A REMARK ON FULLY NONLINEAR, CONCAVE ELLIPTIC EQUATIONS

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0. INTRODUCTION AND STATEMENT OF THE RESULT

In this note we shall be concerned with fully nonlinear elliptic equations of second order of the form

(1) $$F(D^2u) = g(x)$$

for solutions $u(x) \in C^4(\Omega)$, defined in an open subset Ω of $\mathbb{R}^n$ $(n \geqslant 2)$. Here $F \in C^2(\mathbb{R}^{n\times n})$ and $g \in C^2(\Omega)$, with $\mathbb{R}^{n\times n}$ denoting the space of symmetric $n\times n$ matrices $r = [r_{ij}]$. We shall impose the following *assumptions*:

(i) F is uniformly elliptic for u, that is, there exist positive constants λ,Λ such that

$$\lambda|\xi|^2 \leqslant F_{r_{ij}}(D^2u)\xi_i\xi_j \leqslant \Lambda|\xi|^2$$

for all $\xi \in \mathbb{R}^n$.

(ii) F is a concave function on some convex set containing the range of D^2u, so that

$$F_{r_{ij}r_{k\ell}}\eta_{ij}\eta_{k\ell} \leqslant 0$$

for all $\eta = [\eta_{ij}] \in \mathbb{R}^{n\times n}$.

(iii) In addition

$$|g|_{2;\Omega} \leqslant K, \qquad |u|_{2;\Omega} \leqslant M$$

for some constants K,M.

We can now state the result as

THEOREM. *For any* $\Omega' \subset\subset \Omega$, *the Hölder estimate*

$$[D^2u]_{\alpha;\Omega'} \leqslant C$$

holds, where α *depends only on* n , λ , Λ , K , M , *and* C *depends also on* $\mathrm{dist}(\Omega',\partial\Omega)$.

These estimates have been established by Evans [2] and Krylov [7]. They are included in Gilbarg and Trudinger [4] as Theorem 17.14 for equations of the general form

$$F(X,u,Du,D^2u) = 0 . \qquad (2)$$

The proof has been simplified by Trudinger [8],[9]; the main ingredients here are a weak Harnack inequality for non-divergence equations essentially due to Krylov and Safonov (see [8]), and a result from matrix theory of Motzkin and Wasow.

The purpose of the present note is to illustrate a somewhat different approach. The main result is that the a priori estimates can be proved directly without invoking the non-constructive lemma of Motzkin and Wasow. At the Miniconference on Nonlinear Analysis we used Green's function techniques, developed by Hildebrandt and Widman, which incorporate a Giaquinta and Guisti-type lemma (see e.g. [3],[5],[6]). However, by employing divergence techniques, the Hölder estimates also depend on bounds for the second derivatives of F . In order to include the *Bellman equation*, we prefer to present the ideas in the context of non-divergence methods, close to Trudinger's approach. Our approach has also been inspired by Caffarelli's work [1]. We finally mention the other important example, namely the *Monge-Ampère equation*, which can be treated in a possibly more satisfactory manner via Green's function techniques. This, and also the general case (2), will be developed in a forthcoming paper.

ACKNOWLEDGEMENT. I am indebted to Professor E. Heinz, the starting point of this work basically being discussions with him, and I gratefully acknowledge the support of the Department of Mathematics, IAS, at the Australian National University, Canbèrra, where the research was carried out. It is a pleasure to thank Professors D. Robinson, L. Simon and N. Trudinger for their warm hospitality during my stay in Canberra.

1. PROOF OF THE THEOREM

Let $\gamma \in \mathbb{R}^n$ be a directional vector. By differentiating (1) twice with respect to γ, we obtain

$$F_{r_{ij}} D_{ij} D_\gamma u = D_\gamma g ,$$

$$F_{r_{ij}} D_{ij} D_{\gamma\gamma} u + F_{r_{ij} r_{k\ell}} D_{ij} D_\gamma u \, D_{k\ell} D_\gamma u = D_{\gamma\gamma} g ,$$

so that

$$F_{r_{ij}} D_{ij} D_{\gamma\gamma} u \geqslant D_{\gamma\gamma} g ,$$

by the concavity of F. Let $B_{2R} = B_{2R}(x_0) \subset \Omega$, $0 < R \leqslant 1$. The weak Harnack inequality [4], Theorem 9.22, will be applied to $M_{\gamma,2R} - D_{\gamma\gamma} u$, where

$$M_{\gamma,R} = \sup_{B_R} D_{\gamma\gamma} u ,$$

to yield

$$(3) \qquad \left(⨍_{B_R} (M_{\gamma,2R} - D_{\gamma\gamma} u)^p \, dx \right)^{1/p} \leqslant C\{M_{\gamma,2R} - M_{\gamma,R} + R\|D_{\gamma\gamma} g\|_{L^n(B_{2R})}\} ,$$

where p and C are positive constants depending only on n, Λ/λ. Here

$$⨍_{B_R} v \, dx = \frac{1}{|B_R|} \int_{B_R} v \, dx .$$

Denote by e_k $(k = 1,\dots,n)$ the standard unit vectors in $\mathbb{R}$ and let

$$\Gamma = \{e_k, (e_k \pm e_\ell)/\sqrt{2} ; \quad k,\ell = 1,\dots,n, \ k \neq \ell\} .$$

