HOLOMORPHIC REPRESENTATIONS OF $SL(2,\mathbb{R})$ AND QUANTUM SCATTERING THEORY

J.V. Corbett

1. Quantum Scattering

The notation that we use will be essentially that of Reed and Simon [1]. The Hamiltonian operator that describes a system of N particles that interact via two-body potentials is

$$H = H_o + V$$

In the centre of mass coordinate system,

$$H_o = -\sum_{j=1}^{N} \Delta_j/2m_j + \Delta_{c.m.}/2M_{c.m.}$$

$$V = \sum_{i<j} V_{ij}(x_i - x_j) \ , \quad V_{ij} : \mathbb{R}^3 \to \mathbb{R}$$

on the Hilbert space $\mathcal{H}_0 = L^2(\mathbb{R}^{3(N-1)})$. In this work we will impose two conditions on the V_{ij}

(1) each $V_{ij}(y)$ is Δ_y - compact

(2) each $\underset{\sim}{y} \cdot \underset{\sim}{\nabla} V_{ij}(\underset{\sim}{y})$ is Δ_y - bounded.

The first condition ensures that H is self-adjoint on the domain $\mathcal{D}(H) = \mathcal{D}(H_0)$. The second condition will be needed later.

As the N particles may be found in a variety of bound subsystems each moving freely with respect to the others we need some notation.

A cluster decomposition D_k is a partition of $\{1,2, \ldots, N\}$ into k subsets $\{C_j\}_{j=1}^{k}$.

Intercluster potential I_D is the sum of all potentials V_{ij} linking different clusters in D.

The cluster Hamiltonian $H_D = H - I_D$

$$= H_D^0 + \sum_{j=1}^{k} H(C_j)$$

where H_D^0 is the sum of the kinetic energies of the centres of mass of the k clusters minus the kinetic energy of the centre of mass of the total system, and for each j, $H(C_j)$ is the sum of the kinetic energies of the particles in the cluster C_j minus the kinetic energy of the centre of mass of the cluster C_j plus the sum of all the potentials V_{ij} that link particles in the cluster C_j.

The total Hilbert space $\mathcal{H}_o$ can be decomposed for each cluster decomposition D_k into a tensor product of an outer Hilbert space $\mathcal{H}^o_D$ and an inner $\mathcal{H}^i_D$, $\mathcal{H}_o = \mathcal{H}^o_D \otimes \mathcal{H}^i_D$, with $\mathcal{H}^o_D = L^2(\mathbb{R}^{3(k-1)})$ and $\mathcal{H}^i_D = \bigotimes_{j=1}^{k} \mathcal{H}(C_j)$ is the Hilbert space of the internal motion of the cluster C_j.

A channel α, $\alpha = \{D, \eta_\alpha, E_\alpha\}$, is a cluster decomposition D together with a prescription of the bound state, $\eta_\alpha = \prod_{j=1}^{k} \eta^j_\alpha$, $H(C_j)\eta^j_\alpha = E^j_\alpha \eta^j_\alpha$. $E_\alpha = \sum_{j=1}^{k} E^j_\alpha$ is called the threshold energy of the channel α. If a cluster C_j contains only one particle we take $E^j_\alpha = 0$, $\eta^j_\alpha = 1$.

Wave operators

A cluster wave operators $\Omega^\pm_D : \mathcal{H}_o \to \mathcal{H}_o$ is defined as the strong limit, $\Omega^\pm_D = \text{s-lim}_{t\to\mp\infty} e^{iHt} e^{-iH_D t}$

Channel wave operators $\Omega^\pm_\alpha : \mathcal{H}^o_{D(\alpha)} \to \mathcal{H}_o$ are maps from the outer Hilbert space of the clusters in the channel α, to the Hilbert space $\mathcal{H}$; they are given by maps

$$\Omega^\pm_\alpha = \Omega^\pm_{D(\alpha)} \eta_\alpha$$

where $\Omega^\pm_{D(\alpha)}$ are cluster wave operators

i.e. for any $u \in \mathcal{H}^o_{D(\alpha)}$,

$$\Omega^\pm_\alpha \mu = \Omega_{D(\alpha)}(u \otimes \eta_\alpha)$$

Theorem (Hack) [1]

The cluster wave operators $\Omega_D{}^{\pm}$ exist for each cluster decomposition D if each $V_{ij} \in L^2(\mathbb{R}^3) + L^p(\mathbb{R}^3)$ with $2 \leq p < 3$.

