AUTOMORPHISMS OF WEIGHTED MEASURE ALGEBRAS

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In [5], [6] and [8] we studied the automorphisms of the Volterra algebra on $[0,1]$ and of radical weighted convolution algebras on the half line. The multiplier algebra of each of these algebras can be identified with a measure algebra [4, Thm. 1.4], [11], and played an important role in those studies. In this paper we study the automorphisms of weighted measure algebras. We show that the extensions of the automorphisms of a weighted convolution algebra to its multiplier algebra are the only automorphisms of the multiplier algebra. From this fact we obtain information about the structure of the automorphisms of the measure algebras and for certain classes of weights a complete description of the automorphisms becomes available.

The significance of the measure algebras in the study of homomorphisms and derivations is also apparent in [3], [4], [5], [6], [8], [9] and [10].

Let $\mathbf{R}^+$ denote the non–negative real numbers. By a *radical algebra weight* on $\mathbf{R}^+$, we mean a positive, continuous, submultiplicative function w on $\mathbf{R}^+$, satisfying $w(0) = 1$ and $w(x)^{1/x} \to 0$ as $x \to \infty$. If w is such a weight, and if the Banach space $L^1(\mathbf{R}^+, w)$, consisting of (equivalence classes of) Lebesgue measurable functions f on $\mathbf{R}^+$ satisfying

$$\|f\| = \int_0^\infty |f(x)|w(x)dx < \infty$$

is given the convolution product

$$(f * g)(x) = \int_0^x f(x-y)g(y)dy,$$

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then $L^1(\mathbf{R}^+, w)$ is a commutative radical Banach algebra. Let $M(\mathbf{R}^+, w)$ be the space of Radon measures μ on $\mathbf{R}^+$ satisfying

$$\|\mu\| = \int_{\mathbf{R}^+} w(t)d|\mu|(t) < \infty$$

and let $C_0(1/w)$ be the space of all continuous functions f on $\mathbf{R}^+$ such that $f/w \in C_0(\mathbf{R}^+)$ with the norm $\|f\| = \|f/w\|_\infty$. Then $M(\mathbf{R}^+, w)$ can be identified with the dual of $C_0(1/w)$ by the pairing

$$\langle \mu, f \rangle = \int f d\mu.$$

For any μ, ν in $M(\mathbf{R}^+, w)$, the convolution product of μ and ν can be defined by

$$(\mu * \nu)(\varphi) = \iint \varphi(x+y)d\mu(x)d\nu(y) \quad (\varphi \in C_0(1/w)).$$

With this product $M(\mathbf{R}^+, w)$ becomes a Banach algebra. The mapping $f \mapsto f dx$ defines an isometric embedding of $L^1(w)$ into $M(w)$ and identifies $L^1(w)$ with a closed ideal of $M(w)$.

Recall that if the Banach space $L^1[0, a]$ is given the convolution product

$$(f * g)(x) = \int_0^x f(x-y)g(y)dy,$$

then $L^1[0, a]$ becomes a radical Banach algebra, the Volterra algebra of $[0, a]$. Let $M[0, a)$ be the space of all complex regular Borel measures on $[0, a)$ and let $C_0[0, a)$ be the Banach space of all continuous functions f on $[0, a)$ such that $\lim_{x \to a^-} f(x) = 0$, with $\|f\| = \sup |f(x)|$. Then we have $(C_0[0, a))^* = M[0, a)$, and convolution product $*$ in the latter can be defined by

$$\int_{[0,a)} \psi(x)d(\mu * \nu)(x) = \int_{[0,a)} \int_{[0,a-y)} \psi(x+y)d\mu(x)d\nu(y) \quad (\psi \in C_0[0, a)).$$

The Banach algebra $M[0, a)$ can be identified with the multiplier algebra of $L^1[0, a]$ as in [11], where for every $\mu \in M[0, a)$, $\rho_\mu : f \mapsto f * \mu$ is a multiplier on $L^1[0, a]$ and every multiplier arises in this way.

We first give a characterization of compact multipliers on $L^1[0, a]$. An abstract version of the following theorem is in [7].

