EXISTENCE OF WILLMORE SURFACES

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For compact surfaces Σ embedded in $\mathbb{R}^n$, the Willmore functional is defined by

$$F(\Sigma) = \frac{1}{2}\int_{\Sigma} |\underline{\underline{H}}^2|$$

where the integration is with respect to ordinary 2-dimensional area measure, and H is the mean curvature vector of Σ (in case $n = 3$ we have $|\underline{\underline{H}}| = |\kappa_1 + \kappa_2|$, where κ_1, κ_2 are principal curvatures of Σ). In particular $F(S^2) = 8\pi$.

For surfaces Σ without boundary we have the important fact that $F(\Sigma)$ is invariant under conformal transformations of $\mathbb{R}^n$; thus if $\tilde{\Sigma} \subset \mathbb{R}^n$ is the image of Σ under an isometry or a scaling ($x \mapsto \lambda x$, $\lambda > 0$) or an inversion in a sphere with centre not in Σ (e.g. $x \mapsto x/|x|^2$ if $0 \notin \Sigma$) then

(1) $$F(\Sigma) = F(\tilde{\Sigma}) .$$

(See [WJ], [LY], [W] for general discussion.)

For each genus $g = 0, 1, 2, \ldots$ and each $n \geq 3$ we let

$$\beta_g^n = \inf F(\Sigma) ,$$

where the inf is taken over compact genus g surfaces without boundary embedded in $\mathbb{R}^n$. We note some inequalities concerning the numbers β_g^n. Firstly we claim

$$8\pi \le \beta_g^n < 16\pi \tag{2}$$

with equality on the left if and only if $g = 0$ (indeed $\mathcal{F}(\Sigma) \ge 8\pi$, with equality if and only if Σ is a round sphere - see the simple argument of [W]). The right-hand-side inequality in (2) was pointed out to the author by Pinkall [P] and (independently) by Kusner [K]. Both these authors noted that a simple area comparison argument shows that the genus g minimal surfaces Σ_g constructed in S^3 by Lawson [L] have area $< 8\pi$. It then follows (using the appropriate conformal invariance of the Willmore functional between general Riemannian 3-manifolds) that $\mathcal{F}(\tilde{\Sigma}_g) < 16\pi$, where $\tilde{\Sigma}_g$ is the stereographic image of Σ_g in $\mathbb{R}^3$. Another inequality concerning the numbers β_g^n is as follows: if $e_g = \beta_g^n - 8\pi$ ($= \beta_g^n - \beta_0^n$) then

$$e_g \le \sum_{j=1}^{q} e_{\ell_j} \tag{3}$$

for any integers $q \ge 2$ and $\ell_1, \ldots, \ell_q$ with $\sum_{j=1}^{q} \ell_j = g$. To see this we simply note that by taking a genus ℓ_j surface $\Sigma_k^{(j)}$ with $\mathcal{F}(\Sigma_k^{(j)}) \le \beta_{\ell_j}^n + 1/k$ and by making an inversion of $\Sigma_k^{(j)}$ in a suitable sphere we obtain $\tilde{\Sigma}_k^{(j)}$ with $\mathcal{F}(\tilde{\Sigma}_k^{(j)}) \le \beta_{\ell_j}^n + 1/k$ which is C^2 close to S^2 except near some preassigned spherical cap of S^2; near this spherical cap $\tilde{\Sigma}_k^{(j)}$ looks like a spherical cap with ℓ_j handles. Then by cutting out these spherical caps with handles and sewing them back into a copy of S^2 with q spherical caps removed, we get a genus g surface $\tilde{\Sigma}_k$ with

$$F(\tilde{\Sigma}_k) \le 8\pi + \sum_{j=1}^{q} e_{\ell_j} + \varepsilon_k , \quad \varepsilon_k \downarrow 0 \text{ as } k \to \infty ,$$

and hence (3) is established by letting $k \to \infty$.

It is of course tempting to conjecture that the stereographic image $\tilde{\Sigma}_g$ of the Lawson surfaces $\Sigma_g \subset S^3$ (mentioned above) actually minimizes F (so that we would have $F(\tilde{\Sigma}_g) = \beta_g^3$) . The evidence for this in case $g = 1$ seems to be building up (see [LY], [BR]) but as yet it has not been established.

One of the main results of this paper is that $\forall\, n \ge 3$ there *exists a* compact embedded real analytic torus T in $\mathbb{R}^n$ with $F(T) = \beta_1^n$. For arbitrary genus $g \ge 2$ the result is almost as clear-cut; we prove that *there is a genus* g *embedded real analytic surface* Σ *in* $\mathbb{R}^n$ *with* $F(\Sigma) = \beta_g^n$ *unless equality holds in* (3) *for some choice of* $q \ge 2$, $\ell_1, \ldots, \ell_q$, $\sum_{j=1}^{q} \ell_j = g$, *in which case we can construct by the cut-and-paste procedure used to establish* (3) *a minimizing sequence explicitly in terms of lower genus minimizers for* F . It is not clear at the moment whether or not equality *can* hold in (3); certainly since $\beta_g^n < 16\pi$ by (2), it is clear that equality *cannot* hold if $\beta_\ell^n \ge 12\pi \;\forall\, \ell = 1, \ldots, g-1$. (At the moment it is not known whether or not $\beta_\ell^n \ge 12\pi$ though.)

The proof of the above existence results is outlined in §§1 - 4 below. In §5 we give some existence results for immersions minimizing the Willmore functional in a general Riemannian manifold N . More detailed proofs than those sketched in §§1 - 4 will be given in [SL]; however we do here try to give enough detail so that the reader already familiar with the basic background from Geometry/PDE will be able to complete most of the arguments needed. For convenience we take $n = 3$ in the discussion of §§1-4; the generalizations to $n > 3$ are straightforward. We henceforth set $\beta_g = \beta_g^3$.

§1. Lemmas valid for arbitrary compact $\Sigma \subset \mathbb{R}^3$

All lemmas in this section are proved (in $\mathbb{R}^n$) in [SL]; here we merely make some remarks indicating the general method of proof. In each of the lemmas c denotes a fixed constant independent of Σ. First we state two lemmas giving bounds on $\operatorname{diam}\Sigma$ for compact Σ embedded in $\mathbb{R}^3$.

Lemma 1. *If $\partial\Sigma = \emptyset$ and Σ is connected, then*

$$\sqrt{2|\Sigma|/F(\Sigma)} \le \operatorname{diam}\Sigma \le c\sqrt{|\Sigma|F(\Sigma)} .$$

Here $|\Sigma|$ denotes the area of Σ.

