

O. BY WAY OF INTRODUCTION

Amounts (the extensive quantities), modelled by set functions, are in general the primary quantities and states (densities or the intensive quantities), modelled by point functions, the derived ones. Therefore, mathematical models constructed in terms of integrals have conceptual and often also technical priority with respect to ones constructed in terms of derivatives.

A relatively detailed description – with references but without proofs – of a mathematical model of homogeneous isotropic diffusion and of its superposition with a process of creation and/or destruction, illustrates this point. It also gives us an opportunity to introduce problems for which the classical (Lebesgue) integration structure is inadequate and to make a suggestion about the nature of this inadequacy.

So, this chapter represents what is commonly, but inaccurately, called the motivation for the material presented in the subsequent chapters. Also, in Section G, the nature of that material is briefly described and so, the way of approaching the problems introduced in this chapter is indicated. This chapter does not form a part of the systematic exposition though; no reference to it is made in the subsequent chapters.

A. In this section, a mathematical model of homogeneous isotropic diffusion is described.

Let E be the Banach space of all real or complex Borel measures in \mathbb{R}^3 , that is, real or complex valued σ -additive set functions whose domain is the σ -algebra, \mathcal{B} , of all Borel sets in \mathbb{R}^3 . The norm, $\|\varphi\|$, of an element φ of E is the total variation of φ . By $BL(E)$ is denoted the algebra of all bounded linear operators on E . By I is denoted the identity operator on E .

Now, assume that the space, represented as \mathbb{R}^3 , is filled with a solvent into which some soluble substance was added. The distribution of that substance is represented by a (real) positive element of E . Its value on the whole space, which is equal to its norm, is the total amount of the substance added.

For every $t \geq 0$, $x \in \mathbb{R}^3$ and $B \in \mathcal{B}$, let the number $k_t(x, B)$ have the following interpretation: If, at the time 0, a unit amount of the diffusing substance is placed at the point x , then, at the time t , the amount of that substance found in the set B is precisely $k_t(x, B)$. Consistently with this interpretation we assume that

(i) $k_0(x, B) = \delta_x(B)$, for every $x \in \mathbb{R}^3$ and $B \in \mathcal{B}$, that is, $k_0(x, B) = 1$, if $x \in B$, and $k_0(x, B) = 0$, if $x \notin B$;

(ii) for every $t \geq 0$ and every $x \in \mathbb{R}^3$, the set function $B \mapsto k_t(x, B)$, $B \in \mathcal{B}$, represents a probability measure on \mathcal{B} , that is, a non-negative element of E such that $k_t(x, \mathbb{R}^3) = 1$;

(iii) for every $t \geq 0$ and every $B \in \mathcal{B}$, the function $x \mapsto k_t(x, B)$, $x \in \mathbb{R}^3$, is \mathcal{B} -measurable.

The set function $B \mapsto k_t(x, B)$, $B \in \mathcal{B}$, of the requirement (ii) is the distribution of the diffusing substance at the time t provided that a unit amount of the substance is situated at the point x at the time 0. So, the requirement (ii) respects the principle of the conservation of mass. By (i), the requirement (iii) is automatically satisfied for $t = 0$. Without imposing some condition, such as (iii), on the studied kernel not even the most basic analytic techniques would be applicable to it and it would be difficult to interpret it as describing any physical process. On the other hand, the condition (iii) suffices for drawing useful conclusions from the principles of the conservation of mass and of the superposition.

So, assume that at time 0 the distribution of the diffusing substance is represented by the measure $\varphi \in E$, $\varphi \geq 0$. For a fixed $B \in \mathcal{B}$ and $t \geq 0$, let $\mu(X) = \mu_{\varphi, t, B}(X)$ be the amount of the substance which, at the time 0, was in a set $X \in \mathcal{B}$ and at the time t , is found in the set B . Then the principles of superposition and conservation of mass applied to the given situation imply that μ is an additive set function such that

$$\varphi(X) \inf\{k_t(x, B) : x \in X\} \leq \mu(X) \leq \varphi(X) \sup\{k_t(x, B) : x \in X\},$$

for every $X \in \mathcal{B}$. It then follows, from (ii) and (iii), that

$$\mu(X) = \int_X k_t(x, B) \varphi(dx) ,$$

for every $X \in \mathcal{B}$. In particular, $\mu(\mathbb{R}^3)$ is the total amount of the diffusing substance found in the set B at the time t . Hence,

(iv) if the distribution of the diffusing substance at the time 0 is represented by a measure $\varphi \in E$, then the distribution of this substance at a time $t \geq 0$ is represented by the measure $\psi \in E$ given by

$$(A.1) \quad \psi(B) = \int_{\mathbb{R}^3} k_t(x, B) \varphi(dx) ,$$

for every $B \in \mathcal{B}$.

For every $t \geq 0$, let $S(t) : E \rightarrow E$ be the map such that, for every $\varphi \in E$, the element $\psi = S(t)\varphi$ of E is given by (A.1). Then, by (i), $S(0) = I$. Furthermore, by (ii), $S(t)$ is a continuous linear map of E into E of norm equal to 1.

Now we restrict our attention to a time-homogeneous, space-homogeneous and isotropic diffusion. The time-homogeneity is expressed by the condition that $S(t+s) = S(t)S(s)$, for every $s \geq 0$ and $t \geq 0$, that is, the map $t \mapsto S(t)$, $t \geq 0$, from $[0, \infty)$ into $\text{BL}(E)$, is a semigroup of operators. It means that the conditions of diffusion, that is, the properties of the environment and the diffusing substance influencing the diffusion, do not change in time. By (A.1), it can be stated explicitly by requiring that

$$\int_{\mathbb{R}^3} k_{t+s}(x, B) \varphi(dx) = \int_{\mathbb{R}^3} k_t(x, B) \int_{\mathbb{R}^3} k_s(y, dx) \varphi(dy) ,$$

for every $\varphi \in E$. This requirement is of course equivalent to the statement that

(v) the equality

$$(A.2) \quad k_{t+s}(x, B) = \int_{\mathbb{R}^3} k_t(y, B) k_s(x, dy)$$

holds for every $s \geq 0$, $t \geq 0$ and $B \in \mathcal{B}$.

The requirement of the space-homogeneity means that the properties of diffusion are the same around every point of the space \mathbb{R}^3 . Expressed formally,

(vi) for every $t \geq 0$, there is a measure $\kappa_t \in E$ such that $k_t(x, B) = \kappa_t(B-x)$, for every $x \in \mathbb{R}^3$ and every $B \in \mathcal{B}$.

Recall that $B - x = \{y - x : y \in B\}$ for any $B \subset \mathbb{R}^3$ and $x \in \mathbb{R}^3$. By (ii), κ_t is a probability measure in \mathbb{R}^3 , that is a non-negative element of E such that $\kappa_t(\mathbb{R}^3) = 1$, for every $t \geq 0$. If the requirement (vi) is in force, then the equality (A.2) takes the form

$$(A.3) \quad \kappa_{t+s}(B) = \int_{\mathbb{R}^3} \kappa_t(B-x) \kappa_s(dx),$$

for every $s \geq 0$, $t \geq 0$ and $B \in \mathcal{B}$.

The isotropy means that the diffusion is the same in every direction. In formal terms, it reduces to the requirement that

(vii) for every $t \geq 0$, the measure κ_t is invariant with respect to the rotations of the space \mathbb{R}^3 about the origin.

