BOOLEAN ALGEBRAS OF PROJECTIONS OF UNIFORM MULTIPLICITY ONE

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A classical result of J. von Neumann states that if T is a bounded selfadjoint operator in a separable Hilbert space, then the following four algebras are the same: (i) the algebra of all bounded Borel functions of T; (ii) the weakly closed operator algebra generated by T; (iii) the uniformly closed operator algebra generated by the projections in the resolution of the identity for T; (iv) the bicommutant of T.

In Banach spaces (or, more generally, locally convex Hausdorff spaces) the analogues of selfadjoint operators are scalar-type spectral operators with real spectrum. The extent to which von Neumann's bicommutant theorem carries over to such operators T in locally convex spaces X is discussed in the survey article [6]. If X is a Banach space, then the algebras (i) - (iii) always coincide [6; Theorem 1]. The question of whether they also agree with (iv) is still not satisfactorily resolved in general. For instance, it is known that if the resolution of the identity for T has uniform multiplicity one, then the algebras (i) - (iii) do coincide with (iv), [6; Theorem 1]. Unfortunately, this condition is not necessary [6; Remarks 6 & 7]. For non-normable spaces X, even the equality of (i) - (iii), when appropriately formulated, is no longer valid in general. However, if the weakly closed operator algebra generated by the resolution of the identity of T is algebraically isomorphic to $C(\Lambda)$, for some completely regular Hausdorff space Λ, then equality does hold, [6; Theorem 2]. Under this restriction, it again turns out that uniform multiplicity one is a sufficient condition for the equality of (i) - (iv).

In practice it may be difficult to determine whether a given Boolean algebra (briefly, B.a.) of projections has uniform multiplicity one. However, it is often easier to establish whether or not a cyclic vector exists. Since the existence of a cyclic vector implies uniform multiplicity one, this gives at least some hope of testing for uniform multiplicity one in specific examples. Unfortunately, it is easy to produce examples of Boolean algebras of

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projections with uniform multiplicity one which are not cyclic. For instance, this is the case for the B.a. of all diagonal projections in $\ell^p(\Gamma)$, $1 \leq p < \infty$, with Γ an uncountable set. However, such examples are always exhibited in non-separable spaces which leads to the question of whether these are the only spaces in which this can happen.

The purpose of this note is to examine this question. It is shown that there do indeed exist Boolean algebras of projections in separable spaces which are of uniform multiplicity one but not cyclic. However, such a space must necessarily be non-metrizable. That is, in a separable (and complete) metrizable locally convex space a (σ-complete) B.a. of projections is of uniform multiplicity one if and only if it is cyclic. In the Appendix we make some remarks, particularly relevant to non-metrizable spaces, concerning the relation between Boolean algebras of projections and spectral measures, thereby making the connection with scalar-type spectral operators.

Whenever X is a locally convex Hausdorff space, always assumed to be quasicomplete, let $L(X)$ denote the space of all continuous linear operators from X into itself. It is always assumed that $L(X)$ is sequentially complete for the strong operator topology. The concept of a B.a. of projections is standard. A B.a. in $L(X)$ is called equicontinuous if it is an equicontinuous subset of $L(X)$. The notions of σ-completeness and completeness of a B.a. used by Bade in the Banach space setting [1] are algebraic and topological and consequently extend themselves immediately to the locally convex setting; see [8], for example, and also the Appendix.

Given a B.a. $\mathcal{M} \subset L(X)$ and an element $x \in X$, let $\mathcal{M}[x]$ denote the closed subspace of X generated by $\{Bx; B \in \mathcal{M}\}$. A complete, equicontinuous B.a. $\mathcal{M} \subset L(X)$ is said to be of uniform multiplicity one if, for every $x \in X$, there is a projection $B \in \mathcal{M}$ whose range is precisely $\mathcal{M}[x]$. If there exists an element x such that $\mathcal{M}[x] = X$, then $\mathcal{M}$ is called cyclic and x is called a cyclic vector for $\mathcal{M}$.

THEOREM. *Let X be a separable Fréchet (locally convex) space. A σ-complete Boolean algebra $\mathcal{M} \subset L(X)$ is of uniform multiplicity one if and only if it is cyclic.*

Proof. The metrizability of X ensures that $\mathcal{M}$ is equicontinuous [8; Proposition 1.2] and the separability of X implies that $\mathcal{M}$ is actually a complete B.a.; see the proof of

Corollary 4.7 in [8].

Suppose that $\mathcal{M}$ has uniform multiplicity one. Let d be a translation invariant metric generating the topology of X. Let $\{x_n\}_{n=1}^{\infty}$ be a countable, dense set in X. By Proposition 2.7 of [3] there exists in X the structure of a complex Riesz space such that X is a locally solid, topologically complete Riesz space which is Dedekind complete, has Lebesgue topology and such that the B.a. of all band projections coincides with $\mathcal{M}$. Let $|\cdot|$ denote the corresponding Riesz space order in X.