THEOREM 1. *A multiplier* $\rho_\mu : f \mapsto f * \mu$ $(f \in L^1[0,a])$ *is compact if and only if* $\mu \in L^1[0,a]$.

Proof. Let ρ_μ be compact, and let (e_n) be a bounded approximate identity of $L^1[0,a]$. Then there exists a subsequence (e_{n_k}) of (e_n) such that $(e_{n_k} * \mu)$ converges in norm to an element $\nu \in L^1[0,a)$. On the other hand (e_{n_k}) converges weak–* to δ_0. Thus $\mu = \nu \in L^1[0,a]$. Now we show conversely, that for every element $f \in L^1[0,a]$, ρ_f is compact. It suffices to prove this for $f \equiv 1$, as this is a topological generator of $L^1[0,a]$. Since

$$g * 1(x) = \int_0^x g(t)dt$$

and it is well known that the operator

$$Tg(x) = \int_0^x g(t)dt$$

is compact on $L^1[0,a]$, the theorem is proved.

THEOREM 2. *If* θ *is an automorphism of* $M[0,a)$, *then* $\theta(L^1[0,a]) = L^1[0,a]$. *Thus, there exists a quasinilpotent derivation* q *and a complex number* λ *such that* $\theta = e^{\lambda X} e^q$, *where* X *is the derivation* $d(X\mu)(t) = td\mu(t)$.

Proof. Let $f \in L^1[0,a]$. Then by Theorem 1, ρ_f is compact. Now we show that $\rho_{\theta(f)}$ is compact. Let $(M)_1$ be the unit ball of $M[0,a)$. Then

$$((M)_1 * \theta(f))^- = (\theta(\theta^{-1}(M_1) * f))^- = \theta((\theta^{-1}(M_1) * f)^-).$$

Thus $\rho_{\theta(f)}$ is compact by the compactness of ρ_f. It follows from Theorem 1 that $\theta(f) \in L^1[0,a]$. To prove the last statement of the theorem we note that if $\theta|L^1[0,a]$ denotes the restriction of θ to $L^1[0,a]$, then it is an automorphism of $L^1[0,a]$. Thus there exists a quasinilpotent derivation q and a complex number λ such that $\theta|L^1[0,a] = e^{\lambda X} e^q$, see [6]. Now if we denote the unique extension of q to $M[0,a)$ by the same symbol, then $e^{\lambda X} e^q$ defines an automorphism of $M[0,a)$ which coincides with the restriction of θ to $L^1[0,a]$. Since automorphisms of $M[0,a)$ leave $L^1[0,a]$

invariant, they are continuous in the strong operator topology (so) of $M[0,a)$. Thus to complete the proof it suffices to show that $L^1[0,a]$ is so–dense in $M[0,a)$. To this end, let $\mu \in M[0,a)$. Then for every $f \in L^1[0,a]$, we have $\mu * f = \lim \mu * e_n * f$, where (e_n) is a bounded approximate identity of $L^1[0,a]$. Since $\mu * e_n \in L^1[0,a]$, the proof is complete.

The following theorem is an extension of [2, Theorem 3.6]. Recall that $\alpha(f)$ is the infimum of supp(f).

THEOREM 3. *Let w be a radical weight function. If $\mu \in M(w)$, and if $\delta > 0$ is such that*

$$\limsup \left(\frac{\|\mu^{*n}\|}{w(\delta n)} \right)^{1/n} < \infty,$$

then $\alpha(\mu) \geq \delta$.

Proof. Let f be a nonzero element of $L^1(w)$ with $\alpha(f) = 0$. Then we have $\mu * f \in L^1(w)$, and by Titchmarsh's convolution theorem $\alpha(\mu) = \alpha(\mu) + \alpha(f) = \alpha(\mu * f)$. Since

$$\limsup \left(\frac{\|(\mu * f)^n\|}{w(\delta n)} \right)^{1/n} \leq \lim \|f^n\|^{1/n} \limsup \left(\frac{\|\mu^n\|}{w(\delta n)} \right)^{1/n} = 0,$$

we have $\alpha(\mu * f) \geq \delta$ by [2, Theorem 3.6]. Hence $\alpha(\mu) \geq \delta$, and the theorem is proved.