If we add to all these requirements also a certain requirement of continuity, then the maps $S(t) : E \rightarrow E$, $t \geq 0$, describing the process of diffusion, are determined up to a positive parameter - the diffusion constant - which characterizes the speed of this process. In fact, the following theorem, due to G.A. Hunt, holds.

THEOREM 0.1. *Let κ_t , $t \geq 0$, be rotationally invariant probability measures on \mathbb{R}^3 such that the equality (A.1) holds for every $s \geq 0$, $t \geq 0$ and $B \in \mathcal{B}$. Assume that, for every $\epsilon > 0$,*

$$\lim_{t \rightarrow 0+} \frac{1}{t} \kappa_t(\{x : |x| \geq \epsilon\}) = 0.$$

Then, either $\kappa_t = \delta_0$ for every $t \geq 0$, or there exists a constant $D > 0$ such that $\kappa_0 = \delta_0$ and

$$(A.4) \quad \kappa_t(B) = \frac{1}{(4\pi Dt)^{3/2}} \int_B \exp\left[-\frac{|x|^2}{4Dt}\right] dx$$

for every $t > 0$ and every $B \in \mathcal{B}$.

This theorem appeared in greater generality in [25]. It is also presented in Section 2 of Chapter IV of H. Heyer's book [22]. A convenient proof, of such degree of generality that corresponds to the formulation given here, can be found in the notes, [53], on Brownian motion by E. Nelson.

Given a $D > 0$, let

$$(A.5) \quad p(t, x) = \frac{1}{(4\pi Dt)^{3/2}} \exp\left[-\frac{|x|^2}{4Dt}\right]$$

for every $t \geq 0$ and $x \in \mathbb{R}^3$. The formula (A.4) says that the function $x \mapsto p(t, x)$, $x \in \mathbb{R}^3$, is the density of the measure κ_t , for every $t \geq 0$. The function p itself is the solution of the Cauchy problem

$$\dot{u}(t, x) = D\Delta u(t, x), \quad t \geq 0, \quad x \in \mathbb{R}^3; \quad \lim_{t \rightarrow 0+} \int_B u(t, x) dx = \delta_0(B), \quad B \in \mathcal{B}.$$

It is useful to note, for the indicated physical interpretation, that the dimension (unit of measurement) of the constant D is the reciprocal of the unit of time. The values of the measures κ_t , $t \geq 0$, are dimensionless numbers. In fact, if a measure $\varphi \in E$ represents the distribution of the diffusing substance at time 0, then its values are given in a unit of mass. Further, at any time $t \geq 0$, the distribution of the substance is represented by the measure $\psi = S(t)\varphi$, where

$$\psi(B) = \int_B \kappa_t(B-x) \varphi(dx)$$

for every $B \in \mathcal{B}$, and the values of ψ are of course too given in that unit of mass. Consequently, the values of p are given in the reciprocal of a unit of volume.

Using the notation (A.5), the semigroup $S : [0, \infty) \rightarrow \text{BL}(E)$, describing the considered physical process of diffusion, can be expressed in the following concise form: $S(0) = I$ and

$$(A.6) \quad (S(t)\varphi)(B) = \int_B \left[\int_{\mathbb{R}^3} p(t, x-y) \varphi(dy) \right] dx,$$

for every $t > 0$, $\varphi \in E$ and $B \in \mathcal{B}$.

B. Now we describe a mathematical model of a chemical reaction.

We have in mind the following (idealized) situation. The space, \mathbb{R}^3 , is filled with a medium (solvent) in which another substance is distributed. The distribution of this substance is represented by a non-negative or sometimes an arbitrary real valued element of E . The substance reacts with the environment or is in an unstable state so that it changes and thereby increases or decreases in amount. At the same time, we assume that the concentration is so small that the reaction does not alter the environment. On the other hand, we assume that the reaction-rate is proportional to the concentration of the reacting substance and admit that the coefficient of the proportion varies with place and possibly also with time.

To arrive at a formal description of such a process, we assume that, for every $t \geq 0$, an operator $T(t) \in \text{BL}(E)$ is given which has the following meaning. If a measure $\varphi \in E$ represents the distribution of the reacting substance at the time 0, then $T(t)\varphi$ represents the distribution of the reacting substance at the time $t \geq 0$.

Consistently with this interpretation, we assume that $T(0) = I$, the identity operator.

The assumption that the reaction-rate is proportional to the concentration of the reacting substance is then expressed by assuming that a function V on $[0, \infty[\times \mathbb{R}^3$ is given such that

$$(B.1) \quad (\dot{T}(t)\varphi)(B) = \int_B V(t, x) (T(t)\varphi)(dx)$$

for every $t \geq 0$, every $\varphi \in E$ and every $B \in \mathcal{B}$.

Besides (B.1) we assume that

$$(B.2) \quad \lim_{t \rightarrow 0+} T(t)\varphi = \varphi$$

for every $\varphi \in E$.

The conditions (B.1) and (B.2) strongly suggest the presence of the exponential function about. It actually enters formally in the following way.

For every $B \in \mathcal{B}$, let $P(B) \in \text{BL}(E)$ be the operator defined by

$$(P(B)\varphi)(X) = \varphi(B \cap X) = \int_X \chi_B(x) \varphi(dx)$$

for every $\varphi \in E$ and every $X \in \mathcal{B}$.

Then, clearly,

$$(i) \quad P(\mathbb{R}^3) = I;$$

$$(ii) \quad P(B \cap C) = P(B)P(C) \text{ for every } B \in \mathcal{B} \text{ and } C \in \mathcal{B};$$

(iii) if $\varphi \in E$ and $B_j \in \mathcal{B}$, $j = 1, 2, \dots$, are pair-wise disjoint sets whose union is the set B , then

$$P(B)\varphi = \sum_{j=1}^{\infty} P(B_j)\varphi.$$

Given a \mathcal{B} -measurable function W on \mathbb{R}^3 , we denote by

$$P(W) = \int_{\mathbb{R}^3} W(x) P(dx)$$

the operator, whose domain is the set of all measures $\varphi \in E$ such that W is φ -integrable, such that

$$(P(W)\varphi)(X) = \int_X W(x) \varphi(dx) = \int_{\mathbb{R}^3} W(x) (P(dx)\varphi)(X)$$

for every $X \in \mathcal{B}$.

In plain Slovak, $P(W)\varphi$ is the indefinite integral of the function W with respect to φ interpreted of course as an element of the space E . For this reason, some authors, in their depravity, denote $P(W)$ simply as W , i.e. $W = P(W)$. So, $W\varphi$ then stands for the indefinite integral of W with respect to φ . If φ is absolutely continuous, then so is $P(W)\varphi$ and the density of $P(W)\varphi$ is equal to the (point-wise) product of W and the density of φ .

Note that the domain of the operator $P(W)$ is a vector subspace of E . If W is bounded then the domain of $P(W)$ is the whole of E and $\|P(W)\| = \sup\{\|W(x)\| : x \in \mathbb{R}^3\}$.

It is immediate that

- (i) $P(cW) = cP(W)$ for any number c and a measurable function W ;
- (ii) $P(W_1 + W_2) \supset P(W_1) + P(W_2)$ for any measurable functions W_1 and W_2 ; and
- (iii) $P(W_1 W_2) \supset P(W_1)P(W_2)$ for any measurable functions W_1 and W_2 .

