# 18 Uniqueness of the Catenoid

Let $X : M = S_k - \{p_1, \cdots, p_n\} \hookrightarrow \mathbf{R}^3$ be a complete minimal surface, where $S_k$ is a closed Riemann surface of genus k. We say that $X$ has genus k.

The catenoid has genus zero by this definition. We have already proved that the catenoid is the only embedded complete minimal surface of total curvature $-4\pi$ and is the only minimal surface which is a rotation surface. Schoen [74] proved that the catenoid is the only complete minimal surface with exactly two annular ends and finite total curvature. Thus the catenoid has many special features which describe it uniquely.

In 1989, López and Ros proved the following remarkable theorem [49].

**Theorem 18.1** *The catenoid is the only embedded genus zero non-planar minimal surface of finite total curvature.*

The proof of Theorem 18.1 is a combination of the flux formula and the maximum principle at infinity. We will give a proof here adapted from [71].

Another key ingredient in the proof of Theorem 18.1 is deformation. Suppose that $X : M = S_k - \{p_1, \cdots, p_n\} \hookrightarrow \mathbf{R}^3$ is a minimal surface. If for any loop $\Gamma \subset M$, $\mathbf{Flux}(\Gamma)$ is a vertical vector, i.e., parallel to $(0, 0, 1)$, then we say that $X$ has *vertical flux*. By Proposition 17.1, we see that $X$ has vertical flux if and only if for any loop $\Gamma$,

$$\int_\Gamma \eta = 0, \quad \text{and} \quad \int_\Gamma g^2\eta = 0, \tag{18.81}$$

where $g$ and $\eta$ are the Weierstrass data for $X$.

Let $\lambda \in (0, \infty)$ and $\eta_\lambda = \lambda^{-1}\eta$, $g_\lambda = \lambda g$. Consider the corresponding Enneper-Weierstrass representation,

$$\omega_1^\lambda = \frac{1}{2}(\lambda^{-1} - \lambda g^2)\eta; \quad \omega_2^\lambda = \frac{i}{2}(\lambda^{-1} + \lambda g^2)\eta; \quad \omega_3^\lambda = g\eta = \omega_3.$$

If $X$ has vertical flux, then we have a family of well defined minimal surfaces, deformations of the original surface, given by

$$X^\lambda = \Re \int (\omega_1^\lambda, \omega_2^\lambda, \omega_3^\lambda). \tag{18.82}$$

Note that the third coordinate function of $X^\lambda$ does not depend on $\lambda$.

A point $p \in M$ such that $g(p) = 0$ or $\infty$ is called a *vertical point* of $X$. Since $g_\lambda = \lambda g$, if $p$ is a vertical point of $X$ then $p$ is also a vertical point of $X^\lambda$, and vice versa. We first investigate the behaviour of $X^\lambda$ when $p$ is a vertical point.

**Lemma 18.2** *Suppose that $X$ has vertical flux and is non-planar. If $p$ is a vertical point, then $X^\lambda$ is not an embedding when $\lambda$ is sufficiently large or sufficiently small.*

**Proof.** First we assume that $g(p) = \infty$. Let $p \in D$ be a coordinate disk in $M$ and $z(p) = 0$. Without loss of generality, we may assume that $g(z) = z^{-k}$ on $D$ and $k > 0$. By Theorem 14.1, $\eta$ should have a zero of order $2k$ at $p$, so we can write $\eta = z^{2k}h(z)dz$, where $h$ is holomorphic on $D$ and $h(0) \neq 0$. Make a change of coordinate on $D$ by $\zeta = \lambda^{-1/k}z$, then

$$g_\lambda(z) = \lambda z^{-k} = \zeta^{-k}, \quad \text{and} \quad \eta_\lambda = \lambda^{1+1/k}\zeta^{2k}h(\lambda^{1/k}\zeta)d\zeta.$$

