COMPLEMENTATION PROBLEMS CONCERNING THE RADICAL OF A COMMUTATIVE AMENABLE BANACH ALGEBRA

P. C. Curtis, Jr.

In 1986 Bachelis and Saeki, [1], showed that if $\mathfrak{A}$ is a commutative Banach alegbra, with identity and non-zero radical R, which in addition satisfies the following condition

$$A : \mathrm{sp}\{x \in \mathfrak{A}^{-1} : \sup_{n \in \mathbb{Z}} \|x^n\| < \infty\}^- = \mathfrak{A},$$

then there does not exist a closed subalgebra $\mathfrak{B}$ complementary to the radical R (or complementary to any closed ideal I of $\mathfrak{A}$ contained in R).

In [2] R. J. Loy and the present author extended these results in the following way to commutative Banach algebras satisfying either of the following weaker generating conditions.

$$B : \mathrm{sp}\{x \in \mathfrak{A}^{-1} : \|x^n\|\|x^{-n}\| = o(n)\}^- = \mathfrak{A}$$

$$C : \mathrm{sp}\{x \in \mathfrak{A} : \|e^{nx}\|\|e^{-nx}\| = o(n)\}^- = \mathfrak{A}$$

THEOREM 1. *Let $\mathfrak{A}$ be a commutative Banach algebra with identity which satisfies either of the condition B or C. If φ and ψ are continuous homomorphisms of $\mathfrak{A}$ into the commutative Banach algebra $\mathfrak{B}$ such that*

$$(\varphi-\psi)(\mathfrak{A}) \subset \mathit{rad}\,\mathfrak{B},$$

then $\varphi = \psi$.

It follows immediately that if $\mathfrak{A}$ is commutative satisfying B or C, and rad $\mathfrak{A} \equiv R \neq 0$, then $\mathfrak{A}$ cannot have the strong Wedderburn property, that is, there cannot exist a closed subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ with $\mathfrak{C} \simeq \mathfrak{A}/R$ and $\mathfrak{A} = \mathfrak{C} \oplus R$. A similar result holds if I is any closed ideal of $\mathfrak{A}$ contained in R. On the other hand, if $\mathfrak{B}$ is a commutative Banach algebra which satisfies $\mathfrak{B} = \mathfrak{C} \oplus I$, where $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{B}$ continuously isomorphic to $\mathfrak{A}$, and I is a closed ideal of $\mathfrak{B}$

contained in rad $\mathfrak{B}$, then for the given ideal I this decomposition is unique.

As an application of their result, Bachelis and Saeki observed that if E is a compact set not of spectral synthesis in a non-discrete locally compact abelian group G, where A(G) is the Fourier algebra on G, and $I_0(E) = \{f \in A(G) : f = 0$ in a neighbourhood of $E\}^-$, then $A(G)/I_0(E)$ satisfies condition A. Thus such algebras fail to have a strong Wedderburn decomposition. In this case, Rad $(A(G)/I_0(E)) = I(E)/I_0(E)$ where $I(E) = \{f \in A(G) : f(E) = \{0\}\}$. These considerations give rise to the following question. If G is a locally compact, non-compact abelian group, and E is a closed, but not compact, set of non-synthetis in G, does $A(G)/I_0(E)$ fail to have a strong Wedderburn decomposition? More generally does Theorem 1 hold for $A(G)/I_0(E)$?

If G has connected dual, the Beurling-Helson Theorem, [8, 4.7.3], shows that the only measures $\mu \in M(G)$ satisfying $\sup_{n\in\mathbb{Z}} \|\mu^n\| < \infty$ are unimodular point masses, hence condition A cannot hold for $A(G)^+$ where $A(G)^+$ is the algebra A(G) with unit adjoined. Thus for compact E in G, $A(G)/I_0(E)$ satisfies condition A, even though $A^+(G)$ may not. At least for the real line $\mathbb{R}$ one can get around this problem because the appropriate analogue of condition C does indeed hold.

If the Banach algebra $\mathfrak{A}$ has no unit and $a \in \mathfrak{A}$, set $u(a) = \sum_{k=1}^{\infty} \frac{a^k}{k!}$ and let $\mathfrak{S} = \{a \in \mathfrak{A} : (1+\|u(na)\|)(1+\|u(-na)\|) = o(n)\}$.

THEOREM 2. *Let $\mathfrak{A}$ be a commutative Banach algebra, and φ, ψ be continuous homomorphisms of $\mathfrak{A}$ into the commutative Banach algebra $\mathfrak{B}$. If $a \in \mathfrak{S}$ and $\varphi(a) - \psi(a) \in$ rad $\mathfrak{B}$, then $\varphi(a) = \psi(a)$. Consequently, if sp $\mathfrak{S}^- = \mathfrak{A}$, then $\varphi = \psi$.*

Proof: The proof is basically the same as that of [2, Theorem 5.1]. Adjoin an identity e to $\mathfrak{A}$, and to $\mathfrak{B}$ if necessary, and assume $\|e\| = 1$. Define $\varphi(e) = \psi(e) = e \in \mathfrak{B}^+$. Then $r = \varphi(a) - \psi(a) \in$ Rad $\mathfrak{B}^+$ since Rad $\mathfrak{B}$ = Rad $\mathfrak{B}^+$. Let $b = e + u(a) = \exp(a)$ and $z = \psi(b) - \varphi(b)$. Then

$$\begin{aligned}\|(e+\varphi(b)^{-1}z)^n\| &= \|\varphi(b^{-n})\psi(b^n)\| \\ &\le \|\varphi\|\ \|\psi\|\ \|e+u(-na)\|\ \|e+u(na)\| \\ &\le \|\varphi\|\ \|\psi\|(1+\|u(-na)\|)(1+\|u(na)\| \\ &= o(n)\ .\end{aligned}$$

