \lim_{t\to\infty} t/\phi(t) = 0$. For $k = 0,1,2,\dots,$ and any measurable function f on Ω , having distributional derivatives $D^\alpha f$ up to the order k , we set

$$m_k(f) = \int_\Omega \sum_{|\alpha|\leqq k} \phi(D^\alpha f(x))\,dx .$$

The corresponding m-space is denoted by $W^k L_\phi(\Omega)$ and called *the Sobolev-Orlicz space* (of the order k). For $k=0$, the notation $L_\phi(\Omega) = W^0 L_\phi(\Omega)$ is used and we speak about *Orlicz space* ; in the latter case it is also reasonable to consider any measurable $\Omega \subset \mathbb{R}^n$. The spaces $L_\phi(\Omega)$ and $W^k L_\phi(\Omega)$ are B-spaces.

Early results concerned interpolation of linear operators in Orlicz spaces as we mentioned before; the restricted conditions upon the behaviour of Young functions generated the spaces in question. It was proved that $L_\phi(\Omega)$ is the interpolation space between $L_{\phi_1}(\Omega)$ and $L_{\phi_2}(\Omega)$ when ϕ_i do not increase too rapidly or slowly and $\phi^{-1} = (\phi_1^{-1})^{1-\theta}(\phi_2^{-1})^{\theta}$, $0 < \theta < 1$; this does not include all Orlicz spaces "between L_{ϕ_1} and L_{ϕ_2}".

In several papers there was applied the K-method to Orlicz spaces (see e.g. [B]). It was proved that the method works in several particular cases, namely,

$$(L_1, L\log^+ L)_{\theta,1} = L(\log^+ L)^{\theta}, \quad 0 < \theta < 1,$$

$$(L_1, L\log^+ L)_{1/q,q} = L_{1q}, \quad 1 \leqq q \leqq \infty,$$

$$(L\log^+ L, L_\infty)_{1-1/p,q} \subsetneqq L_{pq}, \quad 1 < p < \infty, \quad 1 \leqq q \leqq \infty.$$

An abstract method suggested by Gagliardo for L_p-spaces (see [Ga]) was adopted to Orlicz spaces in [G-P].

Let us now introduce some useful concepts.

2.2 Definition A modular m on the m-space X is said to satisfy *the Δ_2-condition* if $m(2x) \leqq cm(x)$ for some $c = c(m) > 0$ and each $x \in X$. We shall write $m \in \Delta_2$ if this is the case.

If $m \in \Delta_2$ it is possible to obtain growth conditions which partly substitute the role of the homogeneity of a norm. The function

$$\lambda \to \sup\{m(\lambda x)/m(x) \ ; \ x \in X, x \neq 0\} \ , \ \lambda > 0 \ ,$$

is then finite and submultiplicative so that

$$m(\lambda x) \leqq C \max(\lambda^{p_1}, \lambda^{p_2}) m(x) \ , \ x \in X \ , \ \lambda > 0 \ ,$$

for some p_1 , p_2 , $C > 0$.

Let $\bar{X} \in C(A)$ and let $h : \langle 0,\infty) \to \langle 0,\infty)$ be a concave function. We define

$$\bar{X}_h = \{x \in \Sigma(\bar{X}) \ ; \ x = \sum_{i \in \mathbb{Z}} x_i (\text{in } \Sigma(\bar{X})) \ , \ x_i \in \Delta(\bar{X}) \ ,$$

and there exists $C = C(h) < \infty$ such that

$$\|\Sigma z_i x_i / h(2^i)\|_{X_1} \leqq C \ ,$$

$$\|\Sigma 2^i z_i x_i / h(2^i)\|_{X_2} \leqq C \ ,$$

for any finite sequence $\{z_i\} \subset \mathbb{Z}\}$

and set $\|x\|_{\bar{X}_h} = \inf_{x = \Sigma x_i} C(h)$.

