15 The Halfspace Theorem and The Maximum Principle at Infinity

By Jorge and Meeks' theorem, we know that if we stand at infinity to view a complete minimal surface of finite total curvature, it looks like several planes passing through origin,

We will further discuss the image of such a surface. The basic theorem in this section is the Halfspace Theorem due to Hoffman and Meeks [32], its proof is surprisingly simple.

Theorem 15.1 (Halfspace Theorem) *A connected, proper, possibly branched, non-planar complete minimal surface M in $\mathbf{R}^3$ is not contained in a halfspace.*

Proof. Suppose the theorem is false.

Define $\mathbf{H}_t := \{(x_1, x_2, x_3) \,\big|\, x_3 \geq t\}$, $P_t = \partial \mathbf{H}_t$, $t \in \mathbf{R}$. By a translation and rotation, we may assume that $M \subset \mathbf{H}_0$. Let $T := \sup\{t \,\big|\, M \subset \mathbf{H}_t\}$. If $p \in M \cap P_T$, then P_T is the tangent plane T_pM. By Corollary 4.5, M must be on both sides of P_T, contradicting the fact that $M \subset \mathbf{H}_{T-\epsilon}$, any $\epsilon > 0$. Hence $M \cap P_T = \emptyset$. By a translation, we may assume that $T = 0$.

Let M_ϵ be the downward translation of M, then $M_\epsilon \cap P_0 \neq \emptyset$ for any $\epsilon > 0$. Let $C = C_1$ be the half-catenoid $\{(x_1, x_2, x_3) \,\big|\, x_1^2 + x_2^2 = \cosh^2(x_3),\ x_3 < 0\}$. By choosing $\epsilon > 0$ small enough, we may insure that $M_\epsilon \cap C_1 = \emptyset$ and $M_\epsilon \cap D_1 = \emptyset$, where D_1 is the unit-disk in P_0. Specifically, let $d > 0$ be the distance from M to the disk of radius $R = \cosh(1) > 1$. Outside the cylinder over D_R, C_1 lies below the plane P_{-1}. We will choose $\epsilon < \frac{1}{2}\min\{1, d\}$ small enough so that $M_\epsilon \cap C_1 = \emptyset$ and $M_\epsilon \cap P_0 \neq \emptyset$.

Denote by C_t the homothetic shrinking of C_1 by t, $0 < t \leq 1$. Observing that C_t converges smoothly, away from 0, to $P_0 - \{0\}$ we may conclude from the previous paragraph that $C_t \cap M_\epsilon \neq \emptyset$ for t sufficiently small, that $C_t \cap M_\epsilon$ lies outside the cylinder over D_1 for all t, and that $C_t \cap M_\epsilon = \emptyset$ for t close to 1.

Let $S = \{t \,|\, C_t \cap M_\epsilon \neq \emptyset\}$ and $T = \text{lubS}$. We claim that $T \in S$, i.e., $C_T \cap M_\epsilon \neq \emptyset$, thus $T < 1$.

If T is an isolated point of S, we are done. If not, we can find an increasing sequence $t_n \to T$, with $t_0 > T/2$, such that there exist points $p_n \in C_1$ with $t_n p_n \in C_{t_n} \cap M_\epsilon$. If $p_n = (x_n, y_n, z_n)$, we must have $t_n z_n \geq -\epsilon$ which implies $z_n \geq -\epsilon/t_n \geq -2\epsilon/T$. This means that p_n lies on the bounded closed subset $X_T := \{(x_1, x_2, x_3) \in C_1 \,\big|\, x_3 \geq -2\epsilon/T\}$ and must therefore possess a convergent subsequence. If $\{p_j\}$ is that subsequence and $p_j \to p_0 \in C_1$, then $t_j p_j \in C_{t_j} \cap M_\epsilon$. Since X_T is compact and M is proper, $\{t_j p_j\}$ must have a convergent subsequence in M_ϵ, still denoted by $\{t_j p_j\}$, and by continuity, $Tp_0 \in C_T \cap M_\epsilon$. This proves that $C_T \cap M_\epsilon \neq \emptyset$.

