PARTIAL REGULARITY FOR SOLUTIONS OF VARIATIONAL PROBLEMS*

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We report here on some recent results of the authors [FH1,2] within the context of a general discussion of problems in the calculus of variations. Some results (those in [FH2]) were not included in the delivered lecture.

We will consider minima of functionals $\mathcal{F}$ of the form

$$u \mapsto \mathcal{F}[u] = \int_\Omega F(x,u,Du) \tag{1}$$

where $\Omega \subset \mathbb{R}^n$, Ω is open, and $u : \Omega \to \mathbb{R}^N$. It will always be assumed that F is a *Caratheodory* function, i.e. $F = F(x,u,p)$ is measurable in x for all $(u,p) \in \mathbb{R}^n \times \mathbb{R}^N$ and is continuous in (u,p) for almost all $x \in \Omega$. This ensures that $F(x,u,Du)$ is measurable if u is measurable.

Here will be interested in the general case $n \geq 1$ and $N \geq 1$. If $N = 1$, one can obtain much stronger results, for this we refer to [G1], [G2], [GT], [LU], and [M].

There are two questions of fundamental interest. First, one wants to show (subject to various boundary conditions) the *existence* of minima of $\mathcal{F}$ in suitable function classes. Second, one is interested in the *regularity* (i.e. smoothness) properties of such minimisers.

The existence problem in a general sense is solved as a standard consequence of the following result by Acerbi and Fusco [AF].

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* Lecture delivered by the second author.

Theorem 1 *Suppose* $F = F(x,u,p)$ *is a Caratheodory function. Assume that*

$$0 \le F(x,u,p) \le \lambda(1 + |u|^m + |p|^m)$$

for some $m \ge 1$.

Then the functional $\mathcal{F}$ *is weakly sequentially lower semi-continuous in* $H^{1,m}(\Omega;\mathbb{R}^N)$ *iff* F *is quasiconvex.* □

We say that F is *quasiconvex* if linear functions are local minimisers of the "frozen" functionals corresponding to F . More precisely, F is quasiconvex if for a.e. $x_0 \in \Omega$ and for all $(u_0,p) \in \mathbb{R}^N \times \mathbb{R}^{nN}$ one has

$$\int_\Omega F(x_0,u_0,p) \le \int_\Omega F(x_0,u_0,p + D\phi)$$

for all $\phi \le C_c^\infty(\mathbb{R}^n;\mathbb{R}^N)$.

For further discussion on the existence question we refer to the book [G1] and the references therein.

We now discuss in somewhat more detail the regularity question for minima of $\mathcal{F}$.

Suppose F is a Caratheodory function, $F = F(x,u,p)$ is C^2 in p for all $(x,u) \in \Omega \times \mathbb{R}^N$, and F satisfies the following conditions

$$(2)\quad \begin{cases} \text{(i)} \quad |p|^2 - 1 \le F(x,u,p) \le a(1 + |p|^2) \,, \\ \text{(ii)} \quad |F_{pp}(x,u,p)| \le b \\ \text{(iii)} \quad F_{pp}\xi\xi = F_{p^i_\alpha p^j_\beta}\, \xi^i_\alpha \xi^j_\beta \ge \lambda|\xi|^2 \\ \text{for some } \lambda > 0 \text{ and all } \xi \in \mathbb{R}^{nN} \,, \\ \text{(iv)} \quad (1 + |p|^2)^{-1} F(x,u,p) \text{ is Hölder continuous in } (x,u) \\ \text{uniformly in } p \,. \text{ In other words} \\ \quad |F(x,u,p) - F(y,v,p)| \le c(1 + |p|^2)\omega(|x-y|^2 + |u-v|^2) \end{cases}$$

where $\omega(t) \le t^{\sigma}$, $0 < \sigma \le \frac{1}{2}$, and ω is bounded, non-negative, concave, and increasing on $\{t \ge 0\}$.

Then we have the following result due to Giaquinta and Giusti [GG].

