A General Theorem on Gravitational Metrics in Prolate Spheroidal Coordinates.

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Abstract: It is proved that all stationary and axially symmetric metrics that are rational in prolate-spheroidal coordinates have a simple canonical form. This follows directly from a theorem for 2×2 matrices with multinomial elements and has nothing to do with the field equations.

Many physically meaningful stationary and axially symmetric metrics have been constructed in the last two decades including the Kerr metric[1] and the remarkable series of generalisations by Tomimatso and Sato[3,4]. For most of these the metric coefficients are rational in prolate spheroidal coordinates, x and y, and can therefore be written as

$$ds^2 = (D/K)\left(dx^2/V + dy^2/W\right) + ds_2^2, \tag{1}$$

where

$$ds_2^2 = \left\{Adt^2 + 2Cdtd\phi + Bd\phi^2\right\}/D, \tag{2}$$

$$V = x^2 - 1, \qquad W = 1 - y^2. \tag{3}$$

The functions A, B, C and D are all polynomials, and K is rational and indeed often polynomial as well. It is a simple consequence of the gravitational field equations and the requirement that the metric be asymptotically flat that the determinant of ds_2^2 be equal to $-VW$,

$$\begin{vmatrix} A & C \\ C & B \end{vmatrix} = -VWD^2. \tag{4}$$

This is the key equation for the study of such rational metrics! We will show that this implies that there is a canonical diagonalised representation of ds_2^2.

Theorem: *The two dimensional metric ds_2^2 can be written as*

$$ds_2^2 = (I/\Delta)(\pi dt - \rho d\phi)^2 - (J/\Delta)(\sigma dt - \tau d\phi)^2, \tag{5}$$

where π, ρ, σ, τ and Δ are polynomials, I and J are rational functions, and

$$\Delta = |\pi\tau - \rho\sigma|, \quad IJ = VW, \quad D = \Delta \mathrm{Den}(I)\,\mathrm{Den}(J). \tag{6a, b, c}$$

Proof: Equation (4) can be formally written as

$$AB = C^2 - VWD^2 = \left(C + \sqrt{VW}D\right)\left(C - \sqrt{VW}D\right)$$

but $\sqrt{V}$ and $\sqrt{W}$ do not exist in the field of rationals over x and y. Fortunately, the algebraic curve $z^2 = 1 - x^2$ has a well-known rational parametrisation (corresponding to the use of $t = \tan(\theta/2)$ to evaluate certain integrals). Suppose that new coordinates s and t are defined by

$$\begin{aligned} x &= \left(s^2+1\right)/\left(s^2-1\right), \qquad & \sqrt{V} &= 2s/\left(s^2-1\right), \\ y &= \left(1-t^2\right)/\left(t^2+1\right), \qquad & \sqrt{W} &= 2t/\left(t^2+1\right). \end{aligned}$$

Equation (2) is replaced by

$$ds_2^2 = \left\{\tilde{A}dt^2 + 2\tilde{C}dtd\phi + \tilde{B}d\phi^2\right\}/\tilde{D}, \tag{7}$$

where the new polynomials are all totally even (even in both s and t), and it can be arranged that the denominator and numerator have no common factor. From equation (4),

$$\tilde{A}\tilde{B} = \tilde{C}^2 - 4s^2t^2\left(s^2-1\right)^{-2}\left(t^2+1\right)^{-2}\tilde{D}^2$$

and therefore $\tilde{D}$ must be divisible by $(s^2-1)(t^2+1)$,

$$2\tilde{D} = \left(s^2-1\right)\left(t^2+1\right)\tilde{E}, \tag{8}$$

$$\tilde{A}\tilde{B} = \tilde{C}^2 - s^2t^2\tilde{E}^2 = \left(\tilde{C} + st\tilde{E}\right)\left(\tilde{C} - st\tilde{E}\right). \tag{9}$$

The polynomial $\tilde{C} + st\tilde{E}$ can be decomposed into its irreducible factors,

$$\tilde{C} + st\tilde{E} = \prod(\omega)\prod(\alpha + st\beta)\prod(s\gamma + t\delta)\prod(\kappa + st\lambda + s\mu + t\nu), \quad (10)$$

where we have not attempted to index the individual factors and have separated special cases of the last product into the first three. The Greek coefficients are totally even polynomials in s and t.