Theorem If $\Omega_\alpha{}^{\pm}$ exist, then,

1) Ran $\Omega_\alpha{}^-$ is orthogonal to Ran $\Omega_\beta{}^-$ if $\alpha \neq \beta$, and Ran $\Omega_\alpha{}^+$ is orthogonal to Ran $\Omega_\beta{}^-$ if $\alpha \neq \beta$

2) $\Omega_\alpha{}^{\pm}$ are isometries from $\mathcal{H}_{D(\alpha)}{}^0$ onto $\mathcal{H}_\alpha{}^{\pm} = \text{Ran } \Omega_\alpha{}^{\pm}$.

3) $e^{iHt}(\bigoplus_d \Omega_\alpha{}^{\pm}) = \bigoplus_d \Omega_\alpha{}^{\pm} e^{iH_\alpha t}$

where $H_\alpha = H_{D(\alpha)}{}^0 + E_\alpha$

4) $\bigoplus_\alpha (\text{Ran } \Omega_\alpha{}^{\pm}) \subset \mathcal{H}_{ac}(H)$.

The problem of *asymptotic completeness* is to prove that the following two conditions are satisfied.

1) $\mathcal{H}_c(H) = \mathcal{H}_{ac}(H)$, i.e. $\sigma_{s.c}(H) = \phi$

2) $\mathcal{H}_{ac}(H) = \bigoplus_\alpha (\text{Ran } \Omega_\alpha{}^{\pm})$

If a system is asymptotically complete then the Hilbert space $\mathcal{H}$ for the system can be written as

$$\mathcal{H} = \mathcal{H}_{p.p.}(H) \bigoplus_\alpha (\text{Ran } \Omega_\alpha{}^{\pm})$$

This result has proved to be very difficult to obtain for a general N-body system that is capable of supporting non-trivial channels. However in February 1986 I.M. Sigal and A. Soffer announced in the Bulletin of the A.M.S. [2] that asymptotic completeness holds if the two body potentials V_{ij} satisfy

1) $V_{ij}(y)$ is Δ_y - compact

2) $(1+|y|^2)^{\frac{1+\theta}{2}} (\nabla V_{ij}(y))$ are $\nabla_y \cdot$ - bounded for some $\theta > 0$.

3) $|y|^2 \Delta V_{ij}(y)$ are Δ_y - bounded

4) $(1+|y|^2)^{\mu/2} V_{ij}(y)$ are Δ_y - bounded) , $\mu > 1$

The aim of this paper is to show the usefulness of holomorphic representations of $SL(2,\mathbb{R})$ in the description of the problem of asymptotic completeness. The physical meaning of these representations will be discussed elsewhere. We first quote a theorem that relates asymptotic completeness to holomorphic representations of $SL(2,\mathbb{R})$ on $\mathcal{H}_c(H)$. The second example of their usefulness will be in proving the existence of asymptotic variables for the N-particle system.

2. Holomorphic representations

The representations of $SL(2,\mathbb{R})$ that interest us here occur when the metaplectic representation of the sympletic group is restricted to a subgroup isomorphic to $SL(2,\mathbb{R})$. We will call these representations Weil or holomorphic representations. The metaplectic representation of the symplectic group arises when the symplectic group is considered as a group of outer automorphisms of the irreducible representations of the Heisenberg group. A nice account of these matters, and more, occurs in the review articles of R. Howe [3], the Weil representations of the symplectic group is presented in the paper of Saito [4].