The space $\bar{X}_h$ is an interpolation space between X_1 and X_2 . When applied to Orlicz spaces it gives that,

for $\phi^{-1} = \phi_1^{-1} h(\phi_2^{-1}/\phi_1^{-1})$, it is $L_\phi \subset (L_{\phi_1}, L_{\phi_2})_h$

provided at least one of the function ϕ_i generates a modular satisfying the Δ_2-condition, and that $(L_{\phi_1}, L_{\phi_2})_h \subset L_\phi$ if both ϕ_i define a modular satisfying the Δ_2-condition and $\sup_{t>0} h(\mu t)/h(t) = \mathcal{O}(\max(1,\mu))$ for $\mu \to 0$, $\mu \to \infty$. The restrictions are substantial, however, they can be removed by introducing another abstract method and a more general concept of interpolation properties.

2.3 <u>Definition</u> Let (X_i, m_i), $i = 1,2,$ be m-spaces. A linear mapping $T : X_1 \to X_2$ is said to be *m-continuous* if $m_2(\gamma Tx) \leqq m_1(x)$, $x \in X$, for some $\gamma > 0$. An m-space (X,m) is said to be the *m-interpolation* space between X_1 and X_2 if any linear mapping which maps X_i m-continuously into X_i, $i = 1,2,$ maps also X m-continuously into itself.

Analogously, one can introduce a concept of an *m-interpolation method.*

Let us notice that every m-continuous mapping is continuous in the usual sense with respect to the norms defined by the corresponding modulars. Conversely, every bounded linear operator is clearly m-continuous with respect to norms.

Now, let (X_i, m_i), $i = 1,2$ be m-spaces and σ a measurable positive function on $(0,\infty)$ such that

$$\int_0^\infty \min(1,t)\sigma(t)\,dt < \infty . \tag{2.1}$$

Define

$$\begin{aligned} L(t,x) &= L(t,x,\bar{X}) \\ &= \inf\{m_1(x_1) + t m_2(x_2) \; ; \; x = x_1 + x_2 \, , \; x_i \in X_i \, , \; i = 1,2\} \, , \end{aligned} \tag{2.2}$$

and

$$m_\sigma(x) = \int_0^\infty L(t,x)\sigma(t)\,dt \, ,$$

and denote by $\bar{X}_\sigma$ the corresponding m-space and by $\|\cdot\|_\sigma$ its norm.

It can easily be proved that $\bar{X}$ is an m-interpolation space between X_1 and X_2 and that $(X_1,X_2) \to \bar{X}$ is an m-interpolation method. Moreover, if X_i are B-spaces then $\bar{X}_\sigma$ is a B-space, as well, and

$$\|T\|_{\bar{X}_\sigma \to \bar{Y}_\sigma} \leqq \max_{i=1,2} \|T\|_{X_i \to Y_i} \quad .$$

Basic properties of spaces obtained are surveyed in following theorems. We shall suppose that functions $\sigma, \sigma_1, \sigma_2, \ldots$ satisfy (2.1) and are locally bounded.

2.4 THEOREM (i) *Let* $\sigma(t) = \mathcal{O}(\sigma_1(t))$, $t \to 0$, *and* $\sigma(t) = \mathcal{O}(\sigma_0(t))$, $t \to \infty$. *Then* $\bar{X}_{\sigma_0} \cap \bar{X}_{\sigma_1} \subset \bar{X}_\sigma$ *with m-continuous embedding.*

(ii) *Let* X_2 *be m-embedded into* X_1 *and* $\sigma_1(t) = \mathcal{O}(\sigma_2(t))$, $t \to 0$. *Then* $\bar{X}_{\sigma_2} \subset \bar{X}_{\sigma_1}$ *with m-continuous embedding.*

2.5 THEOREM (i) *Let* $T : X_i \to Y$ *be m-continuous,* $i = 1,2$, *and* $T : X_1 \to Y$ *be compact. Let* σ *be nonincreasing near* $+\infty$, $\limsup_{t\to\infty} t^2\sigma(t) = \infty$.

If $m_2 \in \Delta_2$ *or* $m_Y \in \Delta_2$ *then* $T : \bar{X}_\sigma \to Y$ *is compact.*

(ii) *Let* $T : Y \to X_i$ *be m-continuous,* $i = 1,2$, *and* $T : Y \to X_1$ *be compact. Let* $\lim_{t\to\infty} \sigma(t) = 0$.