Under these new coordinates,

$$\omega_1^\lambda = \frac{1}{2}\lambda^{1+1/k}(\zeta^{2k} - 1)h(\lambda^{1/k}\zeta)d\zeta, \qquad \omega_2^\lambda = \frac{i}{2}\lambda^{1+1/k}(\zeta^{2k} + 1)h(\lambda^{1/k}\zeta)d\zeta,$$

$$\omega_3^\lambda = \lambda^{1+1/k}\zeta^k h(\lambda^{1/k}\zeta)d\zeta.$$

Now we dilate $X^\lambda$ by a homothety of ratio $\lambda^{-(1+1/k)}$, $\tilde{X}^\lambda = \lambda^{-(1+1/k)}X^\lambda$. When $\lambda \to 0$, $\tilde{X}^\lambda$ converges uniformly on compact subsets of $\mathbf{C}$ to the minimal surface $X^0 : \mathbf{C} \hookrightarrow \mathbf{R}^3$ (note that for fixed $z \neq 0$, $\lim_{\lambda\to 0} \lambda^{-1/k}z = \infty$ and for fixed $\zeta$, $\lim_{\lambda\to 0} \lambda^{1/k}\zeta = 0$). $X^0$ is determined by the Weierstrass data for $X^0$,

$$g_0 = \zeta^{-k}, \qquad \eta_0 = h(0)\zeta^{2k}d\zeta.$$

Such data gives a complete non-embedded minimal surface. In fact, by Theorem 11.1, $\eta_0$ should have a pole of order 2 to make $X^0$ an embedding at $\zeta = \infty$, but our $\eta_0$ has a pole of order $2k + 2 > 2$ at $\zeta = \infty$.

Since $\tilde{X}^\lambda$ converges to $X^0$ uniformly on compact subset when $\lambda \to 0$, for $\lambda$ small enough, $\tilde{X}^\lambda$, thus $X^\lambda$, is not an embedding.

When $g(p) = 0$, the proof is similar and when $\lambda$ is large enough, $X^\lambda$ is not an embedding. □

**Exercise :** Give a rigorous proof that $X^\lambda$ is not embedded when $g(p) = 0$ and $\lambda$ is large.

Note that if $X$ has vertical flux and $X|_{D-\{p\}} : D - \{p\} \hookrightarrow \mathbf{R}^3$ is an annular end , then $X^\lambda|_{D-\{p\}} : D - \{p\} \hookrightarrow \mathbf{R}^3$ is also an annular end.

Next we will study the behaviour of $X^\lambda$ at an embedded end of vertical limiting normal.

**Lemma 18.3** *Suppose that $X : M \hookrightarrow \mathbf{R}^3$ is non-planar and has vertical flux. If $E = X|_{D-\{p\}} : D - \{p\} \hookrightarrow \mathbf{R}^3$ is an embedded flat annular end with vertical limiting normal, then $E^\lambda = X^\lambda|_{D-\{p\}} : D - \{p\} \hookrightarrow \mathbf{R}^3$ is not embedded for $\lambda$ large or small enough.*

**Proof.** Let $p \in D$ be a coordinate neighbourhood with $z(p) = 0$. As before, we first assume that $g(p) = \infty$ and so $g(z) = z^{-k}$, $k > 1$ since $E$ is a flat end. By Theorem

14.1, $\eta$ has a zero of order $2k-2$, so $\eta = z^{2k-2}h(z)dz$. Again we make the change of coordinate $\zeta = \lambda^{-1/k}z$ and

$$g_\lambda = \zeta^{-k}, \quad \eta_\lambda = \lambda^{1-1/k}\zeta^{2k-2}h(\lambda^{1/k}\zeta)d\zeta.$$

Arguing as before, we dilate $X^\lambda$ by a homothety of ratio $\lambda^{-(1-1/k)}$, $\tilde{X}^\lambda = \lambda^{-(1-1/k)}X^\lambda$. When $\lambda \to 0$, $\tilde{X}^\lambda$ converges uniformly on compact subsets of $\mathbf{C} - \{0\}$ to the minimal surface $X^0 : \mathbf{C} - \{0\} \hookrightarrow \mathbf{R}^3$. $X^0$ is determined by the Weierstrass data for $X^0$,

$$g_0 = \zeta^{-k}, \quad \eta_0 = h(0)\zeta^{2k-2}d\zeta.$$

Thus by Theorem 14.1, this complete minimal surface has an embedded end at $\zeta = 0$ and a non-embedded end at $\zeta = \infty$, since at $\infty$ $\eta_0$ has a pole of $2k > 2$. Hence when $\lambda$ small enough, $X^\lambda$ is not embedded.