For each cluster decomposition D there is a Weil representation of $SL(2,\mathbb{R})$ on the outer Hilbert space $\mathcal{H}_D^{o}$ of the centres of mass of the clusters in D. If there are k clusters in D, $\mathcal{H}_D^{O} = L^2(\mathbb{R}^{3(k-1)})$ because the overall centre of mass coordinate has been removed. In general these unitary representations are complicated to write out but the corresponding representation of the Lie algebra $sl(2,\mathbb{R})$ of $SL(2,\mathbb{R})$ is easy to present.

The standard basis for $sl(2,\mathbb{R})$ is $\{X_+, X_-, Z\}$, $X_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$,

$X_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The Lie products are $[Z,X_+] = 2X_+$, $[Z_1 X_-] = -2X_-$ and $[X_+,X_-] = Z$. The Casimir operator is Ω_G,

$$\Omega_G = \frac{X_+X_- + X_-X_+}{2} + \frac{Z^2}{4} .$$

The Weil representation π_o of the Lie algebra $sl(2,\mathbb{R})$ is

$$\pi_o(Z) = iA_o = -\frac{i}{2}\sum_{j=1}^{n} (x_j \cdot P_j + P_j \cdot x_j)$$

$$\pi_o(X_+) = iH_o = -\frac{i}{2}\sum_{j=1}^{n} P_j^2/m_j$$

$$\pi_o(X_-) = iR_o = \frac{i}{2}\sum_{j=1}^{n} m_j x_j^2$$

where $P_j = \dfrac{1}{i}\dfrac{\partial}{\partial x_j}$, x_j are the self adjoint operators of differentiation and multiplication by x_j, and m_j are positive numbers.

Furthermore the Casimir operator Ω_G is represented by

$$\pi_0(\Omega_G) = \frac{R_0 H_0 + H_0 R_0}{2} - \frac{A_0{}^2}{4} \quad . \qquad = L^2 + n(n-4)/16$$

where L^2 is the sum of the squares of the generators of the orthogonal group O_n, is the Casimir operator for O_n. Thus the irreducible representations of $SL(2,\mathbb{R})$ that occur here are labelled by the parameters of the irreducible representations of O_n. These representations are holomorphic representations of $SL(2,\mathbb{R})$. [3]

The usefulness of these representations in scattering theory stems from the fact that for each cluster decomposition D_k there is a Weil representation $\pi_{D_k}^{\,o}(g)$ of $SL(2,\mathbb{R})$ on $\mathcal{H}_{D_k} = L^2(\mathbb{R}^{3(k-1)})$ with

$\pi_{D_k}^{\,o}(X_+) = -iH_{D_k}^{\,o}$, $\pi_{D_k}^{\,o}(Z) = iA_{D_k}^{\,o}$ and $\pi_{D_k}^{\,o}(X_-) = iR_{D_k}^{\,o}$.

$H_{D_k}^{0}$ is the kinetic energy of the centres of mass of the k clusters in D_k in the centre of mass frame, $A_{D_k}^{0}$ is the corresponding dilation operator and $R_{D_K}^{0}$ the corresponding Jacobi metric.

This leads to the following theorem [5].

Theorem 1

If the channel wave operators $\Omega_\alpha^\pm$ exist as partial isometrics with orthogonal ranges then the scattering is asymptotically complete if and only if there is a pair of representations $\pi^\pm(g)$ of $SL(2,\mathbb{R})$ on $\mathcal{H}_c(H)$ such that

$$\text{(a)} \quad \pi^\pm\left[\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\right] = e^{-iH^\pm b} \quad \text{for all } b \in \mathbb{R} \text{ with } H^\pm = H - \sum_\alpha E_\alpha P_\alpha^\pm ,$$

$$\text{(b)} \quad \pi^\pm(g)\Big|_{\mathrm{Ran}\Omega^\pm} = \bigoplus_\alpha \pi_\alpha^\pm(g)\Big|_{\mathrm{Ran}\Omega_\alpha^\pm} \quad \text{where each } \pi_\alpha^\pm(g) \text{ is}$$

unitarily equivalent to a Weil representation.