Lemma 2. *In the general case when $\partial\Sigma$ is an arbitrary finite union of smooth disjoint Jordan curves, then, still assuming Σ is connected,*

$$\operatorname{diam}\Sigma \le c\left(\int_\Sigma |A| + |\partial\Sigma|\right) ,$$

where $|A|$ is the length of the second fundamental form of Σ and $|\partial\Sigma|$ denotes the length of $\partial\Sigma$.

In the third lemma we give a result which can be viewed as a variant of a result of Li and Yau (see [LY], Theorem 6]).

Lemma 3. *Suppose Σ is a compact surface without boundary, and $\Sigma \cap B_\rho(0)$ contains two components Σ_1, Σ_2 with $\Sigma_j \cap B_{\theta\rho}(0) \neq \emptyset$ and $|\partial\Sigma_j| \le \beta\rho$, $j = 1, 2$, where $\theta \in (0,1)$ and $\beta > 0$. Then*

$$F(\Sigma) \geq 16\pi - c\beta\theta ,$$

where c *is a fixed constant independent of* Σ , β , θ .

In the proofs of Lemmas 1, 3 we use the first variation identity

$$(*) \qquad \int_\Sigma \operatorname{div}_\Sigma X = -\int_\Sigma X \cdot \nu H ,$$

for any C^1 vector field $X = (X^1, X^2, X^3)$ defined in a neighbourhood of Σ , where ν is a smooth unit normal for Σ . Indeed to prove the equality on the left in Lemma 1, we simply choose $X(x) = x - y$, where y is a fixed element of Σ , and note that in this case $\operatorname{div}_\Sigma X \equiv 2$ on Σ ; then the required inequality follows by using the Hölder inequality on the right side. The proof of the inequality on the right side of Lemma 1 involves a more elaborate use of (*). This time, by taking $X(x) = |x - y|^{-2}\varphi(|x - y|)(x - y)$ with a suitable choice of scalar function φ (approximating the characteristic function of the interval $(-\infty , \rho)$), one gets the identity

$$(**) \qquad \pi + \int_{\Sigma \cap B_\rho(y)} \left(\frac{1}{4} H(x) - \frac{\nu(x) \cdot (x - y)}{|x - y|^2}\right)^2 = \rho^{-2}|\Sigma \cap B_\rho(y)| + \frac{1}{8} F(\Sigma \cap B_\rho(y)) .$$

By selecting suitable disjoint balls $B_\rho(y)$ and summing, the inequality on the right of Lemma 1 follows. (For details see [SL].)

To prove Lemma 3 we note first that there is a version of (**) valid in case $\partial\Sigma \neq 0$ and $\rho \to \infty$; viz. we have the identity

$$(***)\quad \pi + \int_\Sigma \left(\frac{1}{4} H(x) - \frac{\nu(x)\cdot(x-y)}{|x-y|^2}\right)^2 = \int_{\partial\Sigma} \eta \cdot \frac{x-y}{|x-y|^2} + \frac{1}{8} F(\Sigma) ,$$

where $y \in \Sigma$ and η is the normal of $\partial\Sigma$ tangent to Σ and pointing into Σ. We actually apply this identity separately to the two components $\tilde{\Sigma}_1$, $\tilde{\Sigma}_2$ obtained as the image of Σ_1, Σ_2 (as in the statement of Lemma 3) under an inversion in the sphere $B_\rho(0)$. (By a slight perturbation we may assume that $0 \notin \Sigma$.) For the point y we take points y_1, y_2 in $\tilde{\Sigma}_1$, $\tilde{\Sigma}_2$ repsectively with $|y_j| \geq (\theta\rho)^{-1}$. (Such y_j exist because $\Sigma_j \cap B_{\theta\rho}(0) \neq \emptyset$, $j = 1, 2$.)

Concerning the proof of Lemma 2, we note that it is enough to prove

$$\operatorname{diam} \Sigma \leq c \int_\Sigma |A| ,$$

subject to the assumption that $|\partial\Sigma| << \operatorname{diam} \Sigma$. For the proof of this (which is elementary), we refer to [SL] again.

§2. Approximate Graphical Decomposition and Biharmonic Comparison.

Here, as in the previous section, we continue to work with arbitrary compact smooth surfaces Σ embedded in $\mathbb{R}^3$. The following lemma asserts that we can decompose a surface Σ into a union of discs, each of which is well approximated by a graph, in balls where the integral of the length of second fundamental form (i.e. $\int|A|$) is small. In this lemma B_ρ is a ball of radius $\rho > 0$ (ρ given) in $\mathbb{R}^3$ with centre 0, and for given $\sigma \in (3\rho/4, \rho)$ and a given surface $\Sigma \subset \mathbb{R}^3$ we let $(\Sigma \cap B_\sigma)^*$ denote the components of $\Sigma \cap B_\sigma$ which have non-empty intersection with $B_{\rho/2}$. Here and subsequently we adopt the convention that if L is a plane in $\mathbb{R}^3$ with unit normal ν and if u is C^2 on some domain $\Omega \subset L$, then graph $u = \{x + u(x)\nu : x \in \Omega\}$.

Lemma 4. *There is* $\varepsilon_0 > 0$ *(independent of* Σ, ρ *) such that if* $\varepsilon < \varepsilon_0$, *if* $\partial\Sigma \cap B_\rho = \emptyset$, *and if* $\int_{\Sigma \cap B_\rho} |A| \le \varepsilon\rho$, *then the following holds:*

There is a set $S \subset (3\rho/4, \rho)$ *of Lebesgue measure* $\ge \rho/8$ *such that if* $\sigma \in S$ *then* Σ *intersects* ∂B_σ *transversely, and*

$$(\Sigma \cap B_\sigma)^* = \bigcup_{j=1}^{N} D_j ,$$

where $N \ge 1$ *and* $D_1, \ldots, D_N$ *are (topologically) discs, with each* ∂D_j *a component of* $\Sigma \cap \partial B_\sigma$.

Furthermore S *can be selected so that corresponding to each such disc* D_j *there is a plane* L_j *containing* 0, *a connected* C^∞

domain $\Omega_j \subset L_j$ *and a function* $u_j \in C^\infty(\bar{\Omega}_j)$ *such that each component of* $\partial\Omega_j$ *, except possibly for the outmost component, is a round circle, and such that*

$$\text{graph } u_j \subset D_j \text{ , Lip } u_j \le c\varepsilon^{\frac{1}{4}} \text{ , } D_j \sim \text{graph } u_j = P_j \text{ ,}$$

where P_j *is a union of discs* $d_j^{(i)}$ *with* $\bar{d}_j^{(i)} \subset \Sigma \sim \partial B_\sigma$ *and*

$$\sum_{i,j} \text{diam } d_j^{(i)} \le c\varepsilon^{\frac{1}{2}}\rho \text{ , } \sum_{i,j} \text{area } d_j^{(i)} \le c\varepsilon\rho^2 \text{ .}$$

(Note in particular this means that if Γ_j is the outermost component of $\partial\Omega_j$, then $\partial D_j = \text{graph}(u_j|\Gamma_j) \subset \Sigma \cap \partial B_\sigma$.)