If $m_Y \in \Delta_2$ *or* $m_i \in \Delta_2$, $i = 1,2$, *then* $T : Y \to \bar{X}_\sigma$ *is compact.*

Sketch of the proof (i). Let $\{x_n\} \subset \bar{X}_\sigma$, $\|x_n\|_\sigma < 1$, $x_n = x_{n1} + x_{n2} = x_{n1}(t) + x_{n2}(t)$, and

$$m_1(x_{n1}(t)) + tm_2(x_{n2}(t)) \leqq 2L(t,x) \quad .$$

Let $c_1 = \left(\int_0^\infty \min(1,\tau)\sigma(\tau)\,d\tau \right)^{-1}$ and

$$M = M(t) \geqq \max(1, 2c_1 \max(1,t)) \quad .$$

Then $\{Tx_{n1}\}$ is relatively compact in Y for

$$m_1(x_{n1}/M) \leqq 2\max(1,t)L(x_n/M,1)$$
$$\leqq (2c_1/M)\max(1,t)m_\sigma(x_n) .$$

Let $x_m - x_n = x_{mn1} + x_{mn2} = x_{mn1}(t) + x_{mn2}(t)$ be chosen so that

$$tm_2(2^{-1}x_{mn2}(t)) \leqq 2L(t,2^{-1}(x_m-x_n)) ,$$

$$x_{mni} = x_{mi} - x_{ni} = x_{mi}(t) - x_{ni}(t) , \quad i = 1,2 ,$$

$$x_{m1} + x_{m2} = x_m , \; x_{n1} + x_{n2} = x_n , \; x_{mni} \in X_i , \; i = 1,2 .$$

Then

$$L(t,2^{-1}(x_n-x_m))$$
$$\leqq \left(\int_t^\infty \sigma(\tau)\,d\tau\right)^{-1}\int_t^\infty L(\tau,2^{-1}(x_m-x_n))\sigma(\tau)\,d\tau$$
$$\leqq \left(\int_t^\infty \sigma(\tau)\,d\tau\right)^{-1} m_\sigma(2^{-1}(x_m-x_n)) ,$$

which yields

$$m_2(2^{-1}x_{mn2}(t)) = m_2(2^{-1}(x_{m2}-x_{n2})) \leqq 2\left(t\int_t^\infty \sigma(\tau)\,d\tau\right)^{-1} .$$

The last term tends to 0 if $t \to \infty$, and if $m_2 \in \Delta_2$ then $\|x_{m2} - x_{n2}\|_\sigma \to 0$ for $m,n \to \infty$; if $m_Y \in \Delta_2$ we can use the m-continuity of $T : X_2 \to Y$.

Hence, for each $\varepsilon > 0$, $\|Tx_{m2}(t) - Tx_{n1}(t)\|_Y < \varepsilon$ for sufficiently large m, n, and some $t > 0$ so that $\{Tx_n\}$ is relatively compact in Y .

The proof of (ii) is similar.

A stability of the method can be established, too, nevertheless the proof is rather lengthy and will be therefore almost fully omitted. Details can be found in [K].

We shall suppose that X_1, X_2 are m-spaces, again, and we shall call a function ω satisfying (2.1) *admissible* if it is nonincreasing, differentiable and if the function $t \to t^{1+\varepsilon}\omega(t)$, $t > 0$, is nonincreasing and the function $t \to t^{2-\varepsilon}\omega(t)$, $t > 0$, is nondecreasing for some $\varepsilon > 0$ on $(0,\infty)$.

Let ω_1 and ω_2 satisfy (2.1) and denote $\bar{E} = (E_1,E_2)$, $E_i = (X_1,X_2)_{\omega_i} = \bar{X}_{\omega_i}$, $i = 1,2$.

First of all we present a formula for the L-functional (in the form of two inequalities).