Using this machinery, we deduce from (B.1) and (B.2) that

$$(B.3) \quad T(t) = P\left(\exp\left[\int_0^t V(s, \cdot) ds\right]\right),$$

for every $t \geq 0$, which means just that

$$(T(t)\varphi)(X) = \int_X \exp\left[\int_0^t V(s, x) ds\right] \varphi(dx)$$

for every φ belonging to the domain of the operator (0.9) and every $X \in \mathcal{B}$.

More generally, let

$$T(t, s) = P\left[\exp\left[\int_s^t V(r, \cdot) d\rho\right]\right]$$

for any $0 \leq \sigma \leq t$. The interpretation of the operators $T(t, s)$ is clear.

C. In this section we describe a mathematical model of evolution of the distribution of a substance which simultaneously undergoes the processes of diffusion and a chemical reaction.

What we are set up to do is to produce a family of operators $U(t) \in \text{BL}(E)$, $t \geq 0$, which have the following meaning: If $\varphi \in E$ is the distribution, at the time 0, of a substance which diffuses in \mathbb{R}^3 and also is subject to a reaction which causes its creation or destruction, then $U(t)\varphi$ is the distribution of this substance at any time $t \geq 0$.

For the sake of simplicity, we will assume the diffusion to be time-homogeneous, space-homogeneous and isotropic, as in Section A, so that there is a constant $D > 0$ such that the semigroup of operators $S: [0, \infty) \rightarrow \text{BL}(E)$ describing it is given by (A.6), for every $t \geq 0$, $\varphi \in E$ and $B \in \mathcal{B}$. Further, we will assume that the reaction-rate does not change in time so that the process of reaction is described by the semigroup $T: [0, \infty) \rightarrow \text{BL}(E)$, where $T(t) = \exp(tP(V))$, for every $t \geq 0$, and V is a function on \mathbb{R}^3 . This is a special case of the situation discussed in Section B, in particular the formula (B.9), when the function V does not depend on time.

Then, of course, $U(0) = I$. For $t \geq 0$, we can expect that $U(t)$ will be well approximated by the operators of the form

$$(C.1) \quad U_\alpha(t) = S(t-t_n)T(t_n-t_{n-1})S(t_n-t_{n-1}) \dots \\ \dots T(t_3-t_2)S(t_3-t_2)T(t_2-t_1)S(t_2-t_1)T(t_1)S(t_1),$$

where α is a sufficiently fine partition of the interval $[0, t]$ given by the points $0 = t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n \leq t$.

Let us introduce a mathematical structure in which this suggestion can be conveniently explored.

For a given $t \geq 0$, let \mathcal{T}_t be the set of all continuous maps $v: [0, t] \rightarrow \mathbb{R}^3$. The elements of \mathcal{T}_t are usually referred to as paths in \mathbb{R}^3 based on the interval $[0, t]$.

Let \mathcal{K}_t be the family of all sets

$$(C.2) \quad Y = \{v \in \mathbb{T}_t : v(t_j) \in B_j, \quad j = 1, 2, \dots, n\}$$

such that n is a natural number, $0 \leq t_1 < t_2 < \dots < t_{n-1} < t_n \leq t$ and $B_j \in \mathcal{B}$ for every $j = 1, 2, \dots, n$.

Then \mathcal{R}_t is a semiring of sets in \mathbb{T}_t . Let

$$M_t(Y) = S(t-t_n)P(B_n)S(t_n-t_{n-1})P(B_{n-1}) \dots P(B_2)S(t_2-t_1)P(B_1)S(t_1)$$

for every set $Y \in \mathcal{R}_t$ given in the form (C.2). Then $M_t : \mathcal{R}_t \rightarrow L(E)$ is an additive set function.

Before returning to our problem, let us note a point about integration with respect to M_t . Namely, if $0 \leq t_1 \leq t$ and W_1 is a function on \mathbb{R}^3 and if $h_1(v) = W_1(v(t_1))$, for every $v \in \mathbb{T}_t$, then

$$\int_{\mathbb{T}_t} h_1 dM_t = \int_{\mathbb{T}_t} W_1(v(t_1))M_t(dv) = S(t-t_1)P(W_1)S(t_1),$$

provided the function h_1 is M_t -integrable. Similarly, if $0 \leq t_1 < t_2 \leq t$ and W_1 and W_2 are functions on \mathbb{R}^3 and if $h_2(v) = W_1(v(t_1))W_2(v(t_2))$, for every $v \in \mathbb{T}_t$, then

$$\int_{\mathbb{T}_t} h_2 dM_t = \int_{\mathbb{T}_t} W_1(v(t_1))W_2(v(t_2))M_t(dv) = S(t-t_2)P(W_2)S(t_2-t_1)P(W_1)S(t_1),$$

provided the function f is M_t -integrable. And so on.

You may note that we have not yet specified what we mean by integrability with respect to M_t . The presented statements and their obvious inductive extensions simply mean that if the integral with respect to M_t is introduced with the slightest regard to reasonableness, then these formulas must be true. Moreover, the function h_2 , say, should be M_t -integrable on \mathbb{T}_t if the function $(x_1, x_2) \mapsto W_1(x_1)W_2(x_2)$, $(x_1, x_2) \in \mathbb{R}^3 \times \mathbb{R}^3$, is integrable on $\mathbb{R}^3 \times \mathbb{R}^3$ with respect to the additive set function

$$B_1 \times B_2 \mapsto S(t-t_2)P(B_2)S(t_2-t_1)P(B_1)S(t_1), \quad B_1 \in \mathcal{B}, \quad B_2 \in \mathcal{B}.$$

So, the operator (C.1) can be written as

$$\begin{aligned} U_\alpha(t) &= \int_{\mathbb{T}_t} \left[\prod_{j=1}^n \exp(V(v(t_j))(t_j - t_{j-1})) \right] M_t(dv) = \\ &= \int_{\mathbb{T}_t} \left[\exp \left[\sum_{j=1}^n V(v(t_j))(t_j - t_{j-1}) \right] \right] M_t(dv) . \end{aligned}$$

Accordingly, we define

$$(C.3) \quad e_t(v) = \exp \left[\int_0^t V(v(s)) ds \right] ,$$

for every $v \in \mathbb{T}_t$. Then we would expect that

$$(C.4) \quad U(t) = \int_{\mathbb{T}_t} e_t(v) M_t(dv)$$

for every $t \geq 0$. Let us show that this expectation is warranted.

First, the formula (C.4) means to say that if $u(t) = U(t)\varphi$, for any given $\varphi \in E$, then

$$(C.5) \quad u(t) = \int_{\mathbb{T}_t} \left[\exp \left[\int_0^t V(v(s)) ds \right] \right] M_t(dv) \varphi .$$

That is, $u(t)$ is equal to the integral of the function e_t with respect to the E -valued additive set function $Y \mapsto M_t(Y)\varphi$, $Y \in \mathcal{R}_t$. Comments about integration with respect to this set function are postponed into the next section.

