10 Complete Minimal Surfaces, Osserman's Theorem

Let $X : M \hookrightarrow \mathbf{R}^3$ be a surface, $\Lambda^2 = |X_1|^2 = |X_2|^2$, and $\gamma : I \to M$ be a differentiable curve. The arc length of γ is $\Gamma := \int_I |(X \circ r)'(t)|\, dt$. A *divergent path* on M is a piecewise differentiable curve $\gamma : [0, \infty) \to M$ such that for every compact set $V \subset M$ there is a $T > 0$ such that $\gamma(t) \notin V$ for every $t > T$. If γ is piecewise differentiable, we define its arc length as

$$\Gamma := \int_0^\infty |(X \circ r)'(t)|\, dt = \int_0^\infty \Lambda(\gamma(t))\, |r'(t)|\, dt.$$

Note that Γ could be ∞.

Definition 10.1 We say that X is *complete* if for any divergent curve γ, $\Gamma = \infty$.

Remark 10.2 The use of a divergent curve instead of boundary to describe completeness is because that if $M = \mathbf{D}^*$, $\{0\}$ is not a boundary point of M, but is the limit point of a divergent curve.

Note that in case that (M, g) is a non-compact Riemannian manifold and $\partial M = \emptyset$, according to the Hopf-Reno theorem, this definition of completeness is equivalent to each of the following:

1. Any geodesic $\gamma : I \subset \mathbf{R} \hookrightarrow M$ can be extended to a geodesic $\gamma : \mathbf{R} \hookrightarrow M$,
2. (M, d) is a complete metric space, where d is the induced distance from the Riemannian metric g (roughly speaking, $d(p, q)$ = the arc length of the shortest geodesic segment connecting p and q),
3. in (M, d), any bounded closed set is compact.

In general, there are many examples of closed minimal submanifolds $M \hookrightarrow (N, g)$ where (N, g) is a Riemannian manifold. For example, $S^2 \subset S^3$ is minimal. But we have seen that there are no closed minimal surfaces in $\mathbf{R}^3$. Hence in some sense a complete minimal surface without boundary is the closest analogue to a "closed minimal surface in $\mathbf{R}^3$".

Definition 10.3 Let $X : M \hookrightarrow \mathbf{R}^3$ be a complete minimal surface. Remember that the Gauss curvature K is a non-positive function on M, hence the integral of K has a meaning. We define

$$K(M) := \int_M K dA \tag{10.39}$$

to be the *total Gauss curvature* of M.

Let $X : M \hookrightarrow \mathbf{R}^3$ be a surface and K be the Gauss curvature. Let $K^- = \max\{-K, 0\}$, $K^+ = \max\{K, 0\}$, then $K = K^+ - K^-$, $|K| = K^+ + K^-$. We first prove a theorem of A. Huber, the proof shown here belongs to B. White [82].

Theorem 10.4 (Huber [35]) *Let $X : M \hookrightarrow \mathbf{R}^3$ be a non-compact, complete surface. If $\int_M |K^-| dA < \infty$, then $\int_M |K^+| dA < \infty$, and M is homeomorphic to $\bar{M} - \{p_1, \cdots, p_k\}$, where $\bar{M}$ is a compact 2-manifold.*

Proof. Fix $x_0 \in M$, and let

$$\Omega_r = \Omega(r) = \{x \in M \mid d(x, x_0) < r\},$$

where $d(x, y)$ is the geodesic distance from x to y. P. Hartman [23] has shown that $\partial\Omega_r$ is, for almost all r, a piecewise smooth, embedded closed curve. Let θ_i, $i = 1, \cdots, n$ be the exterior angles of $\partial\Omega_r$. By Gauss-Bonnet theorem,

$$\begin{aligned} \int_{\partial\Omega_r} \kappa_g \, ds + \sum_i \theta_i &= 2\pi\chi(r) - \int_{\Omega_r} K \, dA \\ &= 2\pi(2 - 2h(r) - c(r)) - \int_{\Omega_r} K \, dA, \end{aligned} \tag{10.40}$$

where $\chi(r)$, $h(r)$, and $c(r)$ are the Euler characteristic, number of handles, and the number of boundary components, respectively, of Ω_r.