2.6 <u>LEMMA</u> (i) *Let* $C \geq 1$, $t > 0$, $\xi_t > 0$ *and* ω_i, $i = 1,2$ *be admissible. Then there exists* $\tilde{C} > 0$ *such that*

$$L(t,x/8C,\bar{E}) \leq \tilde{C}\left[\int_0^{\xi_t} L(s,x,\bar{X})\omega_1(s)\,ds + t\int_{\xi_t}^{\infty} L(s,x,\bar{X})\omega_2(s)\,ds\right], \quad x \in \Sigma(\bar{X}) .$$

(ii) *Let* ω_i, $i = 1,2,$ *be admissible and let the function* $\omega_3(t) = \omega_1(t)/\omega_2(t)$, $t > 0,$ *be such that* $t \to t^{-\delta}\omega_3(t)$ *is nondecreasing on* $(0,\infty)$ *for some* $\delta > 0$. *Then for each* $C \geq 1$ *there exists* $\tilde{C} > 0$ *such that*

$$\int_0^{\omega_3^{-1}(t)} L(s,x/2C,\bar{X})\omega_1(s)\,ds + t\int_{\omega_3^{-1}(t)}^{\infty} L(s,x/2C,\bar{X})\omega_2(s)\,ds$$

$$\leq \tilde{C}\, L(t,x,\bar{E}), \quad x \in \Sigma(\bar{X}) .$$

2.7 <u>THEOREM</u> *Let* ω_1, ω_2, λ *be admissible and let* $\omega_3 = \omega_1/\omega_2$ *satisfy the condition from the previous lemma. Then there exists an admissible function* θ *such that*

$$(\bar{X}_{\omega_1}, \bar{X}_{\omega_2}) = \bar{X}_\theta \quad .$$

Moreover, if

$$\Lambda(t) = \int_t^\infty \lambda(s)\,ds \ , \quad t > 0 \ ,$$

then $\theta(t) \sim \omega_1(t) \quad \Lambda(\omega_3(t))$.

Sketch of the proof   Let m_λ be the modular in $\bar{E}_\lambda = (\bar{X}_{\omega_1}, \bar{X}_{\omega_2})_\lambda$. Lemma 2.6 gives

$$m_\lambda(x) = \int_0^\infty \lambda(t) \int_0^{\omega_3^{-1}(t)} L(s,x,\bar{X})\omega_1(s)\,ds\,dt$$

$$+ \int_0^\infty t\lambda(t) \int_{\omega_3^{-1}(t)}^\infty L(s,x,\bar{X})\omega_2(s)\,ds\,dt$$

$$= K_1 + K_2 \ .$$

After a change of variables

$$K_1 = \int_0^\infty [\omega_1(t) \int_t^\infty \lambda(\omega_3(s))\omega_3'(s)\,ds] \ L(t,x,\bar{X})\,dt \ ,$$

$$K_2 = \int_0^\infty [\omega_2(t) \int_0^t \lambda(\omega_3(s))\omega_3'(s)\omega_3(s)\,ds] \ L(t,x,\bar{X})\,dt \ .$$

Asymptotic properties of functions ω_i and λ give

$$\omega_1(t) \int_t^\infty \lambda(\omega_3(s))\omega_3'(s)\,ds \sim \omega_2(t) \int_0^t \lambda(\omega_3(s))\omega_3'(s)\omega_3(s)\,ds \ ,$$

i.e.

$$\omega_3(t) \ \Lambda(\omega_3(t)) \sim \int_0^\infty \frac{d}{ds} [-\Lambda(\omega_3(s))]\omega_3(s)\,ds \ .$$

Analogous considerations will prove that the function $\theta(t) = \omega_1(t)\ \Lambda(\omega_3(t))$ is admissible.

§3. CONNECTIONS WITH L_p-SPACES AND SOME APPLICATIONS

First of all, let us show how the method applies to Orlicz spaces.

Let ϕ_1, ϕ_2 be Young functions and $\Omega \subset \mathbb{R}^n$ be measurable. Then for any σ satisfying (2.1) we have

$$(L_{\phi_1}(\Omega), L_{\phi_2}(\Omega))_\sigma = L_\phi(\Omega) ,$$

where

$$\phi(t) \sim \phi_1(t)k(\phi_2(t)/\phi_1(t)) \tag{3.1}$$

and

$$k(t) = \int_0^\infty \min(1,t\tau)\sigma(\tau)d\tau . \tag{3.2}$$

An analogous result can be proved for weighted Orlicz spaces generated by the same Young function.