Now, assuming $t \geq 0$ given, let

$$(C.6) \quad f(s, v) = V(v(s)) \exp \left[\int_0^s V(v(r)) dr \right] ,$$

for every $s \in [0, t]$, and $v \in \mathbb{T}_t$. Then

$$\int_{\mathbb{T}_t} f(s, v) M_t(dv) \varphi = S(t-s) P(V) u(s) ,$$

for every $s \in [0, t]$, and

$$\int_0^t f(s, v) ds = \exp \left[\int_0^t V(v(s)) ds \right] - 1 ,$$

for every $v \in \mathbb{T}_t$. Therefore, by the Fubini theorem,

$$\begin{aligned} u(t) - S(t)\varphi &= \int_{\mathbb{T}_t} \left[\exp \left[\int_0^t V(v(s)) ds \right] - 1 \right] M_t(dv) \varphi = \\ &= \int_{\mathbb{T}_t} \left[\int_0^t f(s, v) ds \right] M_t(dv) \varphi = \int_0^t \left[\int_{\mathbb{T}_t} \varphi(s, v) M_t(dv) f \right] d\sigma = \\ &= \int_0^t S(t-s) P(V) u(s) ds , \end{aligned}$$

or

$$(C.7) \quad u(t) = S(t)\varphi + \int_0^t S(t-s) P(V) u(s) ds .$$

If the function V is not 'too large', then, for any $t > 0$, the measure $u(t) \in E$, given by (C.5), is absolutely continuous (with respect to the Lebesgue measure in \mathbb{R}^3). (This is of course obvious for $V = 0$.) If we then abuse the notation and denote by $x \mapsto u(t, x)$, $x \in \mathbb{R}^3$, the density of $u(t)$, we can re-write the integral equation (C.7) as

$$u(t, x) = \int_{\mathbb{R}^3} p(t, x-y) \varphi(dy) + \int_0^t \int_{\mathbb{R}^3} p(t-s, x-y) V(y) u(s, y) dy ds , \quad x \in \mathbb{R}^3 ,$$

which represents the initial-value problem

$$(C.8) \quad \begin{aligned} u_t(t, x) &= D\Delta u(t, x) + V(x)u(t, x) , \quad t \geq 0 , \quad x \in \mathbb{R}^3 ; \\ \lim_{t \rightarrow 0+} \int_B u(t, x) dx &= \varphi(B) , \quad B \in \mathcal{B} . \end{aligned}$$

Our original problem of the superposition of diffusion and a chemical reaction is most commonly formulated as this initial-value problem.

It is clear that formula (C.5) has certain advantages against the integral equation (C.7) and the problem (C.8). For it represents $u(t)$ in a form which allows

various calculations and estimations which are not possible directly from (C.7) or (C.8). Secondly, (C.5) may have a good meaning also when (C.7) or (C.8) do not have a solution (in some sense) or cannot even be written down.

D. Generalizations, considered in [31] and [34], of the situation discussed in the previous section give us the opportunity to introduce problems for which the classical integration theory is inadequate.

Let E be an arbitrary Banach space, $\text{BL}(E)$ the space of bounded linear operators on E and $S: [0, \infty) \rightarrow \text{BL}(E)$ a continuous semigroup of operators. So,

- (i) $S(t+s) = S(t)S(s)$, for every $s \geq 0$ and $t \geq 0$;
- (ii) $S(0) = I$, the identity operator; and
- (iii) $\lim_{t \rightarrow 0+} S(t)\varphi = \varphi$, for every $\varphi \in E$.

Let Λ be a locally compact Hausdorff space, $\mathcal{B}(\Lambda)$ the σ -algebra of Baire sets in Λ and $P: \mathcal{B}(\Lambda) \rightarrow \text{BL}(E)$ a σ -additive spectral measure. That is to say,

- (i) $P(B \cap C) = P(B)P(C)$, for every $B \in \mathcal{B}(\Lambda)$ and $C \in \mathcal{B}(\Lambda)$;
- (ii) $P(\Omega) = I$; and
- (iii) the set function $B \mapsto P(B)\varphi$, $B \in \mathcal{B}(\Lambda)$, is σ -additive, for every $\varphi \in E$.

For a Baire function W on Λ , we denote by

$$P(W) = \int_{\Lambda} W dP$$

the operator such that

$$P(W)\varphi = \int_{\Lambda} W(x)P(dx)\varphi$$

for every $\varphi \in E$ for which the right-hand side exists as integral with respect to the E -valued measure $B \mapsto P(B)\varphi$, $B \in \mathcal{B}(\Lambda)$. (A standard reference for integration with respect to spectral measures and also with respect to Banach valued measures is the monograph [14] of N. Dunford and J.T. Schwartz.) The operator $P(W)$ is bounded, that is, belongs to $\text{BL}(E)$, if and only if the function W is essentially bounded. In general, $P(W)$ is a densely defined closed operator on E .

Given a Baire function V on Λ , assume that the function $\exp V$ is essentially bounded on Λ . Then, for every $t \geq 0$, the function $\exp(tV)$ too is essentially bounded. In that case, let

$$T(t) = P(\exp(tV)),$$

for every $t \geq 0$. The resulting map $T: [0, \infty) \rightarrow \text{BL}(E)$ is a continuous semigroup of operators such that $P(V)$ is its infinitesimal generator. That is

$$P(V)\varphi = \lim_{t \rightarrow 0+} \frac{1}{t} (T(t)\varphi - \varphi)$$

for every φ in the domain of $P(V)$. Then we also write

$$T(t) = \exp(tP(V))$$

for every $t \geq 0$, as customary in the theory of continuous semigroups.

The semigroups S and T are interpreted as describing two evolution processes in which an element φ of the space E is transformed, during a time-interval of duration $t \geq 0$, into the element $S(t)\varphi$ and $T(t)\varphi$, respectively. Our problem is to determine the element of the space E into which a given element φ evolves in a time $t \geq 0$ if both these processes go on simultaneously. In other words, we wish to construct a semigroup U which describes the superposition of the processes described by the semigroups S and T .

This problem is traditionally formulated in terms of differential equations. Let

$$A\varphi = \lim_{t \rightarrow 0+} \frac{1}{t} (S(t)\varphi - \varphi)$$

for every $\varphi \in E$ for which this limit exists. The operator A , the infinitesimal generator of the semigroup S , is not bounded in general.

So, we are seeking a semigroup whose infinitesimal generator is $A + P(V)$, that is, a solution of the initial-value problem

$$(D.1) \quad \dot{U}(t) = AU(t) + P(V)U(t), \quad t \geq 0; \quad U(0+) = I.$$

In other words, we look for the fundamental solution of the differential equation

$$\dot{u}(t) = Au(t) + P(V)u(t), \quad t \geq 0$$

with the unknown E -valued function, u , right-continuous at 0 .

This problem is non-trivial because, strictly speaking, it is not even unambiguously formulated. The point is that the operator $A + P(V)$ is not necessarily the infinitesimal generator of a continuous semigroup of operators. On the other hand, this operator may have an extension which is an infinitesimal generator, but such an extension may not be unique. It is conceivable that the obvious generalizations of the objects introduced in the previous section would be helpful in clarifying the issues involved in this problem and in solving it.

For a $t \geq 0$, let Υ_t be a 'sufficiently rich' set of maps $v: [0, t] \rightarrow \Lambda$, to be called paths in Λ . Let \mathcal{R}_t be the family of all sets (C.2) for arbitrary $n = 1, 2, \dots$, $0 \leq t_1 < t_2 < t_3 < \dots < t_{n-1} < t_n \leq t$ and $B_j \in \mathcal{B}(\Lambda)$, $j = 1, 2, \dots, n$. Let

$$M_t(Y) = S(t-t_n)P(B_n)S(t_n-t_{n-1})P(B_{n-1}) \dots P(B_2)S(t_2-t_1)P(B_1)S(t_1)$$

for any such set Y .