Theorem 2 *Suppose* F *is as in* (1) *and* (2). *Suppose* $u \in W^{1,2}_{loc}(\Omega;\mathbb{R}^N)$ *is a local minimum for* F *(i.e.* $F[u] \le F[u+\phi]$ *for all* $\phi \in W^{1,2}(\Omega;\mathbb{R}^N)$ *with* $\operatorname{spt}\phi \subset\subset \Omega$ *). Then there exists an open set* $\Omega_0 \subset \Omega$ *such that* $u \in C^{1,\alpha}_{loc}(\Omega_0)$ *for some* $0 < \alpha < 1$ *and such that* $\mathcal{H}^n(\Omega \sim \Omega_0) = 0$. *Moreover,*

$$\Omega_0 = \{x_0 \in \Omega : \limsup_{r\to 0} |(Du)_{x_0,r}| < \infty \tag{3}$$

$$\text{and } \liminf_{r\to 0} \fint_{B(x_0,r)} |Du - (Du)_{x_0,r}|^2 = 0\} . \square$$

The theorem is proved by ultimately establishing a local decay estimate in Ω_0 of the form

$$\fint_{B(x_0,\rho)} |Du - (Du)_{x_0,\rho}|^2 \le c\rho^{2\alpha} \tag{4}$$

as $\rho \to 0$, for some $\alpha > 0$. The key idea is to compare u with the minimum v in $B(x_0,r)$ of the frozen functional

$$w \mapsto \int_{B(x_0,r)} F(x_0,(u)_{x_0,r}, Dw)$$

with boundary condition

$$w \in u + W^{1,2}_0(B(x_0,r)) .$$

In particular, one uses the fact that w, being a solution of a constant coefficient equation, satisfies a decay condition analogous to (4). Finally, one uses results of Companato [cf. [Gl, Chapter III]) to deduce the Hölder continuity of u in Ω_0 from (4).

It is an open question whether one can improve the dimension of the singular set $\Omega \sim \Omega_0$. For particular classes of functionals this is indeed the case. On the other hand, one cannot generally expect everywhere regularity, as well-known counterexamples show. Again we refer to [G1] for further discussion.

Aside from the question of the dimension of the singular set, there are some other gaps between the existence results which follow from Theorem 1 and the (partial) regularity results of Theorem 2.

In particular, the *convexity condition* of (2)(iii) implies quasiconvexity but not conversely; see [M, Chapter 4.4] and [G1, Chapter IX.2]. However, it has recently been shown that if one replaces (2)(iii) by the requirement of *strict quasiconvexity* (see below), then one again has partial Hölder continuity of first derivatives of local minimisers.

One says that F is *strictly quasiconvex* if there exists $\gamma > 0$ such that for a.e. $x_0 \in \Omega$, for all $(u_0,p) \in \mathbb{R}^N \times \mathbb{R}^{nN}$, and for all $\phi \in C^1_0(\mathbb{R}^n;\mathbb{R}^N)$, one has

$$\text{(2)(iii)}^* \qquad \int_\Omega [F(x_0,u_0,p) + \gamma|D\phi|^2] \le \int_\Omega F(x_0,u_0,p+D\phi) \ .$$

The following theorem was proved by Evans [E] in case F depends only on p , and then later for general F by Fusco and Hutchinson [FH1] and also by Giaquinta and Modica [GM].

Theorem 3 *Under the same hypothesis as Theorem 2, but with* (2)(iii) *replaced by* (2)(iii)*, *we have that if* $u \in W^{1,2}(\Omega;\mathbb{R}^N)$ *is a local minimum then* $u \in C^{1,\alpha}_{loc}(\Omega_0)$, *for some* $0 < \alpha < 1$ *and some open* Ω_0 *satisfying* $H^n(\Omega \sim \Omega_0) = 0$.

The proof in [E] was by means of a "blow-up" argument. The key new point was to establish the following Caccioppoli type estimate assuming (2)(iii)* rather than (2)(iii):

$$\int_{B(x_0,r/2)} |Du-\xi|^2 \le c(L)\ r^{-2} \int_{B(x_0,r)} |u-a-\xi(x-x_0)|^2$$

provided $B(x_0,r) \subset \Omega$, $a \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{nN}$, and $|\xi| \le L$.

One is naturally tempted to extend the result in [E] (where F depends only on p) to general functionals F (depending on x and u , as well as p) as follows. Suppose u is a local minimiser of F . Try to obtain an estimate in Ω_0 of the form

$$(5) \qquad \fint_{B(x_0,\rho)} |Du-(Du)_{x_0,\rho}|^2 \le c\rho^{2\alpha}$$

by first estimating $\int_{B(x_0,r)} |Du-Dv|^2$, where v minimises $\int_{B(x_0,r)} F(x\ ,(u)_{x_0,r},Dv)$ subject to $v \in u+W_0^{1,2}(B(x_0,r))$, and then by combining this with the estimate (5), with u replaced by v , which estimate is proved in [E].