As in [2, Theorem 5.1] an application of Hille's theorem, [5, 4.10.1], yields that $z = 0$, and $\varphi(b) = \psi(b)$. Since $\varphi(a) = \psi(a) + r$, $\exp \psi(a) = \psi(\exp a) = \varphi(\exp a) = \exp(\varphi(a)) = \exp\psi(a)\cdot\exp r$, and therefore $\exp r = e$ in $\mathcal{B}$. Consequently,

$$u(r) = r \sum_{k=0}^{\infty} \frac{r^k}{(k+1)} = 0\ .$$

Since the second factor in $u(r)$ is invertible, $r = 0$, and the result follows.

To show that for the real line $\mathbb{R}$, $A(\mathbb{R}) = \mathrm{sp}\,\mathfrak{S}^-$ we need the following result.

PROPOSITION 3. *Let* $\mathfrak{T} = \{h : h$ *is piecewise linear, real valued and continuous on* $\mathbb{R}$ *with compact support*$\}$. *Then* $\mathrm{sp}\,\mathfrak{T}^- = A(\mathbb{R})$, *and for* $h \in \mathfrak{T}$, $\|u(inh)\| = O(\log n)$.

Proof. Firstly, it is a theorem of Kahane, [7, p.75], that for h piecewise linear and real valued on the circle $\mathbb{T}$, then $\|e^{inh}\|_{A(\mathbb{T})} = O(\log n)$. Secondly, the piecewise linear function on $\mathbb{T}$ are norm dense in $A(\mathbb{T})$, [4, p. 74], and those with support in $[-\pi+\delta,\pi-\delta]$, where $0 < \delta < \pi$, are norm dense in the set of those functions from $A(\mathbb{T})$ with support in this interval. Lastly, if $h \in A(\mathbb{T})$ has its support in $[-\pi+\delta,\pi-\delta]$, then there exists positive constants C_1, C_2 depending only on δ such that

$$C_1\|h\|_{A(\mathbb{R})} \le \|h\|_{A(\mathbb{T})} \le C_2\,\|h\|_{A(\mathbb{R})}\ ,$$

c.f. [8, Theorem 2.7.6]. Now if $h \in A(\mathbb{R})$ with compact support and if for some $a > 0$ we define $g(at) = h(t)$, then g is piecewise linear if h is, and $\|g\|_{A(\mathbb{R})} = \|h\|_{A(\mathbb{R})}$. Therefore, for $h \in \mathfrak{T}$, $\|u(inh)\| = O(\log n)$ and $\mathrm{sp}\,\mathfrak{T}^- = A(\mathbb{R})$ as required.

COROLLARY 4. *If* E *is a closed set of non-synthesis on the real line* $\mathbb{R}$, *and* $\mathfrak{A} = A(\mathbb{R})/I_0(E)$, *then* $\mathfrak{A}$ *does not have the strong Wedderburn property.*

Since for Banach algebras $\mathfrak{A}$ satisfying condition B or C , the strong Wedderburn property never holds, one may ask under what conditions must the radical fail to have a closed complementary subspace. Conditions A , B or C are not sufficient to guarantee this, since there are examples of algebras generated by their idempotents where the radical is finite dimensional (c.f. [3]).

However, if $\mathfrak{A} = A(G)/I_0(E)$, where E is a closed set of non-synthesis, then for most, perhaps all, known examples $\left[I(E)/I_0(E)\right]^2 \neq I(E)/I_0(E)$, and in this case $I(E)/I_0(E)$ cannot have a closed complementary subspace in $\mathfrak{A}$. The critical property of $\mathfrak{A}$ that is being used is that A(G) and its factor algebras are all amenable. (See [2] for a discussion of commutative amenable Banach algebras.) The following is an illustration of this phenomenon.

THEOREM 5. *Let $\mathfrak{A}$ be a commutative semi-simple Banach algebra with unit which is regular and amenable. Assume for some* $a \in \mathfrak{A}$ and $\mu \in \mathfrak{A}^*$, $\mu \neq 0$,

$$\int_{-\infty}^{\infty} \|e^{ita}\mu\|_{\mathfrak{A}^*} |t| \, dt < \infty .$$

Then for some $\lambda \in \mathbb{R}$, the closed ideals I_1 , I_2 , generated by $\lambda + a$ and $(\lambda + a)^2$ respectively, are distinct. Furthermore, I_1/I_2 has no complementary subspace in $\mathfrak{A}/I_2$.

Proof. The first statement in the well known theorem of Malliavin, c.f.[7, p.231]. Since $\mathfrak{A}$ is regular, I_1 and I_2 have the same hull and I_1/I_2 is the radical in $\mathfrak{A}/I_2$. If $\mathfrak{A}/I_2 = \mathfrak{M} \oplus I_1/I_2$ for some closed subspace $\mathfrak{M}$, then since $\mathfrak{A}$, and hence $\mathfrak{A}/I_2$, are amenable, it follows that I_1/I_2 must have a bounded approximate identity (c.f. [2, theorem 3.7]). This is clearly impossible since $I_2 = \overline{(I_1)^2}$.

An interesting question is whether the radical in an amenable Banach algebra ever can have a bounded approximate identity. No such example is known to the author.

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Department of Mathematics
University of California, Los Angeles
Los Angeles, CA 90024-1555
U.S.A.