Then a heuristic argument, similar to that presented in the previous section, suggests that the operators $U(t)$ can be expressed by the means of the Feynman-Kac type formula:

$$(D.2) \quad U(t) = \int_{\Upsilon_t} \left[\exp \left[\int_0^t V(v(r)) dr \right] \right] M_t(dv),$$

for every $t \geq 0$. In fact, an integral equation for U can be derived in an manner precisely analogous to that of deriving (C.7). Namely, assume that $t \geq 0$, that the function e_t is given by (C.3) for every $v \in \Upsilon_t$ and that the function f is given by (C.6) for every $s \in [0, t]$ and $v \in \Upsilon_t$. Then

$$\int_{\Upsilon_t} f(s, v) M_t(dv) = S(t-s)P(V)U(s)$$

for every $s \in [0, t]$ and

$$\int_0^t f(s, v) ds = \exp \left[\int_0^t V(v(r)) dr \right] - 1 ,$$

for every $v \in \mathbb{T}_t$. Therefore, by the Fubini theorem (!),

$$\begin{aligned} U(t) - S(t) &= \int_{\mathbb{T}_t} \left[\exp \left[\int_0^t V(v(r)) dr \right] - 1 \right] M_t(dv) = \\ &= \int_{\mathbb{T}_t} \left[\int_0^t f(s, v) ds \right] M_t(dv) = \int_0^t \left[\int_{\mathbb{T}_t} f(s, v) M_t(dv) \right] ds = \\ &= \int_0^t S(t-s) P(V) U(s) ds . \end{aligned}$$

The obtained integral equation,

$$(D.3) \quad U(t) = S(t) + \int_0^t S(t-s) P(V) U(s) ds ,$$

replaces the initial-value problem (D.1).

It goes without saying that once we have a solution of the problem (D.1), then we have solutions of the initial-value problems

$$(D.4) \quad \dot{u}(t) = Au(t) + P(V)u(t) , \quad t \geq 0 ; \quad u(0+) = \varphi ,$$

for all $\varphi \in E$. Indeed, it suffices to put $u(t) = U(t)\varphi$, for every $t \geq 0$, where U is a solution of (D.1). On the other hand, the point of the formula (D.2), or the formula (C.5) for a given $\varphi \in E$, is that $U(t)$ or $u(t)$ could possibly be defined by these formulas even when the initial-value problems (D.1) or (D.4) do not have a solution or perhaps could not even be meaningfully formulated.

The question then arises whether the formulas (C.5) and (D.2) can be put on a solid footing. Or, rather, whether a formal framework can be erected in which these formulas have a good meaning and the conditions for a legitimate use of the operations leading to them can be formulated.

Now, integration with respect to the $\text{BL}(E)$ -valued set function M_t is reduced to integration with respect to the E -value set functions $Y \mapsto M_t(Y)\varphi$, $Y \in \mathcal{R}_t$, for every $\varphi \in E$.

Accordingly, the equality (D.2) is defined to mean that

$$U(t)\varphi = \int_{\mathbb{T}_t} \left[\exp \left[\int_0^t V(v(r)) dr \right] \right] M_t(dv) \varphi ,$$

for every $\varphi \in E$, where the integral is understood with respect to the E -valued set function. This reduction is analogous to integration with respect to spectral measures. It has the advantage that one may attempt the construction of a solution of the problem (D.4), for some $\varphi \in E$, by the means of the formula (C.5) and thus avoid the fundamental solution. In fact, it is conceivable that the integral (C.5) may exist for some $\varphi \in E$ while the integral (D.2) does not.

So, there remains the problem how to integrate with respect to the E -valued set functions $Y \mapsto M_t(Y)\varphi$, $Y \in \mathcal{R}_t$, for $\varphi \in E$.

EXAMPLE 0.2. In the case when S is the diffusion semigroup (see Section A) and T the creation/destruction process semigroup (see Section B), the means for an easy solution of this problem are provided by the Wiener measure. In fact, given a set $Y \in \mathcal{R}_t$ of the form (C.2), the number

$$\begin{aligned} w(Y) &= \int_{\mathbb{R}^3} \int_{B_{n-1}} \int_{B_{n-1}} \cdots \int_{B_2} \int_{B_1} p(t-t_n, y-x_n) p(t_n-t_{n-1}, x_n-x_{n-1}) \cdots \\ &\quad \cdots p(t_2-t_1, x_2-x_1) p(t_1, x_1) dx_1 dx_2 \cdots dx_{n-1} dx_{n-1} dx_n dy = (M_t(Y) \delta_0)(\mathbb{R}^3) , \end{aligned}$$

where the kernel p is given by (A.5), is equal to the Wiener measure (or variance 2D per unit of time) of the set Y . To be sure, w is a probability measure on the whole of the σ -algebra \mathcal{S}_t generated by the family of sets \mathcal{R}_t .

Now, for a set $Y \subset \mathbb{T}_t$ and $x \in \mathbb{R}^3$, let $Y-x$ be the set of all paths $s \mapsto v(s) - x$, $s \in [0, t]$, such that $v \in Y$. Let $w_x(Y) = w(Y-x)$ for every $Y \in \mathcal{R}_t$ and $x \in \mathbb{R}^3$. Then w_x is a probability measure on \mathcal{S}_t such that $w_x(Y) = (M_t(Y) \delta_x)(\mathbb{R}^3)$ for every $Y \in \mathcal{R}_t$. Furthermore, if $\varphi \in E$, let

$$w_\varphi(Y) = \int_{\mathbb{R}^3} w_x(Y) \varphi(dx) = \int_{\mathbb{R}^3} w(Y-x) \varphi(dx)$$

for every $Y \in \mathcal{S}_t$. Then w_φ is a real or complex valued σ -additive measure on \mathcal{S}_t such that $w_\varphi(Y) = (M_t(Y)\varphi)(\mathbb{R}^3)$ for every $Y \in \mathcal{R}_t$. Hence, if μ is the variation of the measure φ , then w_μ is a finite positive measure on \mathcal{S}_t such that $|(M_t(Y)\varphi)(B)| \leq w_\mu(Y)$ for every $Y \in \mathcal{R}_t$ and every $B \in \mathcal{B}(\mathbb{R}^3)$. So, if $\varphi \in E$ is real-valued then the norm of the element $M_t(Y)\varphi$ of E , that is, the total variation of this measure, is not greater than $w_\mu(Y)$, for any $Y \in \mathcal{R}_t$. If $\varphi \in E$ is complex then the norm of $M_t(Y)\varphi$ is not greater than $2w_\mu(Y)$, say, for every $Y \in \mathcal{R}_t$. Consequently, there exists a unique continuous linear map $i_\varphi: \mathcal{L}^1(w_\mu) \rightarrow E$ such that $i_\varphi(f) = M_t(Y)\varphi$, whenever f is the characteristic function of a set $Y \in \mathcal{R}_t$. Therefore, we may declare a function f on Υ_t to be integrable with respect to the E -valued set function $Y \mapsto M_t(Y)\varphi$, $Y \in \mathcal{R}_t$, if it is w_μ -integrable and define

$$\int_{\Upsilon_t} f(v) M_t(dv) \varphi = i_\varphi(f)$$

for every $f \in \mathcal{L}^1(w_\mu)$.