Let us now turn our attention to connections between L_p and Orlicz spaces. Actually, interpolation methods enable us to derive various properties of Orlicz spaces very easily from those of L_p-spaces. Let ϕ be a Young function, the corresponding modular satisfying the Δ_2-condition. It can be shown that there exist numbers $p_i \in \langle 1,\infty)$, $i = 1,2,$ and a concave function k such that

$$\phi(t) \sim t^{p_0} k(t^{p_1-p_0}) .$$

In fact, this is an easy consequence of the fact that the function $\lambda \to \sup_{t>0} \phi(\lambda t)/\phi(t)$ is submultiplicative (cf. §2 after Def. 2.2) so that

$$\phi(\lambda t) \leqq C \max(t^{q_0}, t^{q_1})\phi(t) \tag{3.3}$$

with some $1 \leqq q_0 \leqq q_1 < \infty$.

For instance, this immediately gives a generalization of the Mikhlin-Lizorkin multiplier theorem (cf. [T]).

3.1 THEOREM *Let* ϕ *be a Young function satisfying* (3.3) *with some* $1 < q_0 \leqq q_1 < \infty$. *Let* $M : \mathbb{R}^n \to \mathbb{R}^n$ *be such that*

$$|D^\alpha M(x)| \leqq C|x|^{-\alpha} , \quad x \neq 0 ,$$

for each multi-index α , $|\alpha| \leqq [1+n/2]$. *Then* M *is a Fourier multiplier in* $L_\phi(\mathbb{R}^n)$, *i.e. the mapping*

$$f \to (F^{-1}M) * f \tag{3.4}$$

is continuous from $L_\phi(\mathbb{R}^n)$ *into itself.*

For the proof, it is sufficient to realize that the mapping (3.4) is m-continuous in each L_p with respect to the modular $f \to \|f\|^p_{L_p}$, $1 < p < \infty$.

Using the well-known technique of Bessel potentials and extension theorems it is easy to prove

3.2 THEOREM *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with Lipschitz boundary or* $\Omega = \mathbb{R}^n$ *and* $k \in \mathbb{N}$. *Let* ϕ_1 *and* ϕ_2 *be Young functions satisfying the condition* (3.3) *with powers greater than* 1 . *Then, for each* σ *satisfying* (2.1) *we have*

$$(W^k L_{\phi_1}(\Omega) , W^k L_{\phi_2}(\Omega))_\sigma = W^k L_\phi(\Omega) ,$$

where ϕ *is given by* (3.1) *and* (3.2).

Let us note that both Sobolev-Orlicz potential spaces with Young functions not satisfying the condition of the last theorem and also Besov-Orlicz spaces (i.e. spaces based on Orlicz spaces instead of on L_p) has not yet been included into the theory as far as the author is informed.

Finally, let us demonstrate the power of interpolation techniques by another example using §2.

Let ϕ be a Young function satisfying the condition (3.3) with $1 < p_1 \leqq p_2 < n$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary or $\Omega = \mathbb{R}^n$. Then $W^1L_\phi(\Omega) = (W^1_{p_1}(\Omega) , W^1_{p_2}(\Omega))_h$ for a suitable concave function h and $\phi^{-1}(t) \sim t^{1/p_1} h(t^{1/p_2-1/p_1})$. According to the well-known Hardy-Littlewood-Sobolev imbedding theorem we have $W^1_{p_i}(\Omega) \subset L_{q_i}(\Omega)$, where $1/q_i = 1/p_i - 1/n$, $i = 1,2$. It follows immediately that $W^1L_\phi(\Omega) \subset L_{\phi*}(\Omega)$ with $(\phi*)^{-1}(t) \sim t^{1/p_1-1/n} h(t^{1/p_2-1/p_1}) = t^{-1/n}\phi^{-1}(t)$. Cf. [D-T].

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