Notation. For every $a \geq 0$ let $J_a = \{\mu \in M(w) \ : \ \alpha(\mu) \geq a\}$. These are the so–called "standard ideals" in the measure algebra.

LEMMA 1. *Suppose θ is an automorphism of $M(w)$. Then for every $a \geq 0$, $\theta(J_a) = J_a$.*

Proof. Let $\beta(x) = \alpha(\theta(\delta_x))$, $(x \in \mathbf{R}^+)$. Then

$$\begin{aligned} \beta(x+y) &= \alpha(\theta(\delta_{x+y})) = \alpha(\theta(\delta_x) * \theta(\delta_y)) \\ &= \alpha(\theta(\delta_x)) + \alpha(\theta(\delta_y)) = \beta(x) + \beta(y), \end{aligned}$$

by Titchmarsh's convolution theorem. Next we show that β is right continuous at every $x \in \mathbf{R}^+$. Since β is additive, it suffices to show this for $x = 0$. Let $x_n \to 0$ and let $f \in L^1(w)$. Then,

(1) $$\theta(\delta_{x_n} * f) \overset{\|\cdot\|}{\to} \theta(f).$$

Since the sequence $(\theta(\delta_{x_n}))$ is bounded it has a subnet $(\theta(\delta_{x_i}))$ weak–* convergent to a measure μ say. So

(2) $$\theta(\delta_{x_i}) * \theta(f) \overset{w-*}{\to} \mu * \theta(f).$$

From (1) and (2) and Titchmarsh's convolution theorem it follows that $\mu = \delta_0$. The above argument in particular shows that every weak–* convergent subnet of $(\theta(\delta_{x_n}))$ converges to δ_0. Thus

$$\theta(\delta_{x_n}) \overset{w-*}{\to} \delta_0.$$

Whence $\beta(x_n) = \alpha(\theta(\delta_{x_n})) \to 0$. For otherwise there exists $\epsilon > 0$ such that $\alpha(\theta(\delta_{x_n})) > \epsilon$ for infinitely many values of n. Now if f is a continuous function with $f(0) = 1$ and $\operatorname{supp} f \subset [0, \epsilon]$, then $\langle \delta_0, f\rangle = 1$, while $\langle \theta(\delta_{x_n}), f\rangle = 0$ for infinitely many values of n, which contradicts

$$\theta(\delta_{x_n}) \overset{w-*}{\to} \delta_0.$$

Hence $\beta(x_n) \to 0$, and the function β is continuous. Thus there exists a constant A_θ such that $\beta(x) = A_\theta x$.

Next we show that $A_\theta > 0$. If not, then for every $x \in \mathbf{R}^+$, $\alpha(\theta(\delta_x)) = 0$. We first show that this implies that $\alpha(\theta(f)) = 0$, for every function $f \in L^1(w)$ whose support is compact and $\alpha(f) = 0$. By [1, Theorem 2] we have $(L^1(w) * f)^- = L^1(w)$ for any such f. Thus, if $\alpha(\theta(f)) = k > 0$, then for every $g \in L^1(w)$, we have $\alpha(\theta(g)) \geq k$. Now an application of the Banach–Alaoglu theorem together with Titchmarsh's convolution theorem shows that $\theta(\mu) = \text{weak--}^* \lim \theta(\mu * e_n)$, where (e_n) is a bounded approximate identity of $L^1(w)$. This implies $\alpha(\theta(\mu)) \geq k$, which contradicts the assumption that θ is onto. Thus $\alpha(\theta(f)) = 0$. Next assume that $\alpha(f) = m > 0$ and supp$f \subseteq [m, M]$. Then $f = \delta_m * g$, where $\alpha(g) = 0$ and g has a compact support. Then by Titchmarsh's convolution theorem