Roughly speaking the last part of the lemma says that each of the discs D_j can be expressed as a union of graphs with small gradient, together with some "pimples" P_j , the sum of diameters of the pimples being small.

The proof of Lemma 4 makes use of Lemma 2 of §1, the fact that $|A|^2 = \sum_{j=1}^{3} |\nabla\nu^i|^2$ ($\nu = (\nu^1, \nu^2, \nu^3)$ is a smooth unit normal for Σ), and the co-area formula (applied in a manner analogous to [SS, §3]). Here we also need to use the Gauss-Bonnet theorem to eliminate the possibility that instead of *discs* D_j we might get annular regions looking like almost flat discs joined by thin necks. For the details we refer to [SL].

Next we derive an important inequality involving biharmonic functions w .

Lemma 5. *Let* $\Sigma \subset \mathbb{R}^3$ *be smooth embedded,* $\xi \in \mathbb{R}^3$, L *a plane containing* ξ, $u \in C^\infty(\bar{\Omega})$, *where* $\Omega = L \cap B_\rho(\xi) \sim B_\sigma(\xi)$ *for some* $\rho > \sigma > 0$, *and where*

$$\text{graph } u \subset \Sigma \ , \ \text{Lip } u \le 1 \ .$$

Also let $w \in C^\infty(L \cap \bar{B}_\rho(\xi))$ *satisfy*

$$\begin{cases} \Delta^2 w = 0 \quad \text{on} \quad L \cap B_\rho(\xi) \\ w = u \ , \ Dw = Du \quad \text{on} \quad L \cap \partial B_\rho(\xi) \ . \end{cases}$$

Then

$$\int_{L \cap B_\rho(\xi)} |D^2 w|^2 \le c\, \rho \int_\Gamma |A|^2 d\mathcal{H}^1 \ ,$$

where $\Gamma = \text{graph}(u \,|\, L \cap B_\rho(\xi))$, A *is the second fundamental form of* Σ, *and* $\mathcal{H}^1$ *is 1-dimensional Hausdorff measure (arc-length measure) on* Γ; c *is a fixed constant independent of* Σ, ρ, σ.

Remark. Of course there *exists* a w as above, because u is C^∞, so we can use the existence and regularity theory for the Dirichlet problem; the solution w is also clearly *unique*.

Lemma 5 is rather easy to prove once we recall that the function w minimizes $\int_\Omega |D^2 w|^2$ subject to the given boundary conditions. Then (after rescaling so that $\rho = 1$) by the appropriate Sobolev - space trace lemma (see e.g. [TF, 26.5, 26.9 with $m = 2$]), we have, with $G = L \cap B_1(\xi)$ and $\gamma = \partial G = L \cap \partial B_1(\xi)$,

$$\int_G |D^2w|^2 \le c(|u|^2_{H^{3/2}(\gamma)} + |Du|^2_{H^{1/2}(\gamma)})$$

$$\le c \int_\gamma (u^2 + |Du|^2 + |D^2u|^2) \ .$$

Applying the same to $w - \ell$ (ℓ any linear function + constant) we get

$$\int_G |D^2w|^2 \le c \int_\gamma ((u - \ell)^2 + (Du - D\ell)^2 + |D^2u|^2) \ .$$

By selecting ℓ suitably we can then establish that the first two terms on the right are dominated by a fixed multiple of the third. Thus

$$\int_G |D^2w|^2 \le c \int_\gamma |D^2u|^2 \ .$$

Since $|Du| \le 1$ on γ we also have $|D^2u|^2(x) \le c|A|^2(X)$, where X is the point of graph u corresponding to $x \in \gamma$, hence Lemma 5 follows.

§3. Regularity of Measure Theoretic Limits of Minimizing Sequences.

A sequence of compact embedded surfaces $\Sigma_k \subset \mathbb{R}^3$ (with $\partial\Sigma_k = 0$) is called a genus g minimizing sequence for F if genus $\Sigma_k = g \ \forall k$ and if

$$F(\Sigma_k) \le \beta_g + \varepsilon_k , \qquad \varepsilon_k \downarrow 0 .$$

By translation and scaling we can (and we shall) assume

$$0 \in \Sigma_k , \qquad |\Sigma_k| = 1 .$$

Notice that then by Lemma 1 we have a fixed constant $c > 0$ such that

$$(*) \qquad c^{-1} \le \operatorname{diam} \Sigma_k \le c .$$

Our main result here is the following:

Theorem 1. *Given any genus* g *minimizing sequence* Σ_k *as above, there is a subsequence* $\Sigma_{k'}$ *and a compact embedded real analytic surface* Σ *such that* $\Sigma_{k'} \to \Sigma$ *both in the Hausdorff distance sense and in the sense that*

$$\int_{\Sigma_{k'}} f \to \int_{\Sigma} f$$

for each fixed continuous f *on* $\mathbb{R}^3$ *. This* Σ *has genus* $g_0 \le g$ *, and* Σ *minimizes* F *relative to all compact smooth embedded genus* g_0 *surfaces* $\tilde{\Sigma} \subset \mathbb{R}^3$ *.*

Remark. It can of course happen that $g_0 = 0$ (and Σ is a round sphere) even if $g \geq 1$. This is a problem in proving existence of the required genus 1 (or higher genus) minima which we show how to overcome in the next section.

To give an outline of the proof, first note that since $|\Sigma_k| = 1$ we may choose a subsequence $\Sigma_{k'}$ such that the corresponding sequence of measures $\mu_{k'}$, given by $\mu_{k'}(A) = |A \cap \Sigma_{k'}|$ for Borel sets $A \subset \mathbb{R}^3$, converges to a Borel measure μ of compact support. Thus

$$\int_{\Sigma_{k'}} f \to \int_{\mathbb{R}^3} f \, d\mu$$

for each fixed continuous function f in $\mathbb{R}^3$, and (by (*)) the support of μ is compact.