EXAMPLE 0.3. Let $E = L^2(\mathbb{R}^3)$. The Fourier-Plancherel transform of an element $\varphi \in E$ is denoted by $\hat{\varphi}$. Let m be a (strictly) positive number. For every real t , let $S(t): E \rightarrow E$ be the map uniquely determined by the requirement that

$$(S(t)\varphi)^\wedge(\xi) = \exp\left[-\frac{t}{2m} |\xi|^2\right] \hat{\varphi}(\xi)$$

for every $\varphi \in E$ and (almost) every $\xi \in \mathbb{R}^3$. The Plancherel theorem implies that $S(t): E \rightarrow E$ is a unitary operator and the resulting map $S: \mathbb{R} \rightarrow \text{BL}(E)$ is a continuous group of operators.

For every $t \neq 0$ and $x \in \mathbb{R}^3$, let

$$(D.5) \quad p(t, x) = \frac{1}{(2\pi i t/m)^{3/2}} \exp\left[im \frac{|x|^2}{2t}\right].$$

Then,

$$(S(t)\varphi)(x) = \int_{\mathbb{R}^3} p(t, x-y)\varphi(y)dy$$

for every $\varphi \in L^1 \cap L^2(\mathbb{R}^3)$. The kernel (D.5) is obtained from (A.5) by substituting $D = 2/mi$.

The group $t \mapsto S(t)$, $t \in (-\infty, \infty)$, is called the Schrödinger group. It is interpreted as the description of the motions (evolutions) of a free non-relativistic quantum mechanical particle of mass m with three degrees of freedom. The states of such a particle are determined by elements of the space E with norm equal to 1. The word 'free' indicates that no external forces are acting on the particle. Then, if the particle is at a state φ at time $t = 0$, then, at any other time $t \in (-\infty, \infty)$, the particle was or will be at the state $S(t)\varphi$.

Let $\Lambda = \mathbb{R}^3$. For every $B \in \mathcal{B}(\mathbb{R}^3) = \mathcal{B}$, let $P(B)$ be the operator of point-wise multiplication by the characteristic function of the set B . Then, clearly, $P: \mathcal{B} \rightarrow \text{BL}(E)$ is a spectral measure. If W is a measurable function on \mathbb{R}^3 , then $P(W)$ is the operator of multiplication by W . Therefore, one usually writes simply W instead of $P(W)$.

Now, let V be a real-valued function on \mathbb{R}^3 interpreted as the potential of the forces acting on the particle. Let

$$T(t) = \exp(-itP(V))$$

for every $t \in (-\infty, \infty)$. The group T describes the fictitious motions of the particle under the influence of the forces with potential V assuming that 'inertial motions' are suspended.

The superposition, U , of these groups S and T describes the real motions of the particle in the force-field of potential V . The group U can be considered the fundamental solution of the equation

$$\dot{u}(t) = \frac{i}{2m} \Delta u(t) - iVu(t), \quad t \in (-\infty, \infty).$$

That is to say, if $\varphi \in E$ and $u(t) = U(t)\varphi$, for every $t \in (-\infty, \infty)$, then

$$(D.6) \quad \dot{u}(t, x) = \frac{i}{2m} \Delta u(t, x) - i V(x) u(t, x), \quad t \in (-\infty, \infty), \quad x \in \mathbb{R}^3; \quad u(0, x) = \varphi(x), \quad x \in \mathbb{R}^3,$$

assuming that φ is represented by a sufficiently smooth function and the potential V is not 'too bad'.

However, there are considerable difficulties associated with the construction of the semigroup U by the means of the formula (D.2) because it is not at all clear how to integrate with respect to the $\text{BL}(E)$ -valued set function M_t . Indeed, for most vectors $\varphi \in E$ and $\psi \in E$, the scalar valued set functions $Y \mapsto \langle \psi, M_t(Y) \varphi \rangle$, $Y \in \mathcal{R}_t$, (the scalar product in E) has infinite variation on every 'non-trivial' set in \mathcal{R}_t .

E. Because of its significance, Example 0.3 deserves further comments.

Although the problems posed by Example 0.3 are much more difficult to handle, historically it precedes Example 0.2. In his Thesis, [15], R.P. Feynman suggested the replacement of the initial-value problem (D.6) by the formula

$$(E.1) \quad u(t, x) = \int_{\mathcal{T}_t} \exp \left[\frac{i}{\hbar} \left[\frac{m}{2} \int_0^t |\dot{v}(s)|^2 ds - \int_0^t V(v(s)) ds \right] \right] \varphi(v(0)) \mathcal{D}(v)$$

(with some insignificant changes of notation) which is to be understood as

$$(E.2) \quad u(t, x) = \lim_{n \rightarrow \infty} \left[\frac{2\pi i \hbar t}{nm} \right]^{-3n/2} \overbrace{\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \cdots \int_{\mathbb{R}^3}}^{n \text{ times}} \exp \left[\frac{i}{\hbar} \left[\frac{m}{2} \sum_{k=1}^n |x_k - x_{k-1}|^2 - \frac{t}{n} \sum_{k=1}^n V(x_k) \right] \right] \varphi(x_0) dx_0 dx_1 \cdots dx_{n-1},$$

where $x_n = x$. The possibilities of an approach to quantum mechanics based on this suggestion are systematically explored in the book [16] by R.P. Feynman and A.R. Hibbs.

The formula (E.1) has a great heuristic value. Its attractiveness to physicists is to a considerable degree based on the fact that, apart from the factor i/\hbar , the argument of the exponential function is equal to the classical action along the trajectory v . This heuristic value seems to be responsible for the resilience of this

formula, its popularity and that of its variants and generalizations, in spite of serious conceptual difficulties associated with it.

The main difficulty presented by (E.1) is that the integration 'with respect to the variable v ' over the space, \mathcal{T}_t , of paths in \mathbb{R}^3 refers to integration with respect to the infinite product of copies of the Lebesgue measure in \mathbb{R}^3 indexed by all the time-instants from the interval $[0, t]$. However, such an infinite-dimensional analogue of the Lebesgue measure does not exist. This is caused by the fact that the measure of the whole space \mathbb{R}^3 is infinite so that the measure of any (presumably measurable) set in \mathcal{T}_t would be either 0 or ∞ . This state of affairs cannot be remedied by admitting into \mathcal{T}_t more (or even all) maps $v: [0, t] \rightarrow \mathbb{R}^3$ besides the continuous ones. This difficulty is intrinsic and directly insurmountable. Therefore, (E.1) cannot be taken as anything more than a suggestive way of writing (E.2)

By interpreting (E.1) as the limit (E.2), the mentioned difficulty is to a certain degree circumvented together with that which is related to the existence of the derivatives $\dot{v}(s)$ for $v \in \mathcal{T}_t$ and $s \in [0, t]$. However, it should be born in mind that the integrals with respect to x_1, x_2, \dots, x_{n-1} are not absolutely convergent because the integrand has constant absolute value. So, one cannot arbitrarily change the order of integrations.

There is considerable literature devoted to definitions of the Feynman integral, interpreted as the limit (E.2), exploiting, roughly speaking, a suitable summability method for the calculation of the finite-dimensional integrals in (E.2) and/or its approximation which facilitates the subsequent passage to the limit.