$$\alpha(\theta(f)) = \alpha(\theta(\delta_m)) + \alpha(\theta(g)) = 0.$$

Hence from $A_\theta = 0$ it follows that $\alpha(\theta(f)) = 0$, for every f with compact support. We draw a contradiction by showing that there exists an f with compact support and with $\alpha(\theta(f)) > 0$. By [2, Theorem 3.2 II] there exists $f \in L^1(w)$ with supp$f \subseteq [1, 2]$, $\alpha(f) = 1$, and

$$\|f^{*n}\| < w(n), \quad n = 1, 2, \ldots.$$

For this f, we then have

$$\limsup \left(\frac{\|\theta(f)^{*n}\|}{w(n)} \right)^{1/n} \leq 1.$$

Hence by Theorem 3, $\alpha(\theta(f)) \geq 1$. From this contradiction it follows that $A_\theta > 0$.

Next we show that for every $\mu \in M(w)$, $\alpha(\theta(\mu)) = A_\theta \alpha(\mu)$. For μ a finite linear combination of point masses this is immediate. For a general $\mu \in M(w)$, we first prove that $\alpha(\theta(\mu)) \geq A_\theta \alpha(\mu)$. Let (μ_i) be a bounded net in $M(w)$ such that $\mu_i \to \mu$ in the strong operator topology with $\alpha(\mu_i) \geq \alpha(\mu)$ and such that each μ_i is a finite linear combination of point masses [3, Lemma 1.3]. The Banach-Alaoglu theorem together with Titchmarsh's convolution theorem implies that

$$\theta(\mu_i) \overset{w-*}{\rightarrow} \theta(\mu).$$

Now if $\alpha(\theta(\mu)) < A_\theta \alpha(\mu)$, then we choose b such that $\alpha\theta(\mu) < b < A_\theta \alpha(\mu)$ and we let g be a continuous function with supp $g \subset [\alpha(\theta\mu)), b]$ and

$$\int g(x) d\theta(\mu)(x) \neq 0.$$

Since $A_\theta \alpha(\mu) \leq \alpha\theta(\mu_i)$, we have

$$\int g(x) d\theta(\mu_i)(x) = 0.$$

Then

$$0 \neq \int g(x) d\theta(\mu_i)(x) = \lim \int g(x) d\theta(\mu_i)(x) = 0.$$

From this contradiction we conclude

$$\alpha\theta(\mu) \geq A_\theta \alpha(\mu).$$

Now, let $f \in L^1(w)$ have compact support. Then $h = f * \delta_{-\alpha(f)} \in L^1(w)$, $\alpha(h) = 0$, and an earlier argument shows that $\alpha(\theta(h)) = 0$. Hence

$$\alpha(\theta(f)) = \alpha(\theta(h)) + \alpha(\theta(\delta_{\alpha(f)})) = A_\theta \alpha(f).$$

Now suppose $f \in (L^1(w))\backslash\{0\}$ and $\alpha(f) = c$. Let $f_1 = f\chi_{[c,c+1]}$, $f_2 = f\chi_{(c+1,\infty)}$. Then $f = f_1 + f_2$, and so $\theta(f) = \theta(f_1) + \theta(f_2)$. By the conclusion of the previous paragraph we have $\alpha(\theta(f_1)) = A_\theta c$ and $\alpha\theta(f_2) \geq A_\theta(c+1) > A_\theta c$. Therefore,

$$\alpha(\theta(f)) = \min\{\alpha\theta(f_1), \alpha\theta(f_2)\} = A_\theta c = A_\theta \alpha(f).$$

Finally, if $\mu \in M(w)\backslash\{0\}$, then for $f \in (L^1(w)\backslash\{0\})$, the above gives

$$\alpha(\theta(\mu)) + \alpha(\theta(f)) = \alpha(\theta(\mu * f)) = A_\theta \alpha(\mu * f) = A_\theta \alpha(\mu) + A_\theta \alpha(f).$$

Since we have already shown that $\alpha(\theta(f)) = A_\theta \alpha(f)$, it follows that $\alpha(\theta(\mu)) = A_\theta \alpha(\mu))$.