In spt μ (the support of μ) we say ξ is a *bad* point (relative to a preassigned number $\varepsilon > 0$) if

$$\lim_{\rho \downarrow 0} \left(\liminf_{k' \to \infty} \int_{\Sigma_{k'} \cap B_\rho(\xi)} |A_{k'}|^2\right) > \varepsilon^2,$$

where A_k is the second fundamental form of Σ_k. Evidently, since $\frac{1}{2}\int_{\Sigma_k} |A_k|^2 = F(\Sigma_k) - 2\pi(2 - 2g)$, by the Gauss-Bonnet theorem, $\frac{1}{2}\int_{\Sigma_k} |A_k|^2$ is bounded and an obvious argument then shows that there are at most finitely many bad points for each $\varepsilon > 0$. By taking a subsequence again (denoted subsequently simply by Σ_k) we can actually assume

$$\lim_{\rho \to 0} \left(\lim_{k\to\infty} \int_{\Sigma_k \cap B_\rho(\xi)} |A_k|^2\right) > \varepsilon^2$$

for the finitely many bad points $\xi = \xi_1, \ldots, \xi_P$ $(P = P(\varepsilon))$.

On the other hand for any $\xi \in \operatorname{spt}\mu \sim \{\xi_1, \ldots, \xi_P\}$ we can select $\rho(\xi) > 0$ such that for $\rho \le \rho(\xi)$ Lemma 4 is applicable to Σ_k in $B_\rho(\xi)$ for infinitely many k. At the same time we have, since $\beta_g < 16\pi$, that we can apply Lemma 3 to deduce that for large enough k and for small enough θ (θ fixed, independent of k, ε, ξ), only one of the discs $D_j^{(k)}$, say $D_1^{(k)}$, given by applying Lemma 4 can intersect the ball $B_{\theta\rho}(\xi_0)$. For ε small enough (which we subsequently assume) it is then clear there is a plane L_k containing ξ and a set $T_k \subset (\frac{1}{2}\theta\rho, \theta\rho)$ with $|T_k| \ge \frac{\theta\rho}{4}$ and such that, for $\rho_k \in T_k$, there is a connected domain $\Omega_k \subset L_k$, with each component of $\partial\Omega_k$ circular and with outermost component $= L_k \cap \partial B_{\rho_k}(\xi)$, and a $C^\infty(\bar{\Omega}_k)$ function u_k with

(1) $\operatorname{graph} u_k \subset D^{(k)}$, $\rho^{-1}|u_k| + \operatorname{Lip} u_k \le c\varepsilon^{\frac{1}{4}}$, $D^{(k)} \sim \operatorname{graph} u_k = P_k$,

where P_k is a union of discs d_k^i with $\bar{d}_k^i \subset D^{(k)} \sim \Gamma_k$, $\Gamma_k = \operatorname{graph}(u_k \mid L_k \cap \partial B_{\rho_k}(\xi))$, and where $D^{(k)}$ is the intersection of the disc $D_1^{(k)}$ with the truncated cylinder $\{x + \lambda\nu_k : \lambda \in (-1, 1), x \in L_k \cap B_{\rho_k}(\xi)\}$ (ν_k = normal of L_k). (Notice that automatically $D^{(k)}$ is a topological disc by (1).)

Then we can apply Lemma 5 to obtain a biharmonic function w_k such that

$$\int_{L_k \cap B_{\rho_k}(\xi)} |D^2 w_k|^2 \le c \int_{\Gamma_k} |A_k|^2 .$$

Letting $\tilde{A}_k$ be the second fundamental form of graph w_k, we then in particular have

$$\int_{\text{graph } w_k} |\tilde{A}_k|^2 \le c \int_{\Gamma_k} |A_k|^2 .$$

On the other hand Σ_k is a minimizing sequence for the functional $\mathcal{F}_1(\Sigma) = \frac{1}{2}\int_\Sigma |A|^2$, and hence the $C^{1,1}$ composite surface $\tilde{\Sigma}_k = (\Sigma_k \sim D^{(k)}) \cup \text{graph } w_k$ satisfies

$$\mathcal{F}(\tilde{\Sigma}_k) \ge \mathcal{F}(\Sigma_k) - \varepsilon_k \quad , \quad \varepsilon_k \downarrow 0 ,$$

so that

$$\int_{\text{graph } w_k} |\tilde{A}_k|^2 \ge \int_{D^{(k)}} |A_k|^2 - \varepsilon_k .$$

Thus we conclude that for infinitely many k

$$\int_{\Sigma_k \cap B_{\rho_k}(\xi)} |A_k|^2 \le c \int_{\partial D^{(k)}} |A_k|^2 + \delta_k ,$$

where $\delta_k \downarrow 0$. Since ρ_k was selected arbitrarily from the set T_k of Lebesgue measure $\ge \frac{1}{4}\theta\rho$ we can arrange that

$$\int_{\partial D^{(k)}} |A_k|^2 \le 4 \int_{\Sigma_k \cap B_{\theta\rho}(\xi) \sim B_{\theta\rho/2}(\xi)} |A_k|^2 ,$$

so that in fact we get, for $\rho \leq \theta\rho(\xi)$ arbitrary, and for infinitely many k (depending on ρ)

$$\int_{\Sigma_k \cap B_{\rho/2}(\xi)} |A_k|^2 \leq c \int_{\Sigma_k \cap B_\rho(\xi) \sim B_{\rho/2}(\xi)} |A_k|^2 + \delta_k ,$$

where $\delta_k \downarrow 0$.

We also need to make the remark that $\rho(\xi)$ above merely had to be chosen so that $\int_{\Sigma_k \cap B_{\rho(\xi)}(\xi)} |A_k|^2 \leq \varepsilon$ for infinitely many k . In particular this means that if $\xi_0 \in \operatorname{spt} \mu \sim \{\xi_1, \ldots, \xi_P\}$, then we may take $\rho(\xi) = \rho(\xi_0)/2$ for any $\xi \in \operatorname{spt} \mu \cap B_{\rho(\xi_0)/2}(\xi_0)$. Thus we see that the following is established:

If we let

$$\psi(\xi, \rho) = \liminf_{k \to \infty} \int_{\Sigma_k \cap B_\rho(\xi)} |A_k|^2 ,$$

then we have for all $\xi_0 \in \operatorname{spt} \mu \sim \{\xi_1, \ldots, \xi_P\}$ and all $\rho \leq \theta\rho(\xi_0)/2$, and all $\xi \in \operatorname{spt} \mu \cap B_{\rho(\xi_0)/2}(\xi_0)$ that

$$\psi(\rho/2, \xi) \leq \gamma\, \psi(\rho, \xi)$$

for some fixed $\gamma \in (0, 1)$, independent of ρ, ξ . Thus

$$(2) \qquad \psi(\rho, \xi) \leq c(\rho/\rho_0)^\alpha \psi(\rho_0, \xi) \leq c(\rho/\rho_0)^\alpha \psi(\rho(\xi_0), \xi_0)$$

for some $\alpha \in (0, 1)$ and for all such $\rho, \xi,$ where $\rho_0 = \theta\rho(\xi_0)/2$.