Since the boundary of C_T lies inside $D_1 \subset P_0$, and that disk is disjoint from M_ϵ, Tp_0 must be an interior point of C_T. Moreover, the fact that $T < 1$ and $C_t \cap M_\epsilon = \emptyset$ for $t > T$ means that C_T meets M_ϵ at Tp_0, but lies locally on one side of M_ϵ near

Tp_0. We conclude that M_ϵ and C_T are tangent to each other at Tp_0 but stay on one side to each other near Tp_0. By Theorem 4.4 (which is also called *maximum principle for minimal surface*) we know that M_ϵ and C_T coincide near Tp_0. By Theorem 4.2, $M_\epsilon = C_T$. A catenoid, however, is not contained in any halfspace. This gives the desired contradiction. □

Corollary 15.2 *Let $X : M \hookrightarrow \mathbf{R}^3$ be an embedded, nonplanar, complete minimal surface of finite total curvature. Then M has at least two annular ends.*

Proof. By Theorem 11.5, X is proper. If M has only one end, then by Theorem 11.8, $X(M)$ is a graph outside a large ball and is asymptotically a catenoid or flat end. Hence $X(M)$ is contained in a halfspace, which forces $X(M)$ to be a plane. □

Remark 15.3 Theorem 15.1 is a kind of maximum principle at infinity. In [45], a version of maximum principle at infinity is proved, which states that if two embedded minimal surfaces with compact boundary and finite total curvature do not intersect, they are a positive distance apart. In [50] a stronger maximum principle at infinity (but it is called the *weak maximum principle at infinity*) in flat three-manifolds is proved, which says:

If two properly immersed minimal surfaces with compact boundaries in a flat three-manifold are disjoint, they stay a bounded distance apart.

The main tool in the proof of this weak maximum principle at infinity is Theorem 15.1.

The classical maximum principle (Remark 4.6) is one of the main tools in the study of minimal surfaces (and is used in an essential manner in the proof of Theorem 15.1). It is crucial in the study of regularity and in the use of barriers in the Plateau problem. Being, fundamentally, a result about elliptic equations, it is not surprising that there are applicable versions of maximum principle for surfaces with variable mean curvature. See for example Hildebrandt [24], where some of the history of this subject is discussed.

As an easy exercise we give a version of maximum principle at infinity.

Proposition 15.4 *Let $M \subset \mathbf{H}$ be a proper, complete minimal surface with compact boundary, where $\mathbf{H}$ is a halfspace. Then the distances satisfy*

$$d(M, \partial\mathbf{H}) = d(\partial M, \partial\mathbf{H}).$$

The proof is left as an exercise. Note that we only need prove that

$$d(M, \partial\mathbf{H}) \geq d(\partial M, \partial\mathbf{H}).$$

Proof. Translating M we will get a point $p \in \mathrm{int}(M_\epsilon) \cap \partial\mathbf{H}$, and by the maximum principle, we have a contradiction. □

Remark 15.5 Theorem 15.1 says that two proper, complete, connected minimal surfaces must intersect each other if one of them is a plane. We call Theorem 15.1 the *Halfspace Theorem.* In fact, there is a stronger version, called the *Strong Halfspace Theorem.* It says that the conclusion of Theorem 15.1 is true without the assumption that one of the surfaces is a plane. A sketch of its proof is as follows: If the Strong Halfspace Theorem is false, then $M_1 \cap M_2 = \emptyset$. Let N be the flat three-manifold with M_1 and M_2 as boundary. The corollary of Theorem 8 in [52] says that there is a plane contained in N, thus we can apply the Halfspace Theorem. The proof of the existence of a plane in N involves the general Douglas-Plateau problem which is beyond our course.

Theorem 15.1 is essentially a three-dimensional theorem. In $\mathbf{R}^n$, $n > 3$, it is false.