Proof I will just give an outline of the proof here. If asymptotic completeness holds then the conditions (a) and (b) follow from the properties of the $\Omega_\alpha^\pm$. If there exists a pair of representations $\pi^\pm(g)$ of $SL(2,\mathbb{R})$ and $\mathcal{H}_c(H)$ such that (a) and (b) hold then $\mathcal{H}_{ac}(H) = \mathcal{H}_c(H)$ and if $K^\pm = \mathcal{H}_c(H) \ominus_\alpha \mathcal{H}_\alpha^\pm$ there exists on $K^\pm$ sub-representation of $\pi^\pm(g)$ that are unitarily equivalent to a direct sum of Weil representations and hence $K^\pm$ must be the direct sum of the ranges of channel wave operators.

Properties of the representations $\pi_\alpha^\pm(g)\mathcal{H}_\alpha^-$

We will only present the results for the representation $\pi_\alpha^-(g)$ on $\mathcal{H}_\alpha^-$. Those for $\pi_\alpha^+(g)$ on $\mathcal{H}_\alpha^+$ are similar.

Proposition 1 For each channel α with cluster decomposition $D = D(\alpha)$, for all $g \in SL(2,\mathbb{R})$ and for $n_t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

$e^{iHt}\overset{o}{\pi}_D(n_t g n_t^{-1})e^{-iHt}$ converges strongly to $\bar{\pi}_\alpha(g)$ on $\bar{H}_\alpha$.

Note: $\overset{o}{\pi}_D(g) = \overset{o}{\pi}_D(g) \otimes I_{\overset{i}{H}_D}$ following the tensor product decomposition

$H = \overset{o}{H}_D \otimes \overset{i}{H}_D$. We will usually suppress the $I_{\overset{i}{H}_D}$.

Proof For all $\phi \in H_\alpha^-$, let $f \in H_\alpha^0$ be such that $\Omega_\alpha^- f = \phi$.

$$\|e^{iHt}\overset{o}{\pi}_D(n_t g n_t^{-1})\, e^{-iHt}\phi - \pi_\alpha^-(g)\phi\|$$

$$= \|e^{iHt}\, e^{-i\overset{o}{H}_D t}\,\overset{o}{\pi}_D(g)\, e^{i\overset{o}{H}_D t}\, e^{-iHt}\phi - e^{iHt}\, e^{-i\overset{o}{H}_D t}\,\overset{o}{\pi}_D(g)\, e^{i\overset{o}{H}_D t}\, e^{-iE_\alpha t} f\eta_\alpha$$

$$e^{iHt}\, e^{-i\overset{o}{H}_D t}\,\overset{o}{\pi}_D(g)\, e^{i\overset{o}{H}_D t}\, e^{-iE_\alpha t} f\eta_\alpha - \Omega_\alpha^-\,\overset{o}{\pi}_D(g) f\|$$

$$\leq \|e^{iH_D t}\, e^{iHt}\phi - e^{-iE_\alpha t}\eta_\alpha f\|$$

$$+ \|(e^{iHt}\, e^{i\overset{o}{H}_D t}\, e^{-iE_\alpha t}\eta_\alpha - \Omega_\alpha^-)\,\overset{o}{\pi}_D(g) f\|$$

The first term converges to zero as $t \to \infty$ because $\phi = \Omega_\alpha^- f$, the second goes to zero as $t \to \infty$ by definition of Ω_α^- because $\overset{o}{\pi}_D(g) f \in \overset{o}{H}_D$ for all g. This proves the assertion.

Let $g(s)$, $s \in \mathbb{R}$, be a one parameter subgroup of $SL(2,\mathbb{R})$ and let $\overset{o}{L}_D$ be the self-adjoint representative of its generator in the representation $\overset{o}{\pi}_D$ and let L_α^- be the self-adjoint representative of its generator in the representation $\pi_\alpha^-(g)$ on H_α^-.