In a somewhat different manner, a rigorous meaning can be given to (E.1) by constructing the superposition U of the semigroups S and T , defined in Example 0.3, through approximation of the operators $U(t)$ by operators of the form (C.1). In fact, H.F. Trotter, [65], and T. Kato, [27], have found conditions under which the limit

$$U(t)\varphi = \lim_{n \rightarrow 0} (S(tn^{-1})T(tn^{-1}))^n \varphi$$

exists for every $\varphi \in E = L^2(\mathbb{R}^3)$. A variant of this approach is used by G.N. Gestrin, in his paper [17].

However, with this approach integration over the function space \mathcal{T}_t is to some extent suppressed and with it the heuristic value of (E.1) diminishes. In a way, the same can be said about many definitions of (E.1) using sequential limits. A certain useful compromise in this direction is achieved by E. Nelson in his influential paper [52]. He uses the Trotter-Kato formula to guarantee the construction, by the means of integrals over the space \mathcal{T}_t of continuous paths, of the operators $U_\zeta(t)$ analogous to $U(t)$ but with the mass m replaced by complex numbers ζ with positive imaginary parts. For any $\varphi \in E$, the so obtained E -valued function $\zeta \mapsto U_\zeta(t)\varphi$ is then analytic in the upper complex half-plane and the vector $U(t)\varphi = U_m(\varphi)$ is obtained as the boundary value of this function at m , that is, as the non-tangential limit for $\zeta \mapsto m$, $\text{Im}\zeta > 0$. Unfortunately, the boundary value exists only for almost every m (in the sense of the Lebesgue measure).

Nelson's approach led to considerable insight into the situation, especially in the cases of some badly behaved functions V , but still, it did not solve completely the problem of maintaining the heuristic value of the formula (E.1) and, at the same time, turning it into a sufficiently flexible and reliable analytic tool. It seems that a solution of this problem cannot be tied too closely to the specific properties of the Schrödinger group. A structure or a method is called for which is applicable in a wider class of cases. A hint that such a structure might exist can be derived from the work of Mark Kac. He noted that, if the factor i/\hbar is dropped from the exponent in (E.1), then the integral can be given a perfect meaning in terms of the Wiener measure. (Cf. the exposition in [26] Chapter IV.) Of course, by dropping the factor i/\hbar we switch to a different problem. One of the possible physical meanings of the new problem is described in Section C; to another one is devoted the book [61] of B. Simon (see also its review [54] by E. Nelson).

The 'derivation' of the equation (D.3), or (C.7), shows that an integration scheme which allows 'integration with respect to sufficiently wild set functions of

infinite variation' and for which a Fubini-type theorem holds, would do for the required structure. Such an integration scheme is presented in Chapters 2 and 3.

F. To emphasize that the difficulties observed in Example 0.3 are caused neither by the fact that the underlying space, T_t , is infinite-dimensional nor the fact that the values of the integrator belong to an infinite-dimensional space, in this section we mention a classical case in which both, the underlying space and the space of values, are one-dimensional, none-the-less the same difficulties as in Example 0.3 occur. In fact, if the function g has infinite variation in every non-degenerate sub-interval of the interval $[a, b]$, then the difficulties associated with the (definition, existence, properties,... of the) Stieltjes integral

$$(F.1) \quad \int_a^b f(x) dg(x)$$

are in principle the same as with

$$\int_{T_t} f(v) \langle \psi, M_t(dv) \varphi \rangle ,$$

if E , M_t , φ , ψ etc. have the same meaning as in Example 0.3.

Stieltjes integrals (F.1) are the subject of attention for many reasons. Perhaps the most prominent among them is the exploration of the possibilities of integration with respect to (individual) sample paths of stochastic processes, such as the Wiener process, and of the analysis of the solutions of stochastic differential equations. In spite of marked successes, such as that of H. Sussmann, [63], the progress in this direction seems still not satisfactory.

An interesting approach to integrals (F.1), exploiting the moduli of continuity of the functions f and g , was initiated by L.C. Young in [69]. The best result is due to A. Beurling, [3], who used a most ingenious method for introducing integrals of this type. Unfortunately, Beurling's method is difficult to extend to cases in which the interval $[a, b]$ of the real-line is replaced by a more general space. Secondly, it does

not provide a complete metric in the space of functions integrable with respect to a fixed integrator.

G. The classical theory of absolutely convergent integrals proved to be inadequate in the situations described in previous sections. So, it is desirable to produce a more general theory of integration which would be applicable not only in the classical situations but also in the situations similar to those mentioned above.

Because a generalized theory necessarily lacks certain features of a more specific one, the question arises: which aspects of the classical theory of integration should be considered so essential that also the more general theory must retain them? This question is a result of two related concerns, namely that about the actual erection of the new theory and that of its usefulness. That is, we wish to choose those aspects of the classical theory on the basis of which the new theory could be conveniently developed and, at the same time, would guarantee that the new, more general, theory would be sufficiently powerful in the situations for which it is intended. Such a choice is of course a matter of an interpretation of the integration theory.

A short reflection would reveal that an interpretation which is formulated in terms of a particular method, or procedure, used for introducing integrable functions and/or integral, is not really helpful. Then the most fruitful of the 'objective' interpretations of the integration theory, that is, those which are independent of any such procedure, seems to be one that characterizes the L^1 -space as the completion of simple functions (continuous functions,...) in the L^1 -norm. The point of a particular construction of integral is in showing that the completion is represented by functions on the original underlying space or equivalence classes of such functions.

This interpretation can be further refined by noting that there exist families of functions which generate the L^1 -space and are not necessarily vector spaces. That is, the L^1 -space is the completion of the linear hull of such a family and its norm is the largest norm with a given restriction to the generating family of functions. Characteristic functions of sets belonging to a sufficiently rich family of measurable sets can serve as a typical example. To make this remark more perspicuous, we recall

the following fact concerning the classical integration theory.

Let Ω be an abstract space and \mathcal{Q} a semiring of its subsets. For simplicity, a subset of Ω and its characteristic function are denoted by the same symbol. Let ι be a real-valued (finite) non-negative σ -additive set function on \mathcal{Q} .

A function, f , on Ω is integrable with respect to (the measure generated by) ι if and only if there exist numbers c_j and sets $X_j \in \mathcal{Q}$, $j = 1, 2, \dots$, such that

$$(G.1) \quad \sum_{j=1}^{\infty} |c_j| \iota(X_j) < \infty$$

and

$$(G.2) \quad f(\omega) = \sum_{j=1}^{\infty} c_j X_j(\omega)$$

for every $\omega \in \Omega$ for which

$$(G.3) \quad \sum_{j=1}^{\infty} |c_j| X_j(\omega) < \infty.$$

Moreover, the L^1 -norm,

$$\|f\| = \int_{\Omega} |f| d\iota,$$

of such a function f is equal to the infimum of the sums (G.1) taken for all such choices of the numbers c_j and sets $X_j \in \mathcal{Q}$, $j = 1, 2, \dots$. A proof of this fact is given in Section 2E below. The vector space of all (individual) functions integrable with respect to ι is denoted by $\mathcal{L}(\iota)$.

In the case when Ω is an interval of the real-line, \mathcal{Q} consists of intervals and ι is the Lebesgue measure, it can be easily visualized. This case is commented on in more detail in the Preface to the book [50] of J. Mikusiński and in [33]. More comments can be found in Section 3B. Now we mention just a straightforward but important consequence of this fact.