For each $n = 1, 2, \ldots$, the ball $B_n = \{x \in X; d(x, 0) < 2^{-n}\}$ is open in X and so absorbs the bounded (singleton) set $\{|x_n|\}$. That is, there is $\alpha_n > 0$ such that $\alpha_n|x_n| \in B_n$ or, equivalently, $d(\alpha_n|x_n|,\ 0) < 2^{-n}$. The translation invariance of d implies that

$$d\left(\sum_{j=1}^{n}\alpha_j|x_j|,\ \sum_{j=1}^{m}\alpha_j|x_j|\right) = d\left(\sum_{j=m+1}^{n}\alpha_j|x_j|,\ 0\right) \le \sum_{j=m+1}^{n} 2^{-j},$$

for every $n > m$. By completeness of X there is an element $e \in X$ such that $\sum_{j=1}^{\infty}\alpha_j|x_j| = e$. Then e is a weak order unit in X. Indeed, if $y \in X$ and $|y| \wedge e = 0$, then $|y| \wedge |x_n| = 0$, for every $n = 1, 2, \ldots$ It follows that $|y| \wedge |y| = 0$, by density of $\{x_n\}_{n=1}^{\infty}$ and continuity of the lattice operations. By Proposition 2.4 (vi) of [3] the vector $e \in X$ is cyclic for $\mathcal{M}$.

The existence of a cyclic vector for $\mathcal{M}$ is known to imply that $\mathcal{M}$ has uniform multiplicity one; see Proposition 2.6 of [3] and the remarks prior to it, for example. □

For non-metrizable spaces the situation is different.

EXAMPLE. Let $\mathbb{N} = \{1, 2, \ldots\}$. For each $n = 1, 2, \ldots$, let X_n be the vector subspace of $\mathbb{C}^{\mathbb{N}}$ whose elements have support in $\{1, 2, \ldots, n\}$. Fix $1 \le p < \infty$. Let X_n have the subspace topology from $\ell^p(\mathbb{N})$, for every $n = 1, 2, \ldots$ Equip $X = \cup_{n=1}^{\infty} X_n$ with the (strict) inductive limit topology. Then X is a separable, complete locally convex space which is not metrizable. Let Σ denote the σ-algebra of all subsets of $\mathbb{N}$ and associate with each $E \in \Sigma$ the projection operator $P(E) \in L(X)$ of co-ordinatewise multiplication by χ_E, that is,

$$P(E)x = (\chi_E(1)x_1,\ \chi_E(2)x_2, \ldots), \quad x \in X.$$

Then $\mathcal{M} = \{P(E);\ E \in \Sigma\}$ is a complete, equicontinuous B.a. of projections in X. It is clear that $\mathcal{M}$ has no cyclic vector in X. However, $\mathcal{M}$ is of uniform multiplicity one.

Indeed, if $x \in X$ is non-zero then, with $E_x = \{n \in \mathbb{N};\ x_n \neq 0\}$, the cyclic space $\mathcal{M}[x]$ is precisely the range of $P(E_x)$. $\square$

APPENDIX

The connection between Boolean algebras of projections and spectral measures (for example, resolutions of the identity of scalar-type spectral operators) is well known in the Banach space setting. Recall that a B.a. of projections $\mathcal{M}$ is called Bade complete (σ-complete) if it is complete (σ-complete) as an abstract B.a. with respect to the usual partial order of range inclusion and if, for every family (sequence) $\{B_\alpha\} \subset \mathcal{M}$, it is the case that

$$(\wedge B_\alpha)X = \cap_\alpha B_\alpha X \text{ and } (\vee B_\alpha)X = \overline{sp}(\cup_\alpha B_\alpha X),$$

the closed subspace of X spanned by $\cup_\alpha B_\alpha X$; see [1]. Such Boolean algebras are necessarily uniformly bounded in $L(X)$, [1; Theorem 2.2]. Once this fact is available it follows [1; Lemma 2.3] that $\mathcal{M}$ satisfies the following

Monotone (σ-monotone) property: *Whenever $\{B_\alpha\} \subset \mathcal{M}$ is a monotonic net (sequence) with respect to the order in $\mathcal{M}$, then $\lim_\alpha B_\alpha$ exists with respect to the weak operator topology in $L(X)$ and is an element of $\mathcal{M}$.*

Now, the definition of Bade completeness (σ-completeness) extends immediately to the locally convex setting. Furthermore, whenever $\mathcal{M}$ is a complete (σ-complete) B.a. which is uniformly bounded (i.e. equicontinuous), then the monotone (σ-monotone) property follows [8; Proposition 1.3]. A B.a. satisfying the monotone (σ-monotone) property will be called *mp-complete (mp-σ-complete).* So, every equicontinuous Bade complete (σ-complete) B.a. is mp-complete (mp-σ-complete). An examination of the proof of Theorem 1 in [5] shows that it can be adapted to the setting of non-normable spaces to establish that a mp-complete (mp-σ-complete) B.a. in any locally convex space is actually Bade complete (σ-complete). In particular, for Boolean algebras in Fréchet spaces, the notions of mp-complete (mp-σ-complete) and Bade complete (σ-complete) coincide.