Let $L(r)$ denote the length of $\partial\Omega_r$. P. Hartman [23] has proved that $L(r)$ is absolutely continuous. As proved in [83],

$$L'(r) = 2\pi(2 - 2h(r) - c(r)) - \int_{\Omega_r} K \, dA + \sum_{\theta_i < 0} (2\tan(\theta_i/2) - \theta_i),$$

when $-\pi/2 < \theta_i < 0$, $2\tan(\theta_i/2) - \theta_i < 0$, so

$$L'(r) \le 2\pi(2 - 2h(r) - c(r)) - \int_{\Omega_r} K \, dA.$$

Since M is complete and noncompact, $L(r) > 0$ for all $r > 0$; so

$$\begin{aligned} 0 &\le \limsup L'(r) \\ &\le 2\pi(2 - 2\liminf h(r) - \liminf c(r)) - \int_M |K^+| \, dA + \int_M |K^-| \, dA. \end{aligned} \tag{10.41}$$

Thus the negative quantities on the right-hand side are all finite. Since $h(r)$ is a non-decreasing, integer valued function of r,

$$h(r) = \text{some constant } h \text{ for } r > R.$$

Also, $c(r)$ is integer valued, so we can find a sequence $r_i \to \infty$ with

$$c(r_i) = c = \liminf c(r) < \infty.$$

Let A_i be the union of Ω_{r_i} with those connected components of $M - \Omega_{r_i}$ which happen to be compact. (There may not be any, in which case $A_i = \Omega_{r_i}$.) Let $h(A_i)$ and $c(A_i)$ denote the number of handles and boundary components, respectively, of A_i. Then

$$h = h(\Omega_{r_i}) \le h(A_i) \le h(\Omega_{r_{i+j}}) = h$$

provided j is large enough that $A_i \subset \Omega_{r_{i+j}}$: so

$$h(A_i) \equiv h \tag{10.42}$$

and clearly $c(A_i) \leq c(\Omega_{r_i})$. By passing to a subsequence we may assume

$$c(A_i) \equiv c' \ (\leq c). \tag{10.43}$$

By (10.42) and (10.43), the A_i are homeomorphic, with A_{i+1} obtained from A_i by attaching annuli. The result follows immediately. □

Since for minimal surfaces $K \leq 0$ on M, we know that a complete minimal surface of finite total curvature has finite topology. We are interested in the conformal structure of M. Now since M has finite topology, $M = S_k - (\{p_1, \cdots, p_n\} \cup \bigcup_{i=1}^{l} U_i)$ as a Riemann surface, where $U_i \subset S_k$ is conformally a closed disk. Furthermore, there are disjoint conformal open disks $D_i \subset S_k$, $i = 1, \cdots, n+l$, such that $p_i \in D_i$, $i = 1, \cdots, n$, and $U_i \subset D_{i+n}$, $i = 1, \cdots, l$, and the boundaries ∂D_i are mutually disjoint analytic Jordan curves. See, for example, [1], I 44D and II 3B. Hence each $\overline{D}_{i+n} - U_i$ is conformally a doubly connected plane domain, which must be conformally equivalent to some

$$\tilde{A}_R := \{z \in \mathbf{C} \,|\, 1/R \leq |z| < R\}$$

with $1 < R < \infty$.

Let $\phi : \overline{D}_{i+n} - U_i \to \tilde{A}_R$ be a conformal diffeomorphism. Since X is complete with finite total curvature, $X \circ \phi^{-1}$ is a complete minimal annulus with finite total curvature, where completeness of $X \circ \phi^{-1}$ means for any curve $\gamma : I \to A_R$ diverging to $|z| = R$, the arc length Γ of γ is infinity. We will prove that such a complete minimal annulus does not exist and hence $M = S_k - \{p_1, \cdots, p_n\}$.

Actually, we will prove $M = S_k - \{p_1, \cdots, p_n\}$ by showing that for any $\tilde{A}_R$ there is no complete Riemannian metric which is conformal to the Euclidean metric and has non-positive Gauss curvature and finite total curvature. If there were a closed disk U_i removed from S_k, the induced metric on some $\tilde{A}_R$ by $X \circ \phi^{-1}$ would be a complete Riemannian metric which is conformal to the Euclidean metric and has non-positive Gauss curvature and finite total curvature. Thus we know that it must be the case that $M = S_k - \{p_1, \cdots, p_n\}$.

By the way, since there do exist complete minimal annuli $Y : D^* \hookrightarrow \mathbf{R}^3$, we see that D^* is not conformally equivalent ot any A_R or $\tilde{A}_R$ as mentioned in the last section.

First we give an easy lemma which uses the special structure of $\tilde{A}_R$. The proof is left as an exercise.