When $g(p) = 0$, similar argument gives that when $\lambda$ large enough, $X^\lambda$ is not embedded. □

**Lemma 18.4** *Suppose that $X : M = S_k - \{p_1, \cdots, p_n\} \hookrightarrow \mathbf{R}^3$ is embedded and all ends have vertical normal. If $X$ has vertical flux, then $X^\lambda$ is an embedding for all $\lambda > 0$.*

**Proof.** First note that since $X$ is embedded, at each puncture $p_i$,

$$|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 \cong \frac{1}{|z|^4},$$

where $\phi_i dz = \omega_i$. Thus for any deformation $X^\lambda$, we have

$$|\phi_1^\lambda|^2 + |\phi_2^\lambda|^2 + |\phi_3^\lambda|^2 \cong \frac{1}{|z|^4}.$$

This then tells us that each end of $X^\lambda$ is embedded. By the weak maximum principle at infinity (see Remark 15.3), the distance between any two ends of $X$ is positive. Since the third coordinate of $X^\lambda$ is independent of $\lambda$, any two ends of $X^\lambda$ are eventually disjoint. Thus outside of a compact set $C_\lambda \subset M$, $X^\lambda$ is embedded.

Now let $B := \{\lambda \in (0,\infty) \,\big|\, X^\lambda \text{ is embedded }\}$. We want to prove that $B$ is both open and closed; then by the connectedness of $(0,\infty)$ and $1 \in B$, we know that $B = (0,\infty)$.

Suppose $\lambda_0 \in B$. Since $X^\lambda$ uniformly converges to $X^{\lambda_0}$ on compact sets when $\lambda \to \lambda_0$, and each $X^\lambda$ is embedded outside of a compact set, it follows for $\lambda$ near $\lambda_0$ that $X^\lambda$ is embedded.

Now suppose that $\{\lambda_n\} \subset B$ and $\lambda_n \to \lambda$ when $n \to \infty$. If $X^\lambda$ is not embedded, then there are $x$ and $y \in M$ such that $x \neq y$ and $X^\lambda(x) = X^\lambda(y)$. Let $D_1$ and $D_2$ be disjoint closed disk type neighbourhoods of $x$ and $y$ respectively, such that $X^\lambda|_{D_i}$ is embedded. Since $X^{\lambda_n}$ converges uniformly on $D_i$ and $X^{\lambda_n}(D_1) \cap X^{\lambda_n}(D_2) = \emptyset$, by shrinking $D_i$ if necessary, $X^{\lambda_n}(D_i)$ are disjoint graphs on the same plane domain, and $\lim_{n\to\infty} X^{\lambda_n}(x) = \lim_{n\to\infty} X^{\lambda_n}(y)$. By the maximum principle (Theorem 4.4 and Remark 4.6), $X^\lambda(D_1) = X^\lambda(D_2)$. This shows that the image $X^\lambda(M)$ is an embedded minimal surface of finite total curvature and $X^\lambda : M \to X^\lambda(M)$ is a finite sheet covering. But outside a compact set, $X^\lambda$ is one to one, so this covering is single sheeted, that is $X^\lambda$ must be embedded. This proves that $B$ is also closed, hence also proves this lemma. □

Now we can prove Theorem 18.1.

**Proof of Theorem 18.1.** Without loss of generality, we may assume that all ends of $X$ have vertical limiting normals. Let $D_i \subset S_0 = \mathbf{C} \cup \{\infty\}$ be disjoint open disks such that $p_i \in D_i$. Then $\partial D_i$ are generators of $H_1(M)$. By (17.80), $X$ has vertical flux on each $\partial D_i$, hence has vertical flux on any loop, i.e., $X$ has vertical flux.

Since $X$ is embedded, by Lemma 18.4 $X^\lambda$ is embedded for any $\lambda \in (0, \infty)$. By Lemma 18.2 and Lemma 18.3, $g \neq 0$ or $\infty$ on $M$ and $X$ does not have flat ends. We claim that $X$ has exactly two catenoid ends.

In fact, since $g \neq 0$ or $\infty$ on $M$, $dX_3$ never vanishes on $M$ where $X = (X_1, X_2, X_3)$. Suppose $X$ has more than two catenoid ends. Let $P_t := \{(x, y, z) \in \mathbf{R}^3 \mid z = t\}$; there is an $N > 0$ such that if $t < -N$ or $t > N$ then $X(M) \cap P_t$ has at least two components. By Morse theory, $X_3^{-1}(-\infty, -N)$ or $X_3^{-1}(N, \infty)$ has at least two components since $X_3$ has no critical points on $M$. Again by Morse theory, $M = X_3^{-1}(\mathbf{R})$ has at least two components, contradicting the fact that $M$ is connected.

By Corollary 17.5, $X$ must have at least two catenoid type ends, so $X$ has exactly two catenoid type ends.

Now by the total curvature formula (13.57), $X$ has total curvature $-4\pi$. By Corollary 14.6, $X$ must be a catenoid. The proof is complete. □