Henceforth $\xi_0 \in \operatorname{spt} \mu \sim \{\xi_1, \ldots, \xi_P\}$ is fixed and we take

$\xi \in \operatorname{spt} \mu \cap B_{\rho(\xi_0)/2}(\xi_0)$ and $\rho \in (0, \theta\rho(\xi_0)/2)$, and let

$$\alpha_k = \alpha_k(\rho, \xi) = \int_{\Sigma_k \cap B_\rho(\xi)} |A_k|^2 \quad (<\varepsilon) ,$$

and let L_k, Ω_k, u_k, ρ_k, d_k^i be as in (1). Also let $\bar{u}_k$ denote an extension of u_k to all of L_k such that

$$\text{(3)} \qquad \rho^{-1} \sup |\bar{u}_k| + \operatorname{Lip} \bar{u}_k \le c\,\varepsilon \quad .$$

Since $\Sigma \operatorname{diam} d_k^i \le c\sqrt{\alpha_k}\rho$ (by Lemma 4), Poincaré's inequality gives

$$\inf_{\lambda \in \mathbb{R}} \int_{\Omega_k} |f - \lambda|^2 \le c\,\rho^2 \int_{\Omega_k} |Df|^2 + c\sqrt{\alpha_k} \sup |f - \lambda|^2 \rho^2 ,$$

with c independent of k . Applying this with $f = D_j u_k$, we have $\eta_k \in L_k$ so that

$$\int_{\Omega_k} |Du_k - \eta_k|^2 \le c\,\rho^2 \int_{\Omega_k} |D^2 u_k|^2 + c\sqrt{\alpha_k}\,\rho^2 \le c\sqrt{\alpha_k}\,\rho^2 .$$

Then, since by Lemma 4 $\Sigma|d_k^i| \le c\sqrt{\alpha_k}\rho^2$, we have

$$\int_{B_{\rho_k}(\xi) \cap L_k} |D\bar{u}_k - \eta_k|^2 \le c\,\rho^2 \sqrt{\alpha_k} ,$$

so finally, by (2), for suitable $\gamma > 0$

$$\text{(4)} \qquad \int_{B_{\theta\rho/2}(\xi) \cap L_k} |D\bar{u}_k - \eta_k|^2 \le c\,\rho^{2+\gamma} .$$

Taking a subsequence so that the L_k converge to L , $\eta_k \to \eta \in L$, and so that (by the Arzela-Ascoli theorem) graph $\bar{u}_k$

converges in the Hausdorff distance sense to graph u , with $u \in \text{Lip}\, L$, $\rho^{-1}\sup|u| + \text{Lip}\, u \le c\,\varepsilon^{\frac{1}{4}}$ and

$$\int_{B_{\theta\rho/2}(\xi)\cap L} |Du - \eta|^2 \le c\,\rho^{2+\gamma} . \tag{5}$$

In measure-theoretic terms (provided we take ε small enough to begin with) this means we have established that for all $\xi \in \text{spt}\,\mu \cap B_{\theta\rho(\xi_0)/2}(\xi_0)$ and for all $\rho < \theta\rho(\xi_0)/4$

$$H^2 \llcorner (\Sigma_k \cap B_\rho(\xi)) = H^2 \llcorner (\text{graph}\ \bar{u}_k \cap B_\rho(\xi)) + \theta_k ,$$

where θ_k is a signed measure with total mass $\le c\,\rho^{2+\gamma}$ and (taking limits in the measure-theoretic sense)

$$\mu \llcorner B_\rho(\xi) = H^2 \llcorner (\text{graph}\ u \cap B_\rho(\xi)) + \theta , \tag{6}$$

where total mass of $\theta \le c\,\rho^{2+\gamma}$ and where u satisfies (5) (with $\eta = \eta(\rho, \xi) \in L$).

In view of the arbitraryness of ρ , ξ it then follows from (5) and (6) that, if ε is small enough, firstly

(7) *the measure* μ *has a unique multiplicity* 1 *tangent plane at each point* $\xi \in \text{spt}\,\mu \cap B_{\theta\rho(\xi_0)/4}(\xi_0)$ *with normal* $\nu(\xi)$ *, such that*

$$|\nu(\xi_1) - \nu(\xi_2)| \le c|\xi_1 - \xi_2|^\gamma , \quad \xi_1, \xi_2 \in \text{spt}\,\mu\cap B_{\theta\rho(\xi_0)/4}(\xi_0) ,$$

and also that then

(8) $$\mu \llcorner B_{\theta\rho(\xi_0)/8}(\xi_0) = \mathcal{H}^2 \llcorner \Sigma ,$$

where Σ is an embedded $C^{1,\gamma/2}$ surface expressible as graph w for some $w \in C^{1,\gamma/2}(U)$, U an open subset of a plane L_0 containing ξ_0 .

On the other hand, since $\int_{\Sigma_k \cap B_\rho(\xi)} H_k^2 \le c\,\rho^\gamma$ and since Σ (with multiplicity 1) is the varifold limit of Σ_k in $B_{\theta\rho(\xi_0)/8}(\xi_0)$, Σ has generalized mean curvature H satisfying

$$\int_{\Sigma \cap B_\rho(\xi)} H^2 \le c\,\rho^\gamma ,$$

for $\xi = x + w(x)\nu_0 \in$ graph w (ν_0 = unit normal of L_0) such that dist $(x, \partial U) > 2\rho$. Since w is a C^1 weak solution of the mean curvature equation

$$\operatorname{div} \left(\frac{Dw}{\sqrt{1 + |Dw|^2}} \right) = H ,$$

it then follows from a standard difference quotient argument (e.g. by the obvious modifications of the argument used in [GT, Theorem 8.8]) that $w \in W^{2,2}_{\ell oc}(U)$ and (by an additional hole-filling argument)

(9) $$\int_{U \cap B_\rho(x)} |D^2 w|^2 \le c\,\rho^\gamma$$

for each $x \in U$ with dist $(x, \partial U) > 0$.