However, one cannot readily estimate $\int_{B(x_0,r)} |Du-Dv|^2$ with v as above, precisely because $F_{p_i^\alpha p_j^\beta}(x,u,p)$ satisfies a Legendre Hadamard condition $F_{p_i^\alpha p_j^\beta}\xi^i\xi^j\ \eta_\alpha\eta_\beta \ge \gamma|\xi|^2|\eta|^2$ rather than a Legendre condition $F_{p_i^\alpha p_j^\beta}\xi^i_\alpha\xi^j_\beta \ge \gamma|\xi|^2$.

This problem is solved in [FH1] by invoking a lemma of Ekeland (cf. [Gl, Theorem 2.3, p.257]), from which one can deduce the existence for any $B(x_0,r) \subset\subset \Omega$ of a function v such that

$$\fint_{B(x_0,r)} |Dv-Du|^2 \le r^{2\alpha}$$

and v minimises the problem

$$f \mapsto \fint_{B(x_0,r)} F(x_0,(u)_{x_0,r},Df) + cr^{\beta} \left(\fint_{B(x_0,r)} |Df-Dv|^2\right)^{\frac{1}{2}}$$

$$f \in v + W_0^{1,2}(B(x_0,r);\ \mathbb{R}^N) ,$$

for some small positive α, β. For further details see [FH1, §4].

The estimate (5) is obtained by means of a "blow-up" argument. Thus one supposes such an estimate is not true, blows up minimisers v_m obtained as above in appropriate balls $B(x_m, r_m)$, and obtains a contradiction by passing to a limit of the v_m.

As remarked above, the results in [E], [GG] and [FH], allow a weakening of the hypotheses of Theorem 2 by replacing convexity by a strengthened form of quasiconvexity. Another natural weakening of the hypotheses of Theorem 2 is to replace the quadratic growth of $F(x,u,p)$ in the variable p, by a growth rate of order $|p|^m$ for some m.

Motivated by a functional given by

$$F(x,u,p) = a(x,u)(1+|p|^m), \quad m \geq 2,$$

we consider the following structural conditions to replace (2) (where $m \geq 2$):

(2) (i)' $\quad |p|^m - 1 \leq F(x,u,p) \leq a(1+|p|^m)$

(ii)' $\quad |F_{pp}(x,u,p)| \leq b(1+|p|^{m-2})$

(iii)' $\quad F_{pp}\xi\xi \geq \lambda(1+|p|^{m-2})\xi\xi$

(iv)' $\quad (1+|p|^m)^{-1} F(x,u,p)$ is Hölder continuous on (x,u) uniformly in p.

Then one can still prove $C^{1,\alpha}$ regularity (some $\alpha > 0$) on an open Ω_0 with $\mathcal{H}^n(\Omega \sim \Omega_0) = 0$, as in Theorem 2. Moreover, one can even replace (2)(iii)' by (2)(iii)'*

(2)(iii)'*
$$\int_\Omega [F(x_0,u_0,p_0) + \gamma(|D\phi|^2 + |D\phi|^m)] \leq \int_\Omega F(x_0,u_0,p_0 + D\phi)$$

for a.e. $x_0 \in \Omega$, for all $(u,p) \in \mathbb{R}^N \times \mathbb{R}^{nN}$, for all $\phi \in C_0^1(\mathbb{R}^n;\mathbb{R}^N)$, and for some $\gamma > 0$; see [E], [FH1] and [GM].

However, if one considers a functional given by (1) and

$$F(x,u,p) = a(x,u)|p|^m , \quad m \ge 2 , \tag{6}$$

one sees that one should replace the structural condition (2)(iii)' by

(2)(iii)'' $$F_{pp}\xi\xi \ge \lambda|p|^{m-2}\xi\xi .$$

Although (partial) regularity results are not known for such a general class of functionals, there are some results. Uhlenbeck [U] has shown *complete* $C^{1,\alpha}$ (some small $\alpha > 0$) regularity for minimisers (even stationary points) of $\int |Du|^m$ in case $m \ge 2$. This has been extended to $m \geqslant 1$ by Tolksdorff [T1]. Moreover, examples show that one cannot expect $C^{1,\alpha}$ regularity for all $0 < \alpha < 1$ (cf. [T2]).

In [FH2], partial regularity was shown for minimisers of functionals corresponding to (6). More generally, we have the following result.

Theorem 4 *Suppose* $u \in W^{1,p}_{loc}(\Omega)$ *is a local minimum for*

$$F[u] = \int_\Omega [G^{\alpha\beta}(x,u)g_{ij}(x,u)\, D_\alpha u^i\, D_\beta u^j]^{p/2} \tag{7}$$

where $p \ge 2$. *Suppose* G *and* g *satisfy*

$$|\xi|^2 \le G\xi\xi \le M|\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^n ,$$

$$|\eta|^2 \le g\eta\eta \le M|\eta|^2 \quad \text{for all} \quad \eta \in \mathbb{R}^N ,$$

and G, g *are* $C^{0,\sigma}$ *on* $\Omega \times \mathbb{R}^n$.