If μ is an additive set function on \mathcal{Q} (which may, possibly, be vector valued) such that $|\mu(X)| \leq \iota(X)$, for every set $X \in \mathcal{Q}$, then there exists a unique linear functional, $\ell = \ell_{\mu}$, on $\mathcal{L}(\iota)$ such that $\ell(X) = \mu(X)$, for every $X \in \mathcal{Q}$, and

$|\ell(f)| \leq \|f\|$, for every function $f \in \mathcal{L}(\iota)$. In particular, the integral with respect to ι is the linear functional, ℓ , on $\mathcal{L}(\iota)$ such that $\ell(X) = \iota(X)$, for every $X \in \mathcal{Q}$, and $|\ell(f)| \leq \|f\|$, for every $f \in \mathcal{L}(\iota)$.

Now, assume that μ is an additive set function on \mathcal{Q} and that there does not exist a finite σ -additive set function, ι , such that $|\mu(X)| \leq \iota(X)$, for every $X \in \mathcal{Q}$. In the previous sections, we have shown that such set functions occur abundantly and are of considerable interest. In such case, μ does not generate a continuous linear functional on any L^1 -space containing the characteristic functions of all sets from \mathcal{Q} . Nevertheless, there may still exist a complete normed space (strictly speaking, a seminormed space), \mathcal{L} , consisting of functions on Ω and containing the characteristic functions of sets belonging to \mathcal{Q} , such that μ can be extended to a continuous linear functional on \mathcal{L} .

So, we may look for a non-negative set function, ρ , on \mathcal{Q} , which is a restriction to \mathcal{Q} (interpreted of course as a family of functions) of the norm on some such space \mathcal{L} , such that $|\mu(X)| \leq \rho(X)$, for every $X \in \mathcal{Q}$. If the space \mathcal{L} is generated by \mathcal{Q} , that is, it is the completion of the linear hull of \mathcal{Q} , and if the norm on \mathcal{L} is the smallest norm coinciding with ρ on \mathcal{Q} , then μ can be uniquely extended to a continuous linear functional on the whole of \mathcal{L} .

Let us turn the tables and call a non-negative set function, ρ , on \mathcal{Q} an integrating gauge, if

$$\rho(X) \leq \sum_{j=1}^{\infty} |c_j| \rho(X_j)$$

for any set $X \in \mathcal{Q}$, numbers c_j and sets $X_j \in \mathcal{Q}$, $j = 1, 2, \dots$, such that

$$X(\omega) = \sum_{j=1}^{\infty} c_j X_j(\omega)$$

for ever $\omega \in \Omega$ satisfying the inequality (G.3).

Given an integrating gauge, ρ , on \mathcal{Q} , let $\mathcal{L}(\rho, \mathcal{Q})$ be the family of all functions, f , on Ω for which there exist numbers, c_j , and sets, $X_j \in \mathcal{Q}$, $j = 1, 2, \dots$, such that

$$\text{G.4)} \quad \sum_{j=1}^{\infty} |c_j| \rho(X_j) < \infty$$

and the equality (G.2) holds for every $\omega \in \Omega$ for which the inequality (G.3) does. For any such function, $f \in \mathcal{L}(\rho, \mathcal{Q})$, let

$$q(f) = \inf \sum_{j=1}^{\infty} |c_j| \rho(X_j),$$

where the infimum is taken over all choices of numbers c_j and sets $X_j \in \mathcal{Q}$, $j = 1, 2, \dots$, satisfying condition (G.4), such that the equality (G.2) holds for every $\omega \in \Omega$ for which the inequality (G.3) does.

It turns out that $\mathcal{L}(\rho, \mathcal{Q})$ is a vector space and q is a norm (strictly speaking, a seminorm) under which the space $\mathcal{L}(\rho, \mathcal{Q})$ is complete and the linear hull of \mathcal{Q} is dense in it. Moreover, if μ is an additive set function on \mathcal{Q} such that $|\mu(X)| \leq \rho(X)$, for every $X \in \mathcal{Q}$, then there exists a unique linear functional, ℓ , on $\mathcal{L}(\rho, \mathcal{Q})$ such that $\ell(X) = \mu(X)$, for $X \in \mathcal{Q}$, and $|\ell(f)| \leq q(f)$, for every $f \in \mathcal{L}(\rho, \mathcal{Q})$.

Now the problem naturally arises of producing a sufficient supply of integrating gauges. Of the various ways of solving this problem, let us mention the following one. If ι is a finite non-negative σ -additive set function on \mathcal{Q} and φ a continuous, increasing and concave function on $[0, \infty)$ such that $\varphi(0) = 0$, then the set function ρ , defined by $\rho(X) = \varphi(\iota(X))$ for every $X \in \mathcal{Q}$, is an integrating gauge on \mathcal{Q} .

To show the usefulness of this construction, let us indicate how it solves the problem of Stieltjes integration with respect to functions of infinite variation. So, let, for example, $\Omega = (0, 1]$, let \mathcal{Q} be the family of all intervals $(s, t]$ such that $0 \leq s \leq t \leq 1$, let g be a function on $[0, 1]$ such that $|g(t) - g(s)| \leq |t - s|^{\frac{1}{2}}$, for any $s \in [0, 1]$ and $t \in [0, 1]$, and let $\mu(X) = g(t) - g(s)$, for any $X = (s, t] \in \mathcal{Q}$. If we define $\rho(X) = (\iota(X))^{\frac{1}{2}}$, for every $X \in \mathcal{Q}$, where ι is the Lebesgue measure, then we obtain an integrating gauge, ρ , on \mathcal{Q} . Now we can define

$$\int_0^1 f d\mu = \int_0^1 f(x) dg(x),$$

for any function $f \in \mathcal{L}(\rho, \mathcal{Q})$, to be the value, $\ell(f)$, of the continuous linear functional, ℓ , on $\mathcal{L}(\rho, \mathcal{Q})$ such that $\ell(X) = \mu(X)$, for every $X \in \mathcal{Q}$.

Other applications are presented in Chapter 7 in which we return to the problems described in this chapter. Hopefully, they suffice as an indication that the attention paid to the introduced notions is warranted. Nevertheless, it is natural and convenient to introduce a still more general structure which generalizes simultaneously integrals with respect to σ -additive set functions and Daniell integrals. To do so, it suffices to replace the family of characteristic functions of sets belonging to \mathcal{Q} by any sufficiently rich family, \mathcal{K} , of functions on Ω . It is assumed that a functional, ρ , to be called an integrating gauge, is given on \mathcal{K} such that there exists a complete (semi)normed space $\mathcal{L}(\rho, \mathcal{K})$, consisting of functions on Ω such that the linear hull of \mathcal{K} is dense in $\mathcal{L}(\rho, \mathcal{K})$ and the norm of $\mathcal{L}(\rho, \mathcal{K})$ is the smallest norm whose values on \mathcal{K} coincide with those of ρ . The construction of $\mathcal{L}(\rho, \mathcal{K})$ is of course analogous to that of $\mathcal{L}(\rho, \mathcal{Q})$; it is briefly described in the pre-amble to Chapter 2. The definition of integral is sketched in the pre-amble to Chapter 3. Some of the possibilities inherent in this more general structure are exploited in Chapter 6 which deals with the spectral theory of operators. Not without interest may also be the fact, adverted to in Chapter 3, that many classical function spaces may be defined as instances of the space $\mathcal{L}(\rho, \mathcal{K})$, for suitable choices of \mathcal{K} and ρ .