

ASYMPTOTIC BEHAVIOUR NEAR ISOLATED SINGULAR POINTS
FOR GEOMETRIC VARIATIONAL PROBLEMS

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Isolated singularities for extrema of functionals of "geometric" type have been studied in [SL1], [SL2]. Here we will use the notation of [SL1]. We consider functionals of the form

$$(1) \quad \mathcal{F}(u) = \int_0^\infty \int_\Sigma e^{-mt} \left[F(\omega, u, \nabla u, \frac{\partial u}{\partial t}) + E(\omega, t, u, \nabla u, \frac{\partial u}{\partial t}) \right] d\omega dt$$

where Σ is a compact manifold, m constant $\neq 0$, $\nabla =$ gradient on Σ , and where E has exponential decay with respect to t as $t \uparrow \infty$. Here, u is a C^2 section of a vector bundle V over $\Sigma \times (0, \infty)$.

For these functionals, it is proved in [SL1] that under certain conditions, e.g. that the C^2 norm of u on $\Sigma \times (0, \infty)$ is finite, $F(\omega, z, p, q)$ convex in p , $q \cdot F_q(\omega, z, p, q) \geq |q|^2$ for $|q| \leq 1$ and $q \cdot F_q(\omega, z, p, q) > 0$ for $q \neq 0$, and $F(\omega, z, p, 0)$ real analytic in (z, p) , an extremum u of (1) has a limit as $t \uparrow \infty$. However, the method of proof does not yield estimates for the rate of convergence to the limit, except in special circumstances.

We also consider the functionals

$$\mathcal{F}_\Sigma(u) = \int_\Sigma F(\omega, u, \nabla u, 0) d\omega .$$

We write $e^{-mt} M$, M_Σ for the Euler-Lagrange operators of \mathcal{F} , \mathcal{F}_Σ respectively (see [SL1, I§2]). As in [SL1, I§4], we write \mathcal{L} , L for the linearisation at 0 of M , M_Σ . We call \mathcal{L} , L Jacobi field operators, and solutions of $\mathcal{L}v = 0$ ($Lv = 0$) are called Jacobi fields (on Σ). The conditions in [SL1, I§2] imply that M , M_Σ are second order, quasilinear elliptic operators.

In general, there are non zero Jacobi fields on Σ . Suppose u_s is a 1-parameter family of solutions of $M_\Sigma(u_s) = 0$ with $\partial/\partial s(u_s)|_{s=0} = v$. Then $L_\Sigma v = 0$, so v is a Jacobi field. A Jacobi field on Σ is called *integrable* if it is the initial velocity of such a family u_s .

The special circumstances when the proof in [SL1, II§6] shows that extrema of (1) converge to their limits at exponential rates are those when all Jacobi fields on Σ are integrable. We will show that if the Jacobi fields on Σ are not all integrable, then there may be an extrema of (1) which converges to its limit at only some power rate. The method of proof is to use the contraction mapping principle starting at such a Jacobi field to construct a solution. This idea is used in [SL1, I§7] to construct solutions with exponential decay rates, based on a method of [CHS]. However, different estimates are needed to establish power rate decay.

We wish to solve the equation $Mu = 0$. Solutions of $M_\Sigma(u) = 0$ provide solutions of $Mu = 0$ which are independent of t . We suppose $M_\Sigma(0) = 0$. Write $K = \ker L$ and $P : L^2(\Sigma) \rightarrow K$. Note that $k = \dim K$ is finite since L is elliptic. Now $M_\Sigma(u) = 0$ is equivalent to

- (2) (a) $(I-P)M_{\Sigma}(u) = 0$
 (b) $P M_{\Sigma}(u) = 0$.

Apply the implicit function theorem [NL§2.7] to the analytic map $(I-P)M_{\Sigma}(u)$, whose linearisation has kernel = $\{0\}$ in $C^2(\Sigma, V)$ to deduce the existence of an analytic function $\Phi : B_{\rho}(0) \subseteq K \rightarrow \text{nbhd of } 0$ in $(I-P)C^2(\Sigma, V)$ such that

$$(I-P)M_{\Sigma}(v+\Phi(v)) = 0 .$$

Hence, in a neighbourhood of 0 in $C^2(\Sigma, V)$, the set of solutions of $M_{\Sigma}(u) = 0$ corresponds to the analytic set in K defined by $PM_{\Sigma}(v+\Phi(v)) = 0$.

Write

$$H(v) = PM_{\Sigma}(v+\Phi(v)) , \quad h(v) = \mathcal{F}_{\Sigma}(v+\Phi(v)) ,$$

so H, h are analytic functions in a neighbourhood of 0 in $K \cong \mathbb{R}^k$.