We now show that w is actually $C^{2,\alpha}$ for some $\alpha > 0$. (Higher regularity, and real-analyticity, of w is standard (see e.g. [MCB]) once we get as far as $C^{2,\alpha}$.) To establish $C^{2,\alpha}$ regularity on u we need the following lemma:

Lemma 6. *Let* $\beta > 0$, $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$, *and let* $u \in W^{2,2}(\Omega) \cap C^{1,\alpha}(\Omega)$ *satisfy*

$$\int_{\Omega \cap \{x\,:\,|x-\xi|<\rho\}} |D^2u|^2 \le \beta\,\rho^{2\alpha}$$

for each $\xi \in \Omega$ *and* $\rho < 1$. *Suppose further that* u *is a weak solution of the* 4^{th} *order quasilinear equation*

$$\frac{\partial^2}{\partial x^j\,\partial x^s}\left(a^{ijrs}(x, u, Du)\,\frac{\partial^2 u}{\partial x^i \partial x^r}\right) = \frac{\partial f^j}{\partial x^j}$$

where a^{ijrs} *and* f^j *satisfy the following:*

(i) $$\int_{\Omega \cap \{x\,:\,|x-\xi|<\rho\}} \sum_{j=1}^{2} |f^j| \le \beta\,\rho^{\alpha}$$

for each $\xi \in \Omega$ *and* $\rho < 1$,

(ii) $a^{ijrs} = a^{ijrs}(x, \xi, p)$ *is a Lipschitz function on* $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$ *with Lipschitz constant* β *and with* $a^{ijrs}\xi_{ir}\,\xi_{js} \ge \beta^{-1} \sum_{i,r=1}^{2} \xi_{ir}^2$, $|a^{ijrs}| \le \beta$.

Then $u \in W^{3,2}_{\ell oc}(\Omega)$ *and there are* $c = c(\beta)$ *and* $\alpha' = \alpha'(\beta) > 0$ *such that*

$$\int_{\{x\,:\,|x-\xi|<\rho\}} |D^3u|^2 \le c\,\rho^{2\alpha'}$$

for each $\xi \in \Omega$ *with* $\operatorname{dist}(\xi, \partial\Omega) > 2\rho$. *(So* $u \in C^{2,\alpha'}(\Omega)$ *.)*

For the proof of this lemma we refer to [SL]. Here we simply point out that for any $\xi \in \Omega$ we can write the equation in the form

$$\frac{\partial^2}{\partial x^j\,\partial x^s}\left(a_0^{ijrs}\frac{\partial^2 u}{\partial x^i\,\partial x^r}\right) = \frac{\partial f^j}{\partial x^j} + \frac{\partial^2 f^{js}}{\partial x^j \partial x^s} ,$$

where $a_0^{ijrs} = a^{ijrs}(\xi,\ u(\xi)\ ,\ Du(\xi))$ and $f^{js} = (a^{ijrs} - a_0^{ijrs})\frac{\partial^2 u}{\partial x^i \partial x^r}$. One then uses difference quotients and the technical lemma 5.4.2 of [MCB] to establish the required result locally near ξ .

Thus we have sketched the proof of real analyticity of $\Sigma = \text{spt}\ \mu$ away from the finitely many bad points $\xi_1, \ldots, \xi_P$. Since (by lower semicontinuity) $\int_\Sigma |A|^2 < \infty$, one can (essentially by direct modifications of the techniques sketched above) establish that $\int_{\Sigma \cap B_\rho(\xi_j)} |A|^2 \leq c\,\rho^\gamma$ for $\rho \in (0, 1)$ and that Σ is representable as a $C^{1,\gamma/2}$ graph near ξ_j . Then Lemma 6 can again be applied to give $C^{2,\alpha}$ regularity near ξ_j . (See [SL] for details.)

Finally the fact that Σ_k converges to Σ in the Hausdorff distance sense is an easy consequence of the fact (from identity (***) of §1) that each limit point ξ of a sequence $\xi_k \in \Sigma_k$ which is *not* in spt μ must have $\int_{B_\rho(\xi) \cap \Sigma_k} H_k^2 \to \infty$ for each $\rho > 0$; thus there can be no such points ξ because $\mathcal{F}(\Sigma_k)$ is bounded.

§4. Proof of the main fixed genus result in $\mathbb{R}^3$.

Suppose first $g = 1$ and let Σ_k be a sequence of embedded tori with $F(\Sigma_k) \to \beta_1$. Assume we normalize (as in §3) so that $0 \in \Sigma_k$ and $|\Sigma_k| = 1$. Then by Theorem 1 we have a subsequence (still denoted Σ_k) and a real analytic compact embedded surface Σ of genus ≤ 1 which minimizes F relative to all surfaces $\tilde{\Sigma}$ of the same genus as Σ. If Σ is a sphere (genus 0) then it must be a round sphere (because only round spheres minimize F). Thus we are left with the alternatives

$$(1) \quad \begin{cases} \textit{either } \Sigma \text{ is genus } 1 \text{ with } F(\Sigma) = \beta_1 \text{ as required} \\ \textit{or } \Sigma \text{ is a round sphere.} \end{cases}$$

Naturally the second alternative *can* occur; what we want to show is that we can make an appropriate inversion and rescaling to give a new minimizing sequence $\tilde{\Sigma}_k$ of tori for which the limit surface $\tilde{\Sigma}$ definitely satisfies the first alternative in (1).

As a matter of fact we shall show quite generally that if Σ_k is *any* genus g minimizing sequence in the sense of §3 with $g \geq 1$, then there is a new genus g minimizing sequence $\tilde{\Sigma}_k$ converging to a minimizing surface of genus ≥ 1. We briefly sketch how such $\tilde{\Sigma}_k$ is constructed. First, we may assume that the limit surface Σ of the original sequence is a round sphere (otherwise it has genus ≥ 1 and we have nothing further to prove). Since the convergence is in the Hausdorff distance sense, for each k we can find a Jordan curve γ_k with $\gamma_k \cap \Sigma_k = \emptyset$, γ_k not null-homotopic in $\mathbb{R}^3 \sim \Sigma_k$, and

$\alpha_k \to 0$, where

$$\alpha_k := \sup \{\text{dist}\,(\tilde{\gamma}_k, \Sigma_k) : \tilde{\gamma}_k \text{ homotopic to } \gamma_k \text{ in } \mathbb{R}^3 \sim \Sigma_k\} .$$

In view of the definition of α_k one readily checks that there must be a ball $B^{(k)} = B_{\alpha_k}(\xi_k)$ with $B^{(k)} \cap \Sigma_k = \emptyset$ and with $\partial B^{(k)} \cap \Sigma_k$ containing at least two points p_k, q_k with p_k not in the open hemisphere of $\partial B^{(k)}$ with pole q_k . Now let $\tilde{\Sigma}_k$ be the surface obtained as the image of Σ_k by first making a translation taking ξ_k to 0 , then making an inversion in $B_{\alpha_k}(0)$, then scaling $x \mapsto \alpha_k^{-1} x$.