On summing (3) over $\gamma \in \Gamma$, we obtain the following

LEMMA 1. *There exists a* $y_0 \in B_R$ *, and there is a constant* $C > 0$ *depending only on* n *,* Λ/λ *,* K *and* M *, for which the inequalities*

$$\sup_{B_{2R}}(D_{\gamma\gamma}u - D_{\gamma\gamma}u(y_0)) \leq C\{w(2R) - w(R) + R^2\} = Cw^*(R) \tag{4}$$

hold for any $\gamma \in \Gamma$ *. Here*

$$w(R) = \sum_{\gamma\in\Gamma} \operatorname{osc}_{B_R} D_{\gamma\gamma}u$$

and, obviously,

$$w^*(R) = \{w(2R) - w(R) + R^2\} .$$

We proceed to derive (4) for all unit vectors $\gamma \in \mathbb{R}^n$: First note that

$$\left(\fint_{B_R} |D_{\gamma\gamma}u - D_{\gamma\gamma}u(y_0)|^p\, dx\right)^{1/p} \leq Cw^*(R) \tag{5}$$

for $\gamma \in \Gamma$. Hence we have, for $i,j = 1,\ldots,n$,

$$\left(\fint_{B_R} |D_{ij}u - D_{ij}u(y_0)|^p\, dx\right)^{1/p} \leq Cw^*(R) ,$$

and the inequality (5) holds therefore for all unit vectors $\gamma \in \mathbb{R}^n$. The application of the local maximum principle [4], Theorem 9.20, to $D_{\gamma\gamma}u - D_{\gamma\gamma}u(y_0)$ yields

LEMMA 2. *The inequalities*

$$\sup_{B_{R/2}} (D_{\gamma\gamma}u - D_{\gamma\gamma}u(y_0)) \leq C\{(\fint_{B_R} |D_{\gamma\gamma}u - D_{\gamma\gamma}u(y_0)|^p\, dx)^{1/p} + R\|D_{\gamma\gamma}g\|_{L^n(B_R)}\} \leq Cw^*(R)$$

hold for all directions $\gamma \in \mathbb{R}^n$.

Now we can prove

LEMMA 3. *There exist* n *orthogonal directions* $\gamma_1,\ldots,\gamma_n$ *such that*

$$\operatorname*{osc}_{B_{R/2}} D_{\gamma_k\gamma_k} u \leqslant Cw^*(R) .$$

Proof. Using the concavity of F, we see that

$$g(x) - g(y_0) = F(D^2u(x)) - F(D^2u(y_0))$$

$$\leqslant F_{r_{ij}}(D^2u(y_0))(D_{ij}u(x) - D_{ij}u(y_0))$$

for $x \in B_{R/2}$. Hence diagonalizing $[F_{r_{ij}}(D^2u(y_0))]$, i.e., writing

$$F_{r_{ij}}(D^2u(y_0)) = \sum_{k=1}^{n} \lambda_k \gamma_{ik} \gamma_{jk} ,$$

it follows that

$$g(x) - g(y_0) \leqslant \sum_{k=1}^{n} \lambda_k (D_{\gamma_k\gamma_k} u(x) - D_{\gamma_k\gamma_k} u(y_0)) ,$$

where $\gamma_k = (\gamma_{1k},\ldots,\gamma_{nk})$. Thus, for $\ell = 1,\ldots,n$,

$$\lambda_\ell (D_{\gamma_\ell\gamma_\ell} u(y_0) - D_{\gamma_\ell\gamma_\ell} u(x)) \leqslant \sum_{k\neq\ell} \lambda_k (D_{\gamma_k\gamma_k} u(x) - D_{\gamma_k\gamma_k} u(y_0)) + g(y_0) - g(x)$$

$$\leqslant Cw^*(R) ,$$

and the statement of the lemma follows.

On combining Lemmata 2 and 3, we obtain the inequality

$$w(R/2) \leqslant Cw^*(R) = C\{W(2R) - w(R) + R^2\} ,$$

and therefore

$$w(R/2) \leqslant \delta w(2R) + CR^2 ,$$

where $0 < \delta < 1$. The theorem can now be deduced from the calculus lemma 8.23 of [4].

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