Now by an argument similar to that of [4, Lemma 3] it can be shown that $\theta(\delta_x)$ has a non–zero mass at $\alpha(\theta(\delta_x)) = A_\theta x$. Thus

$$\theta(\delta_x) = k(x)\delta_{A_\theta x} + \mu_x \quad (x \in \mathbf{R}^+),$$

where $k(x) \neq 0$, $\alpha(\mu_x) \geq A_\theta x$ and $\mu_x(\{A_\theta x\}) = 0$. From the equations

$$\theta(\delta_{x+y}) = k(x+y)\delta_{A_\theta(x+y)} + \mu_{x+y},$$

$$\begin{aligned}\theta(\delta_{x+y}) &= \theta(\delta_x) * \theta(\delta_y) = (k(x)\delta_{A_\theta x} + \mu_x) * (k(y)\delta_{A_\theta y} + \mu_y) \\ &= k(x)k(y)\delta_{A_\theta(x+y)} + k(x)\delta_{A_\theta x} * \mu_y + k(y)\mu_x * \delta_{A_\theta y} + \mu_x * \mu_y,\end{aligned}$$

and the fact that the measure $k(x)\delta_{A_\theta x} * \mu_y + k(y)\delta_{A_\theta y} * \mu_x + \mu_x * \mu_y$ has zero mass at $A_\theta(x+y)$, it follows that $k(x+y) = k(x)k(y)$. Since k is bounded near 0, it follows that there exists a complex number z, such that $k(x) = e^{zx}$ for every $x \in \mathbf{R}^+$. Thus

$$(3) \qquad \theta(\delta_x) = e^{zx}\delta_{A_\theta x} + \mu_x$$

with $\alpha(\mu_x) \geq A_\theta x$, and $\mu_x(\{A_\theta x\}) = 0$. Similarly, there exists a complex number ζ such that

$$(4) \qquad \theta^{-1}(\delta_x) = e^{\zeta x}\delta_{A_\theta^{-1}x} + \nu_x, \quad (x \in \mathbf{R}^+).$$

with $\alpha(\nu_x) \geq A_\theta^{-1}x$ and $\nu_x(\{A_\theta^{-1}x\}) = 0$. Now an argument similar to that of the proof of [8, Lemma 2] shows that $\zeta = -z$. From (3) and (4) it then follows that

$$\|\theta^{-1}\|^{-1} \leq e^{bx}\frac{w(A_\theta x)}{w(x)} \leq \|\theta\|, \quad (x \in \mathbf{R}^+),$$

where $b = \mathrm{Re}(z)$. Now if $A_\theta > 1$, then we have

$$(5) \qquad \|\theta^{-1}\|^{-1} \leq e^{bx}\frac{w((A_\theta - 1)x)w(x)}{w(x)} \leq e^{bx}w((A_\theta - 1)x),$$

whence

$$(6) \qquad (\|\theta^{-1}\|^{-1})^{1/x} \leq e^b(w((A_\theta - 1)x))^{1/x}.$$

Letting $x \to \infty$ in (6), radicality of w yields a contradiction. The other half of the inequality shows that $A_\theta < 1$ cannot occur. Thus $A_\theta = 1$. This shows that $\alpha[\theta(\mu)] = \alpha(\mu)$. Thus $\theta(J_a) \subset J_a$. Since also $\theta^{-1}(J_a) \subseteq J_a$, we have $\theta(J_a) = J_a$, and the lemma is proved.

THEOREM 4. *Suppose θ is an automorphism of $M(w)$. Then $\theta(L^1(w)) = L^1(w)$.*

Proof. By Lemma 1 we have $\theta(J_a) = J_a$ $(a \geq 0)$. If, for every $\mu \in M(w)$, we identify $\mu + J_a$ with the restriction of μ to $[0, a)$ we get an isomorphism of $M[0, a)$ and $M(w)/J_a$. Now for every $a > 0$, let $\theta_a : M[0, a) \to M[0, a)$ be defined by

$$\theta_a(\mu + J_a) = \theta(\mu) + J_a.$$

Then θ_a is an automorphism of $M[0,a)$. Hence by Theorem 2, $\theta_a(f+J_a) \in L^1(0,a)$, for every a. Hence $\theta(f)+J_a \in L^1(0,a)$ for every a. Since this holds for every $a>0$, it follows that $\theta(f)$ is absolutely continuous with respect to the Lebesgue measure. Hence $\theta(f) \in L^1(w)$, and the proof is complete.