Then $\tilde{\Sigma}_k \subset \bar{B}_1(0)$ and $\tilde{\Sigma}_k \cap \partial B_1(0)$ contains at least two points p_k, q_k with $|p_k - q_k| \geq \sqrt{2}$. Furthermore since diam $\Sigma_k \geq c$ (independent of k) by Lemma 1, and since $\alpha_k \to 0$, it follows that there are points $\eta_k \in \tilde{\Sigma}_k$ with $|\eta_k| \to 0$. On the other hand if $\tilde{\Sigma}$ is the limit surface of (a subsequence of) $\tilde{\Sigma}_k$, then (using the Hausdorff distance sense convergence of $\tilde{\Sigma}_k$ to $\tilde{\Sigma}$) we have that $\tilde{\Sigma}$ contains 0 as well as two distinct points $p, q \in \partial B_1(0)$, and we also have $\tilde{\Sigma} \subset \bar{B}_1(0)$. Thus $\tilde{\Sigma}$ is *not* a round sphere, hence (since it minimizes $\mathcal{F}$ relative to surfaces of genus = genus $\tilde{\Sigma}$, and since only the round spheres minimize $\mathcal{F}$ relative to genus 0 surfaces) we conclude genus $\tilde{\Sigma} \geq 1$ as required.

In view of the alternatives (1) this completes the existence proof for genus 1 . For genus $g \geq 2$ the required result is an easy consequence of the above general result, together with the cutting and pasting procedure used to prove (3) of the introduction.

§5. Existence of Willmore Immersions in Riemannian Manifolds.

Here we briefly discuss existence results for the Willmore functional in case the ambient manifold is a general complete Riemannian manifold of dimension $n \geq 3$ (instead of $\mathbb{R}^n$). Since we have no analogue of (2) of the introduction or of Lemma 3, it is necessary to work with *immersed* rather than embedded surfaces in order to get a good natural existence theory.

First we need to set up some terminology, principally the following definitions, in which

$$f : M \to N$$

is an immersion from a surface $M \in \mathcal{M}$; here we let $\mathcal{M}$ denote the set of compact 2-dimensional manifolds without boundary, and for technical reasons we do not require the elements $M \in \mathcal{M}$ to be connected.

Definition 1 Given $f : M \to N$ as above, $[f]$ will denote the set of immersions $\tilde{f} : M \to N$ which are smoothly homotopic to f .

Thus $\tilde{f} \in [f]$ means that $\tilde{f}$ is an immersion $M \to N$ and that there is a 1-parameter family of maps $\{f_t\}_{t \in [0, 1]}$ with

(i) $f_0 = f$, $f_1 = \tilde{f}$

(ii) the map $(x, t) \in M \times [0, 1] \mapsto f_t(x) \in N$ is smooth.

Definition 2 Given $f : M \to N$ as above, $\overline{[f]}$ is the set of smooth immersions $\tilde{f}$ of some $\tilde{M} \sim \mathcal{B}$ into N, where $\tilde{M} \in \mathcal{M}$ and $\mathcal{B} \subset \tilde{M}$ is a finite (or empty) set of points, such that $\tilde{f}$ extends to give a $C^{1,\alpha}$ branched immersion of all of $\tilde{M}$ into N for some $\alpha > 0$, and such that there exists a sequence φ_k of diffeomorphisms of $\tilde{M} \sim \mathcal{B}$ onto open subsets U_k of M, and a sequence $f_k \in [f]$ with

(i) $\quad f_k \circ \varphi_k \to \tilde{f}$ locally in the C^2 sense on $\tilde{M} \sim \mathcal{B}$,

(ii) $\quad f_k(M \sim U_k) \subset \bigcup_{x \in \mathcal{B}} B_{\varepsilon_k}(\tilde{f}(x))$ for some sequence $\varepsilon_k \downarrow 0$.

Of course $\tilde{M}$ may have more components and fewer handles than M, because if M_k denotes M equipped with the metric pulled back from N by f_k, then (i), (ii) mean that M_k may have necks and handles which shrink to zero as $k \to \infty$.

Remark. By $C^{1,\alpha}$ branched immersion $\tilde{f} : \tilde{M} \to N$ we mean that $\tilde{f}$ is of class $C^{1,\alpha}$, there are only finitely many points y such that the Jacobian of $\tilde{f}$ vanishes, and, at such points y, in suitable local coordinates for $\tilde{M}$ and N, $\tilde{f}$ has a classical branch point of some order $m \geq 1$. Thus there is a plane L through $\tilde{f}(y)$ in N (identified with $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^n$ via local coordinates for N) such that, with M locally identified with $\mathbb{R}^2$ and y corresponding to 0,

$$\tilde{f}(r\cos\theta\ ,\ r\sin\theta) = (r\cos m\theta\ ,\ r\sin m\theta\ ,\ \psi(r\cos\theta\ ,\ r\sin\theta))$$

where $\psi : \mathbb{R}^2 \to \mathbb{R}^{n-2}$ satisfies

$$|\psi(x)| \le c|x|^{1+\alpha} \quad , \quad |d\psi_x| \le c|x|^{\alpha}$$

$$|d\psi_x - d\psi_{\bar{x}}| \le c|x - \bar{x}|^{\alpha}$$

for $|x|$, $|\bar{x}| \le 1$, $x, \bar{x} \in \mathbb{R}^2$.

Next we introduce the class of functionals to be considered here; for smooth compact oriented surfaces Σ (isometrically) embedded in N (possibly with $\partial\Sigma \ne \emptyset$) we consider functionals of the form

$$\mathcal{F}(\Sigma) = \frac{1}{2}\int_{\Sigma}(|A|^2 + \Phi(x, \tau))d\mathcal{H}^2$$

where A is the second fundamental form of Σ , Φ is smooth, τ is a smooth orienting unit 2-vector for Σ (thus at each point $x \in \Sigma$, $\tau(x) = e_1 \wedge e_2$ for some orthonormal basis e_1, e_2 of $T_x\Sigma \subset T_xN$).

$\mathcal{F}$ extends naturally to smooth immersions $f : M \to N$ (where $M \in \mathcal{M}$). For such an immersion

$$(*) \qquad \mathcal{F}(f) = \frac{1}{2}\int_{\text{range} f} \sum_{x \in f^{-1}(y)} (|A(x)|^2 + \Phi(y, \tau(x)))d\mathcal{H}^2(y) ,$$

where $|A(x)|^2$ and $\tau(x)$ are defined for $x \in M$ as the square length of second fundamental form and orienting 2-vector at $y = f(x)$ of the embedded submanifold obtained as the image under f of a small neighbourhood of $x \in M$.

Subject to these agreements, we have the following theorem.

Theorem 2. *Let* $M \in \mathcal{M}$, $f : M \to N$ *be a smooth immersion,* $\alpha := \inf_{\tilde{f} \in [f]} \mathcal{F}(\tilde{f})$, *and suppose there is a sequence* $f_k \in [f]$ *with* $\mathcal{F}(f_k) \to \alpha$, *with* $\limsup_{k \to \infty} \text{area}(f_k) < \infty$, *with* $\bigcup_{k=1}^{\infty} \text{range}\, f_k$ *contained in a compact subset of* N, *and with the sum of diameters of the components of* $\text{range}\, f_k \not\to 0$ *as* $k \to \infty$.