Then $u \in C^{1,\alpha}_{loc}(\Omega_0)$ *for some* $0 < \alpha < 1$ *and some* $\Omega_0 \subset \Omega$, *where* $H^{n-q}(\Omega \sim \Omega_0) = 0$ *for some* $q > p$.

Moreover

$$\Omega_0 = \{x_0 \in \Omega : \limsup_{r\to 0} |(u)_{x_0,r}| < \infty \quad \text{and} \quad \liminf_{r\to 0} r^{p-n} \int |Du|^p = 0\} . \tag{8}$$

□

The main idea in the proof is to first obtain an appropriate decay estimate for minima of functionals of the form

$$u \mapsto \int [G^{\alpha\beta} g_{ij} \, D_\alpha u^i D_\beta u^j]^{p/2} ,$$

where $[G^{\alpha\beta}]$, $[g_{ij}]$ are *constant* inner products on $\mathbb{R}^N$ and $\mathbb{R}^n$ respectively.

By a change of coordinates, one reduces the problem to considering functionals of the form

$$u \mapsto \int |Du|^p ,$$

where $|Du| = (D_\alpha u^i D_\alpha u^i)^{\frac{1}{2}}$. Indeed, we work more generally with solutions of the Euler Lagrange equation

$$\int |Du|^{p-2} \, Du \, D\phi = 0 \tag{9}$$

for all $\phi \in W_0^{1,p}(\Omega;\mathbb{R}^N)$.

One might hope for an estimate on solutions u of (9) which has the form

$$⨍_{B(x_0,\tau R)} |Du - (Du)_{\tau R}|^p \le c\tau^{p\alpha} ⨍_{B(x_0,R)} |Du - (Du)_R|^p$$

for all $B(x_0,R) \subset\subset \Omega$ and $0 < \tau < 1$. However, it is not clear that such an estimate is true. What is done instead in [FH2] is to obtain an estimate of the form

$$\phi(x_0,\tau R) \le c\tau^\alpha \, \psi(x_0,R) \quad \text{for} \quad 0 < \tau < 1 , \tag{10}$$

where one defines

$$\phi(x_0,\rho) = ⨍_{B(x_0,\rho)} |Du - (Du)_{x_0,\rho}|^p + |(Du)_{x_0,\rho}|^{p-2} ⨍_{B(x_0,\rho)} |Du - (Du)_{x_0,\rho}|^2 ,$$

whenever $B(x_0,\rho) \subset\subset \Omega$.

The proof of (10) uses earlier estimates of Uhlenbeck [U].

One finally proves Theorem 4 by comparing a minimum of (7) with a minimum of the problem

$$\begin{cases} v \mapsto \int_{B(x_0,R)} [G^{\alpha\beta}(x_0,(u)_{x_0,R})\, g_{ij}(x_0,(u)_{x_0,R})\, D_\alpha v^i D_\beta v^j]^{p/2} \\ v \in u + W_0^{1,p}(B(x_0,R), \mathbb{R}^N) , \end{cases}$$

where $B(x_0,2R) \subset\subset \Omega$. One combines an estimate of the type (10) (with u there replaced by v) together with an estimate of the form

$$⨍_{B(x_0,R)} |Du - Dv|^p \le c^* R^\varepsilon$$

for some small $\varepsilon > 0$, where here c^* depends on $⨍_{B(x_0,2R)} |Du|^p$ and $|⨍_{B(x_0,R)} u|$.

The resulting estimate which one obtains is

(11) $$\phi(x_0,\tau R) \le c^{**}(\tau R)^\alpha$$

for some small $\alpha > 0$ and all sufficiently small τ , provided $B(x_0,2R) \subset\subset \Omega_0$ where Ω_0 is defined in (8). Here c^{**} has the same dependencies as c^* . By the usual Compariato estimates it follows $u \in C^{1,\alpha}_{loc}(\Omega_0)$.

Finally, we remark that if in (7) the matrix G does not depend on u , and u is a locally *bounded* minimum, then the dimension of the singular set is at most $n-[q]-1$ for some $q > p$ (q does not depend on u), where $[q]$ is the integer part of q . If $n \le q+1$, then u can have at most isolated singularities. The proof is a modification of a similar argument in [GG].

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