Proposition 2 In the limit as t tends to infinity,

$$e^{-iHt}\left(e^{-i\overset{o}{H}_D t}\,\overset{o}{L}_D\,e^{i\overset{o}{H}_D t}\otimes I_{H^i_D}\right)e^{-iHt}$$ converges to L_α^- in the strong resolvent sense on H_α^-.

Proof

This result follows immediately from Trotter's Theorem ([1], vol.I) as everything happens on the subspace H_α^- of H.

It is useful to write out this result for the three basic elements of $sl(2,\mathbb{R})$

(i) $e^{iHt}H_D{}^0\,e^{-iHt}$ converges to $(H-E_\alpha)$ in the strong resolvent sense on H_α^-.

(ii) $e^{+iHt}(A_D{}^0 - 2tH_D{}^0)\,e^{-iHt}$ converges to A_α^- in the strong resolvent sense on H_α^-.

(iii) $e^{iHt}(R_D{}^0 - t\,A_D{}^0 + t^2H_D{}^0)\,e^{-iHt}$ converges to R_α^- in the strong resolvent sense on H_α^-.

Let D be the space of C'-vectors for the representation $\pi_0(g)$ of $SL(2,\mathbb{R})$ on $L^2(\mathbb{R}^{3(N-1)})$. It is well known that $D = D(R_0) \cap D(H_0)$ equipped with a norm $\|u\|_D = \|u\| + \|(H_0+R_0)u\|$.

Theorem 2

Let $H = H_0 + \sum_{i<j} V_{ij}$ be such that each $V_{ij}(y)$ is Δ_y-compact and each $(y\cdot\nabla V_{ij})(y)$ is Δ_y-bounded then for all $\phi \in H_\alpha^- \cap D$,

(a) $t^{-2}(\phi,\overset{o}{R}_D(t)\phi)$, (b) $(2t)^{-1}(\phi,\overset{o}{A}_D(t)\phi)$ (c) $(\phi,\overset{o}{H}_D(t)\phi)$

all converge to $(\phi,\ (H-E_\alpha)\phi)$ as t tends to infinity.

Proof

(c) By the special case (i) of Proposition 2, $H_D^\circ(t)$ converges to $(H-E_\alpha)$ in the strong resolvent sense on $\mathcal{H}_\alpha^-$. Furthermore if $z \in \mathbb{C}$, $\mathrm{Im}\, z \neq 0$ then

$(H-z)^{-1}\,\mathcal{H}_\alpha^- \subset \mathcal{H}_\alpha^-$ and so $\lim\limits_{t\to\infty} \| (H_D^\circ(t) - (H-E_\alpha))\phi\| = 0$ for all

$\phi \in \mathcal{H}_\alpha^- \cap \mathcal{D}(H)$.

(a) and (b) are proved in similar ways and we will only look at (a).

We first show that for $\phi \in \mathcal{H}_\alpha^- \cap \mathcal{D}$, $\phi = \Omega_\alpha^- f$ then $t^{-2}\,|(\phi, R_D^\circ(t)\phi) - (f, e^{iH_D^0 t} R_D^\circ e^{-iH_D^0 t} f)| \to 0$ as $t \to \infty$. This depends upon the fact that if $\phi \in \mathcal{D}$ then $\|R_D^\circ e^{-iHt}\phi\| \leq C_2\,(1+|t|^2\|\phi\|_{\mathcal{D}}$ and

$\|A_D^\circ e^{-iHt}\phi\| \leq C_1(1+|t|)\|\phi\|_{\mathcal{D}}$, for the class of Hamiltonians in the enunciation of the theorem.