Then there is $\tilde{f} \in \overline{[f]}$, *related to* f_k *via diffeomorphisms* φ_k *as in Definition 2, with*

$$\mathcal{F}(\tilde{f}) \leq \alpha ,$$

and $\tilde{f} \in [f]$ *if and only if* <u>*equality*</u> *holds here. In any case* $\tilde{f} : \tilde{M} \sim \mathcal{B} \to N$ *minimizes* $\mathcal{F}$ *relative to all immersions* $g : \tilde{M} \sim \mathcal{B} \to N$ *which are homotopic to* $\tilde{f}$ *via smooth homotopies which fix a neighbourhood of the finite set* $\mathcal{B}$.

Remarks. (1) Notice the assumption $\limsup \text{area}(f_k) < \infty$ is automatically satisfied if Φ is everywhere positive. If N is compact, if $\mathcal{F}$ is the exact Willmore functional (as defined in [WJ]), and if N is locally conformally flat and has positive sectional curvature, then we can always replace f_k by a new sequence $\tilde{f}_k \in [f]$ such that all assumptions on f_k are automatically satisfied (as one easily checks).

(2) The theorem naturally extends to more general classes of functionals; in place of $\mathcal{F}$ we could consider for example functionals of the form $\mathcal{G}(\Sigma) = \int_\Sigma \Phi(x, \tau, A)\, d\mathcal{H}^2$, where A is the second fundamental form of Σ and where Φ is smooth with appropriate convex and "essentially quadratic" dependence on A.

(3) It may be that f is null homotopic (e.g. in case $N = \mathbb{R}^3$ we showed in §4 that there is an embedding f_* of the torus which minimizes the Willmore functional relative to all branched immersions of the torus).

(4) Trivially we can extend the above result to branched immersions of non-orientable surfaces, provided $\Phi(x, \tau) = \Phi(x, -\tau)$, $(x, \tau) \in N \times \Lambda_2(N)$, by using oriented double covers as follows: If M is non-orientable and compact and if $f : M \to N$ is a branched immersion, we let $\bar{M}$ be the oriented double cover of M, $\bar{f}$ the branched immersion: $\bar{M} \to N$ corresponding to f, and let $\mathcal{F}(f) = \frac{1}{2}\mathcal{F}(\bar{f})$. Then we apply Theorem 2 to $\bar{f}$ in order to deduce the appropriate result about f.

(5) One can say more about the regularity of $\tilde{f}$ near the points of $\mathcal{B}$; see [SL].

To prove Theorem 2 we modify the techniques of the previous sections to work in the setting of immersions into N. In particular there are analogues of Lemmas 2 and 4 to such a setting, in addition to local analogues of identities like (**), (***) of §1. One begins by taking a minimizing sequence f_k as in the statement of the theorem, and by defining the associated Borel measures μ_k on N according to

$$\mu_k(A) = \int_{A \cap \text{range } f_k} \theta_k \, d\mathcal{H}^2$$

where θ_k is the multiplicity function for f_k ($\theta_k(y)$ = number of points in the set $f_k^{-1}(y)$), and where $\mathcal{H}^2$ is 2-dimensional Hausdorff measure on N.

We select a subsequence (still denoted f_k) so that μ_k has a limit measure μ . The principal aim (cf. §1-4 above) is to prove that $\operatorname{spt}\mu$ is the image of a branched immersion. As before, for a given $\varepsilon > 0$ we define $\xi \in \operatorname{spt}\mu$ to be a bad point if (with a notation similar to that in (*) above)

$$\lim_{\rho\downarrow 0}\liminf_{k\to\infty}\int_{B_\rho(\xi)\cap \operatorname{range} f_k}\sum_{x\in f_k^{-1}(y)}|A_k(x)|^2\,dH^2(y) > \varepsilon\,.$$

Since $\int_{\operatorname{range} f_k}\sum_{x\in f_k^{-1}(y)}|A_k(x)|^2\,dH^2(y)$ is bounded, it is easy to prove that there are at most finitely many bad points $\xi_1, \ldots, \xi_P$, $P = P(\varepsilon)$.

By using modifications of Lemma 2 and Lemma 4 to the immersed setting, and using again biharmonic comparisons as in §2, it is quite easy to prove that, near each point $\xi \in \operatorname{spt}\mu \sim \{\xi_1, \ldots, \xi_P\}$, the measure μ is the area measure of a finite union of smooth embedded discs. To handle the bad points $\xi_1, \ldots, \xi_P$ it is necessary to use the following lemma. For further details of the proof of Theorem 2 (and of the proof of the following lemma), we refer again to [SL].

Lemma 7. *Suppose* $f : D \sim \{0\} \to \mathbb{R}^n$ *is a smooth immersion, where* D *is the disc* $\{x \in \mathbb{R}^2 : |x| \le 1\}$ *and where* $\mathbb{R}^n$ *is equipped with a smooth metric* g . *Suppose that* $\mathcal{F}(f) < \infty$ *and* area $(f) < \infty$, *that* f *extends continuously to* D , *and that* f *minimizes the functional* $\mathcal{F}$ *relative to all immersions* $\tilde{f} : D \sim \{0\} \to \mathbb{R}^n$ *such that* $\tilde{f} \equiv f$ *in some neighbourhood of* $\partial D \cup \{0\}$.

Then we can reparametrize F *so that it extends as a* $C^{1,\alpha}$

branched immersion of D *into* $\mathbb{R}^n$ *for some* $\alpha > 0$ *; that is, there is a diffeomorphism* φ *of* $D \sim \{0\}$ *onto* $D \sim \{0\}$ *such that* $\lim_{x \to 0} \varphi(x) = 0$ *and such that* $f \circ \varphi$ *extends to be a* $C^{1,\alpha}$ *branched immersion of* D *into* $\mathbb{R}^n$. *In case the multiplicity of the branch point is* 1, *we can select* φ *so that* $f \circ \varphi$ *extends to a* $C^{1,\alpha}$ *embedding.*

In the proof of Lemma 7 one shows that it is possible to select ρ_0 such that $p := \lim_{x \to 0} f(x) \notin f(\partial D_\rho)\ \forall \rho < \rho_0$ and such that the varifold $f_\# |D_\rho|$ has multiplicity m tangent planes at p for some positive integer m independent of ρ, and that then the theorem holds with $f \circ \varphi$ having branch point of order m (and no branch point if $m = 1$).

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