Consider a factor of the type $f(s,t) = \kappa + st\lambda + s\mu + t\nu$. Both $f(s,t)$ and $f(-s,-t)$ are factors of $\tilde{C} + st\tilde{E}$, whilst $f(-s,t)$ and $f(s,-t)$ are factors of its conjugate, $\tilde{C} - st\tilde{E}$. If we compute the product of the first two factors,

$$f(s,t)\,f(-s,-t) = \left(\kappa^2 + s^2t^2\lambda^2 - s^2\mu^2 - t^2\nu^2\right) + 2st\left(\kappa\lambda - \mu\nu\right),$$

then we see that the bracketed terms are all totally even and so the terms in the fourth product in equation (10) can be absorbed into the second product,

$$\tilde{C} + st\tilde{E} = \prod\omega\prod(\alpha + st\beta)\prod(s\gamma + t\delta), \quad (11a)$$

$$\tilde{C} - st\tilde{E} = \prod\omega\prod(\alpha - st\beta)\prod(s\gamma - t\delta). \quad (11b)$$

Now consider a factor of the first type, $\omega(s^2, t^2)$. Since it occurs in both of $\tilde{C} \pm st\tilde{E}$ it is a factor of $\tilde{C}$ and $\tilde{E}$. From equation (9) ω^2 divides the product of $\tilde{A}$ and $\tilde{B}$. If each had a factor ω then both the numerator and the denominator in equation (7) would be divisible by ω. This contradicts our assumption that all such factors have been cancelled out.

Since both $\tilde{A}$ and $\tilde{B}$ are totally even we know that if $\alpha + st\beta$ is a factor of $\tilde{A}$ then so is $\alpha - st\beta$, and if $s\gamma + t\delta$ is then so is $s\gamma - t\delta$. We can therefore divide the factors of equation (11) into those that are factors of $\tilde{A}$ and those that are factors of $\tilde{B}$,

$$\begin{aligned}
\tilde{C} + st\tilde{E} =& \left[\prod\omega\prod(\alpha + st\beta)\prod(s\gamma + t\delta)\right] \\
&\times\left[\prod\acute{\omega}\prod\left(\acute{\alpha} + st\acute{\beta}\right)\prod\left(s\acute{\gamma} + t\acute{\delta}\right)\right], \\
\tilde{A} =& \epsilon\left[\prod\omega\prod(\alpha + st\beta)\prod(s\gamma + t\delta)\right] \\
&\times\left[\prod\omega\prod(\alpha - st\beta)\prod(s\gamma - t\delta)\right], \\
\tilde{B} =& \epsilon\left[\prod\acute{\omega}\prod\left(\acute{\alpha} + st\acute{\beta}\right)\prod\left(s\acute{\gamma} + t\acute{\delta}\right)\right] \\
&\times\left[\prod\acute{\omega}\prod\left(\acute{\alpha} - st\acute{\beta}\right)\prod\left(s\acute{\gamma} - t\acute{\delta}\right)\right],
\end{aligned}$$

where $\epsilon = \pm 1$. If we multiply out the factors in each pair of square brackets we find that there are two distinct possibilities, depending on the parity of the number of factors of the type $(s\gamma \pm t\delta)$ inside each square bracket,

$$\prod \omega \prod (\alpha + st\beta) \prod (s\gamma + t\delta) = \begin{cases} (i) & \epsilon\tilde{\pi} - st\tilde{\sigma}, \\ (ii) & \epsilon s\tilde{\pi} - t\tilde{\sigma}, \end{cases}$$

$$\prod \acute{\omega} \prod \left(\acute{\alpha} + st\acute{\beta}\right) \prod \left(s\acute{\gamma} + t\acute{\delta}\right) = \begin{cases} (i) & \tilde{\rho} + \epsilon st\tilde{\tau}, \\ (ii) & s\tilde{\rho} + \epsilon t\tilde{\tau}, \end{cases}$$

where $\tilde{\pi}, \tilde{\rho}, \tilde{\sigma}, \tilde{\tau}$ are all totally even, and the signs (including factors of ϵ) have been chosen so that the final results will be simple,

$$(i) \quad \tilde{A} = \epsilon\left(\tilde{\pi}^2 - s^2t^2\tilde{\sigma}^2\right), \quad \tilde{B} = \epsilon\left(\tilde{\rho}^2 - s^2t^2\tilde{\tau}^2\right), \quad \tilde{C} = \epsilon\left(\tilde{\pi}\tilde{\rho} - s^2t^2\tilde{\sigma}\tilde{\tau}\right),$$

$$(ii) \quad \tilde{A} = \epsilon\left(s^2\tilde{\pi}^2 - t^2\tilde{\sigma}^2\right), \quad \tilde{B} = \epsilon\left(s^2\tilde{\rho}^2 - t^2\tilde{\tau}^2\right), \quad \tilde{C} = \epsilon\left(s^2\tilde{\pi}\tilde{\rho} - t^2\tilde{\sigma}\tilde{\tau}\right).$$