Also $H(v) = -\nabla^{\mathbb{R}^k} h(v)$. Write h as a power series about 0 :

$$h(v) = h(0) + \sum_{r \geq q > 2} h_r(v) ,$$

where h_r is homogeneous of degree r , and $h_q \neq 0$. Since

$$Lv = 0 , \quad \nabla^2 h(0) = 0 \quad \text{so } q > 2 .$$

All Jacobi fields on Σ are integrable if and only if $H(v) \equiv 0$. Thus we can state the result:

THEOREM: If $h_q/m < 0$ at some point on S^{k-1} , then there is a solution $u(t)$ of (1) on $(0, \infty)$ and $\bar{p} \in K$ such that

$$\sup_{0 < t} (T+t)^{\left(\frac{1}{(q-2)+\epsilon}\right)} \left| u(t) - \bar{p}/(T+t)^{\frac{1}{(q-2)}} \right| < \infty,$$

for some $T > 1$, $\epsilon > 0$.

Remark: If $h_q/m \geq 0$ on S^{k-1} , then there may be a solution $u(t)$ on $(-\infty, 0)$ if $E(w, -t, u, p, q)$ has exponential decay with respect to $|t|$ as $|t| \uparrow \infty$.

Proof: Let $h_q|_{S^{n-1}}$ have a minimum at p , so $\nabla_{S^{n-1}} h_q(p) = 0$, i.e.

$H_q(p) = \nabla_{\mathbb{R}^n} h_q(p) = h_q(p) - p$, with $h_q(p)/m < 0$. Put $\bar{v} = \left[\frac{m}{h_q(p)} \frac{1}{(q-2)} \right]^{\frac{1}{(q-2)}} \left[T+t \right]^{\frac{-1}{(q-2)}} p$, where $T > 1$ is to be chosen. Hence

$$-m \frac{d}{dt}(\bar{v}) + H_q(\bar{v}) = 0.$$

The equation $Mu = 0$ is equivalent to

$$(3) \quad \begin{aligned} (a) \quad & (I-P)M(u) = 0 \\ (b) \quad & PM(u) = 0. \end{aligned}$$

The linearisation of the operator in (a) about 0 is $(I-P)\mathcal{L}$, which can be written $((I-P)w)'' - m((I-P)w)' + L_{\Sigma}w$ (see [SL1, I§4]). The

linearisation of the operator in (b) about $\bar{v} + \Phi(\bar{v})$ can be written

$$(Pw)'' - m(Pw)' + (T+t)^{-1}M(Pw) ,$$

where $M : K \rightarrow K$ is a finite dimensional linear operator. Thus we can rewrite (3) as

$$(4) \quad \begin{aligned} (a) \quad & ((I-P)w)'' - m((I-P)w)' + L_{\Sigma}w = R_{(a)}(\bar{v}, w) \\ (b) \quad & (Pw)'' - m(Pw)' - (T+t)^{-1}MPw = R_{(b)}(\bar{v}, w) . \end{aligned}$$

Define iterative maps $A_{(a)}(w)$ to be the solution of (4)(a) with right hand side $R_{(a)}(\bar{v}, w)$, and $A_{(b)}(w)$ to be the solution of (4)(b) with right hand side $R_{(b)}(\bar{v}, A_{(a)}(w) + Pw)$. Note that $A_{(a)}$ must be used to define $A_{(b)}$. Also, 4(b) is a finite system of ordinary differential equations for which $t = \infty$ is a singular point of the second kind (see [CL, Ch.5]). Special estimates based on asymptotic series need to be used to construct solutions of this equation.

One can prove estimates for these maps which imply that they are contractions in suitable Banach spaces, so there exists a (unique) solution. Certain boundary values can be prescribed on Σ , namely those components allowable in [SL1, I§6], and, depending on p , certain components in $\ker L_{\Sigma}$. For details, see [AD]. q.e.d.

In [SL1, I§3], it is shown that functionals for *energy* and *area* have the form (1), and satisfy appropriate conditions. (One puts $t = -\log|x|$ if the domain is $B_1(0)$.) The above result implies the existence of harmonic maps which converge to tangent maps and minimal

surfaces which converge to tangent cones at rates proportional to powers of $1/\log|x|$ as $|x| \rightarrow 0$, if there are non-integrable Jacobi fields.

In [AD], it is shown that the metric, g , on $\mathbb{R}^n \sim \{0\}$ can be analytically perturbed so that

$$S^{n-1} \longrightarrow \langle \mathbb{R}^n \sim \{0\}, g \rangle$$

is harmonic, but there are non integrable Jacobi fields. From work of [AA], it follows that products of 3 or more spheres of suitable radii have non-integrable Jacobi fields. Also, Nagura [NT] showed that there are non-integrable Jacobi fields on minimal surfaces given by immersions into spheres by harmonic polynomials of high degree. All these examples have codimension at least 2.

Milani [M] showed that there exist minimising currents whose supports are not analytic sets. However, his examples have subanalytic support. (For the definition of subanalytic, see e.g. [HI], [HA]). The minimal surfaces constructed above do not even have subanalytic support, since they converge to tangent cones at rates given by $1/\log|x|$, not $|x|^\alpha$ (see [BR]). However, it is not known whether these are minimising examples, even if the tangent cone is taken to be minimising.

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