COROLLARY 1. *Suppose θ is an automorphism of $M(w)$. Then there is a real number α, a non–negative integer N, and a derivation D on $M(w)$, such that for every $\mu \in M(w)$*

$$\theta(\mu) = \text{weak–}^* \lim_{n\to\infty} e^{i\alpha X}[(e^{NX}e^{D}e^{-(N+1)X})^n e^{nX}](\mu).$$

Proof. By Theorem 4 and [8, Theorem 1], there is a real number α, a non–negative integer N, and a derivation D on $M(w)$, such that for every $f \in L^1(w)$

$$\theta(f) = \lim_{n\to\infty} e^{i\alpha X}[(e^{NX}e^{D}e^{-(N+1)X})^n e^{nX}](f). \tag{7}$$

Now let $\mu \in M(W)$ and $f \in L^1(w)\backslash\{0\}$. Then by (7)

$$\begin{aligned} \theta(\mu * f) &= \lim_{n\to\infty} e^{i\alpha X}[(e^{NX}e^{D}e^{-(N+1)X})^n e^{nX}](\mu * f) \\ &= \lim_{n\to\infty} e^{i\alpha X}\{[(e^{NX}e^{D}e^{-(N+1)X})^n e^{nX}](\mu) \\ &\qquad * [(e^{i\alpha X}e^{NX}e^{D}e^{-(N+1)X})^n e^{nX}]\}(f). \end{aligned} \tag{8}$$

By the uniform boundedness principle the sequence $\langle[e^{i\alpha X}(e^{NX}e^{D}e^{-(N+1)X})^n e^{nX}]\rangle$ is bounded. Hence the sequence $\langle e^{i\alpha X}[(e^{NX}e^{D}e^{-(N+1)X})^n e^{nX}](\mu)\rangle$ has a weak–* convergent subnet converging to a measure ν. From (8) it then follows that

$$\theta(\mu) * \theta(f) = \theta(\mu * f) = \nu * \theta(f),$$

and hence $\theta(\mu) = \nu$ by Titchmarsh's convolution theorem. This shows that the weak–* limit of $\langle e^{i\alpha X}[(e^{NX}e^{D}e^{-(N+1)X})^n e^{nX}](\mu)\rangle$ exists and

$$\theta(\mu) = \text{weak–}^* \lim_{n\to\infty} e^{i\alpha X}[(e^{NX}e^{D}e^{-(N+1)X})^n e^{nX}](\mu)$$

for every $\mu \in M(w)$, and the corollary is proved.

Recall that a weight w belongs to the class $\mathcal{W}^+$, [8] if

(i) for some positive a,

$$P(a) = \sup\left\{x\frac{w(x+a)}{w(x)} : x \in \mathbf{R}^+\right\} < \infty.$$

(ii) $\inf\{a : P(a) < \infty\} > 0$.

For example a weight w satisfying $w(x) = e^{-x\log x}$ for large values of x belongs to $\mathcal{W}^+$. Theorem 3 and [8, Theorem 2.b] yield the following:

COROLLARY 2. *Suppose $w \in \mathcal{W}^+$. Then for every automorphism θ of $M(w)$, there exists a real number α, a non–negative number λ, and a derivation D, such that $\theta = e^{i\alpha X}e^{\lambda X}e^{D}e^{-\lambda X}$.*

REMARK 1. The weak–* limit in the statement of Theorem 4 is in fact a norm limit when $\mu \in L^1(w)$ or when μ is a linear combination of point masses. However we do not know if this is true in general.

REMARK 2. With some minor changes in the proof of Lemma 1 and Theorem 4 it can be shown that if θ is an isomorphism from $M(w_1)$ onto $M(w_2)$ then θ maps $L^1(w_1)$ onto $L^1(w_2)$.

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