The signs are such that

$$\tilde{E} = \tilde{\pi}\tilde{\tau} - \tilde{\rho}\tilde{\sigma}, \qquad \tilde{D} = \left(\tilde{\pi}\tilde{\tau} - \tilde{\rho}\tilde{\sigma}\right)\left(s^2 - 1\right)\left(t^2 + 1\right).$$

Inserting these into equation (2.1),

$$ds_2^2 = \left[\left(\tilde{I}/\tilde{\Delta}\right)\left(\tilde{\pi}dt - \tilde{\rho}d\phi\right)^2 - \left(\tilde{J}/\tilde{\Delta}\right)\left(\tilde{\sigma}dt - \tilde{\tau}d\phi\right)^2\right],$$

where

$$\tilde{\Delta} = \tilde{E} = \left(\tilde{\pi}\tilde{\tau} - \tilde{\rho}\tilde{\sigma}\right), \qquad \tilde{I}\tilde{J} = 4s^2t^2\left(s^2 - 1\right)^{-2}\left(t^2 + 1\right)^{-2}.$$

If we transform back to (x, y) coordinates,

$$s^2 = (x + 1)/(x - 1), \qquad t^2 = (1 - y)/(1 + y),$$

then $\tilde{\pi}, \tilde{\rho}, \tilde{\sigma}, \tilde{\tau}$ become rational functions of x and y with denominators products of powers of $(x - 1)$ and $(1 - y)$. Define a_x (a_y) as the maximum of the x (y) degrees of π and ρ, and similarly b_x (b_y) as the maximum of the x (y) degrees of σ and τ. If we define

$$\begin{aligned} (\pi, \rho) &= (\tilde{\pi}, \tilde{\rho})(x - 1)^{a_x}(y + 1)^{a_y}, \\ (\sigma, \tau) &= (\tilde{\sigma}, \tilde{\tau})(x - 1)^{b_x}(y + 1)^{b_y}, \\ \Delta &= \tilde{\Delta}(x - 1)^{a_x + b_x}(y + 1)^{a_y + b_y} = (\pi\rho - \sigma\tau), \\ I &= \tilde{I}(x - 1)^{b_x - a_x}(y + 1)^{b_y - a_y}, \\ J &= \tilde{J}(x - 1)^{a_x - b_x}(y + 1)^{a_y - b_y}, \end{aligned}$$

then π, ρ, σ, τ and Δ are polynomials and $IJ = VW$. The metric ds_2^2 takes the form of equation (5). Any common factors of π and ρ or of σ and τ should be cancelled out. This will change the factors I and J but will not change the over-all form. Finally, if we expand the quadratic terms in equation (5) to give the form in equation (2) we see that D is given by equation (6c). This completes the proof of the theorem.

In Kerr and Wilson[2] the canonical representations have been given for the first three Tomimatso-Sato solutions. As for most other asymptotically flat solutions, the denominator D is an irreducible polynomial and the rational functions I and J are simple factors of VW. When the T-S parameter $\delta = 3$ the various functions are given by

$$\begin{aligned}
\pi &= q^3W^4 + p^2qV^2\left(6W^2 + 8VW + 3V^2\right) \\
\sigma &= p^3V^4 + pq^2W^2\left(6V^2 + 8VW + 3W^2\right), \\
\rho &= V\sigma + 4pV^3(3V + 4) \\
&\qquad + (2/p)(1 + px)\left[p^2\left(3V^4 + 16V^3 + 16V^2\right) + 3q^2W^4\right], \\
\tau &= W\pi + 4qVW^2(3V + 6W - 4) \\
&\qquad + 8qW^2(1 + px)\left[3V^2 + 3VW + 2V + 6W - 4\right].
\end{aligned}$$

and the diagonal components of the metric by $(I, J) = (W, V)$. These polynomials are far simpler than the original A, B, C, D. For instance, B has over 200 terms for this T-S solution.

Bibliography

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