Now one uses the identity, that holds on $\mathcal{D}$,

$$e^{iH_D^\circ t} R_D^\circ e^{-iH_D^\circ t} = R_D^\circ + t\,A_D^\circ + t^2\,H_D^\circ$$

to obtain that in the limit as $t \to \infty$

$$t^{-2}(\phi, e^{+iHt} R_D^\circ\, e^{-iHt}\phi) = (f,\, H_D^\circ f)$$
$$= (\phi,\, (H-E_\alpha)\phi)$$

as $\phi = \Omega_\alpha^- f$.

Each of the three limits has a physical interpretation or consequence.

$$\lim_{t\to\infty} (\phi,\, H_D^\circ(t)\phi) = (\phi, (H-E_\alpha)\phi) \quad \text{for} \quad \phi \in \mathcal{H}_\alpha^- \cap \mathcal{D}(H) \supset \mathcal{H}_\alpha^- \cap \mathcal{D}$$

says that for such ϕ, $(\phi, H\phi) \le E_\alpha \|\phi\|^2$, and that in the limit as t tends to infinity the total energy augmented by the binding energy of the clusters in the channel α is purely kinetic.

$\lim_{t\to\infty} (\phi, \frac{A_D^\circ(t)}{2t}\phi) = (\phi, (H-E_\alpha)\phi) \ge 0$, means that asymptotically the motion is outgoing as $(\phi, A_D^\circ(t)\phi) > 0$ for t large enough.

$\lim_{t\to\infty} (\phi, \frac{R_D^\circ(t)}{t^2}\phi) = (\phi, (H-E_\alpha)\phi) \ge 0$, means that the radial separation between the clusters in the channel α increases linearly with t for large enough t.

Let $\left\{\zeta_j\right\}_{j=1}$ be the Jacobi coordinates for the centres of mass of the clusters in the channel α, let $\left\{P_j\right\}_{j=1}^{k}$ be their conjugate momenta, $\{M_j\}$ their masses, then we obtain as a corollary, the existence of asymptotic variables in the sense of Enss [6].

Corollary

Let H satisfy the conditions of theorem then for any $\phi \in \mathcal{H}_\alpha^- \cap \mathcal{D}$

$$\lim_{t\to\infty} \|(\zeta_j t^{-1} - P_j M_j^{-1})\phi\| = 0$$

Proof

By Theorem 2,

$$\lim_{t\to\infty} \left(\phi_t, \left(\frac{R_D^\circ}{t^2} - \frac{A_D^\circ}{t} + H_D^\circ\right)\phi_t\right) = 0$$

But the left side is $\|(\zeta_j t^{-1} - P_j M_j^{-1})\phi\|^2$.

This corollary together with the three results of Theorem 2 describe the propagation properties of states in the channels. More work needs to be done before asymptotic completeness is proven.

REFERENCES

[1] M. Reed and B. Simon "Methods of Modern Mathematical Physics" Vol.III. Scattering Theory, Chapter X1.5, Academic Press. NY. (1974) M. Reed and B. Simon I, Vol.I Trőtters Theorem (V111.21).

[2] I.M. Sigal, A. Soffer "Asymptotic Completeness of Short-Range Many-Body [2] Systems". Bulletin A.M.S. 14, 107 (1986).

[3] R. Howe, "On the role of the Heisenberg Group in Harmonic Analysis" Bulletin of A.M.S. Vol.3, Number 2, 1980 pp.821-843. "Dual Pairs in Physics: Harmonic Oscillators Photons, Electrons, and Singletons". Lectures in Applied Mathematics, Volume 21, 1985. American Mathematical Society.

[4] Saito, M. "Représentations unitaires des groups symplectiques". J. Math-Soc Japan 24, 1972, 232.

[5] Corbett, J.V. "Asymptotic Completeness for N-body scattering" Macquarie Mathematics Report. Nov. 1984.

[6] Enss, V. "Quantum Scattering Theory for Two- and Three-Body Systems with Potentials of Short and Long Range", Lecture Notes in Mathematics, 1159, Schrődinger Operators, Springer-Verlag. Berlin. Heidelberg, New York Tokyo.

School of Mathematics and Physics
Macquarie University NSW 2109
Australia