It may be of interest to note that, even in the Banach space setting, the "monotone property" version of completeness and σ-completeness have been adopted as the appropriate definitions by some authors; see [7], for example. Granted that the main feature of complete and σ-complete Boolean algebras should be their realization as the range of a spectral measure, there is good reason for adopting the "monotone property" definition, especially in view of the fact that such Boolean algebras of interest exist which are not equicontinuous; see the Proposition below and the remarks following its proof.

We recall that a spectral measure is a map $P : \Sigma \to L(X)$, where Σ is a σ-algebra of subsets of some set Ω, which is multiplicative (i.e. $P(E \cap F) = P(E)P(F)$, for every $E, F \in \Sigma$), assigns the identity operator to Ω, and is countably additive for the strong operator topology in $L(X)$.

PROPOSITION. *Let $\mathcal{M}$ be a mp-σ-complete Boolean algebra in $L(X)$. Then there exists a σ-algebra of sets Σ and a spectral measure $P : \Sigma \to L(X)$ such that $\mathcal{M} = \{P(E); E \in \Sigma\}$.*

Conversely, the range of any spectral measure is a mp-σ-complete B.a. of projections.

Proof. Realize $\mathcal{M}$ as the range of a finitely additive, multiplicative measure Q defined on the algebra Λ of all simultaneously open and closed subsets of a compact, totally disconnected Hausdorff space Ω. The assignment $E \mapsto Q(E)$ is also an order isomorphism in the sense that $E \leq F$ (with respect to set inclusion) if and only if $Q(E) \leq Q(F)$ (with respect to the usual partial order for commuting projections). It follows from the compactness of Ω and the fact that members of Λ are both closed and open that Q is actually σ-additive on Λ for the strong and hence, also for the weak, operator topology in $L(X)$.

Suppose $\{E_n\} \subset \Lambda$ is an increasing sequence of sets. Then $Q(E_n) \leq Q(E_{n+1})$, for every $n = 1, 2, \ldots$, and so $\{Q(E_n)\}_{n=1}^{\infty} \subset \mathcal{M}$ is monotonic. By assumption, there is $B \in \mathcal{M}$ such that $B = \lim_{n\to\infty} Q(E_n)$ with respect to the weak operator topology. By the Theorem of Extension (v) for vector measures [4] there is a σ-additive measure $P : \Sigma \to L(X)$ such that $P(E) = Q(E)$, for all $E \in \Lambda$; here Σ is the σ-algebra generated by Λ. If Λ_σ is the system of all sets $E = \cup_{n=1}^{\infty} E_n$ where $\{E_n\}_{n=1}^{\infty} \subseteq \Lambda$ is increasing, then $P(E) = \lim_{n\to\infty} Q(E_n)$, for the weak operator topology, and so $P(E) \in \mathcal{M}$, for all $E \in \Lambda_\sigma$. If $\Lambda_{\sigma\delta}$ is the system of all sets $E = \cap_{n=1}^{\infty} E_n$ where $\{E_n\}_{n=1}^{\infty} \subseteq \Lambda_\sigma$ is decreasing, then $P(E) = \lim_{n\to\infty} P(E_n)$, for the weak operator topology, and so $P(E) \in \mathcal{M}$, for all

$E \in \Lambda_{\sigma\delta}$. This argument can be continued via transfinite induction to conclude that the extended measure $P : \Sigma \to L(X)$ assumes all of its values in $\mathcal{M}$. Accordingly, $\{P(E);\ E \in \Sigma\} = \mathcal{M}$.

It remains to check that the so defined measure P is multiplicative. This can be done as in the proof of Proposition 3.6 in [2].

That the range of a spectral measure is an abstract B.a. with the σ-monotone property can be proved as in the Banach space setting; see [5; Lemma 1], for example. It follows simply from the properties of a spectral measure. □

We conclude with some remarks. If the space X is barrelled, then it follows from the Proposition above that any mp-complete (mp-σ-complete) B.a. is necessarily equicontinuous.

Problem. Is a Bade complete (σ-complete) B.a. in a barrelled space necessarily equicontinuous?

In non-barrelled spaces it is easy to exhibit mp-complete (mp-σ-complete) Boolean algebras which are not equicontinuous. For instance, let X denote $L^p([0,1])$, $1 < p < \infty$, equipped with its weak topology. Then X is quasicomplete and $L(X)$ is sequentially complete for the strong operator topology. Let Σ be the σ-algebra of Borel subsets of [0,1] and, for each $E \in \Sigma$, let $P(E) \in L(X)$ be the operator of pointwise multiplication by χ_E. Then $P : \Sigma \to L(X)$ is a spectral measure and hence $\mathcal{M} = \{P(E); E \in \Sigma\}$ is a mp-σ-complete B.a. But, $\mathcal{M}$ is not an equicontinuous subset of $L(X)$.

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