Lemma 10.5 *If $g_{ij} = \lambda^2 \delta_{ij}$ is a complete Riemannian metric on $\tilde{A}_R$, then $\tilde{g}_{ij}(z) := \lambda^2(z)\lambda(1/z)\delta_{ij}$ is a complete Riemannian metric on $\hat{A}_R := \{1/R < |z| < R\}$.*

Next we prove that if $\lambda = e^h$ and h is harmonic, then $\lambda^2 \delta_{ij}$ on $\tilde{A}_R$ cannot be complete.

Proposition 10.6 *Suppose $D \subset \mathbf{C}$ and $g_{ij} = e^{2h}\delta_{ij}$ is a complete Riemannian metric on D. If $\triangle h = 0$, then conformally D is either $\mathbf{C} - \{0\}$ or $\mathbf{C}$.*

Proof. Consider the conformal universal covering $\pi : \tilde{D} \subset \mathbf{C} \to D$. Since π is holomorphic, $\tilde{h}(z) = h(\pi(z))$ is harmonic. Furthermore, $\tilde{g}_{ij} = e^{2\tilde{h}}\delta_{ij}$ is a complete Riemannian metric on $\tilde{D}$. Since $\tilde{D}$ is simply connected, there is a harmonic function $\tilde{k}$ conjugate to $\tilde{h}$. Define a holomorphic function $w : \tilde{D} \to \mathbf{C}$ by

$$w(z) = \int_0^z e^{\tilde{h}(\zeta) + i\tilde{k}(\zeta)} d\zeta.$$

Since $\tilde{D}$ is simply connected, w is well defined.

First we claim that w sends a geodesic into a straight line. In fact, the induced metric by w from the Euclidean metric on $\mathbf{C}$ is $|w'|^2\delta_{ij} = e^{2\tilde{h}}\delta_{ij} = \tilde{g}_{ij}$. Hence the metric $\tilde{g}$ is the first fundamental form of the surface $w : \tilde{D} \to \mathbf{C} \subset \mathbf{R}^3$ and $w : (\tilde{D}, \tilde{g}) \to (\mathbf{C}, \bullet)$ is an isometry. Let γ be a geodesic on $\tilde{D}$, then $w \circ \gamma$ is a plane geodesic of $w(\tilde{D}) \subset \mathbf{C}$, and thus $w \circ \gamma$ must be a straight line segment in $\mathbf{C}$.

Next we prove that w is one to one and onto. Let $\gamma : [0, \infty) \hookrightarrow \tilde{D}$ be a geodesic ray of unit speed on $\tilde{D}$. Then $(w \circ \gamma)'(t) = w'(\gamma(t))\, \gamma'(t) = \rho(t)e^{i\theta(t)} \neq 0$, since w' and γ' are both non-zero. Since γ is unit speed, $1 = |\gamma'(t))|_{\tilde{g}} = |(w \circ \gamma)'(t)| = \rho(t)$. Since $w \circ \gamma$ is a straight line segment, $\theta(t)$ must be a constant, say θ_0. Thus we can write

$$w(\gamma(t)) = w(\gamma(0)) + te^{i\theta_0}.$$

This proves that w sends any geodesic ray one to one and onto a ray in $\mathbf{C}$.

Now by completeness, $\tilde{D}$ is the union of all geodesic rays starting from 0. Since w is locally a conformal diffeomorphism, different geodesic rays starting from 0 are mapped by w one to one and onto different rays starting from $w(\gamma(0)) \in \mathbf{C}$, thus w must be one to one and onto $\mathbf{C}$.

Now $w : \tilde{D} \to \mathbf{C}$ is a conformal diffeomorphism, so $\tilde{D} = \mathbf{C}$. Since the conformal universal covering of D is $\mathbf{C}$, conformally D must be either $\mathbf{C}$ or $\mathbf{C} - \{0\}$. □

Now we have to use the facts that $X : M \hookrightarrow \mathbf{R}^3$ has finite total curvature and $\tilde{A}_R$ is hyperbolic in order to construct a complete metric $e^{2h}\delta_{ij}$ on A_R such that h is harmonic.

Proposition 10.7 *Let $D \subset \mathbf{C}$ be a hyperbolic domain and $g_{ij} = \lambda^2\delta_{ij}$ a Riemannian metric on D, such that $\triangle \log \lambda \geq 0$ and*

$$\int_D \triangle \log \lambda \, dx\, dy < \infty. \tag{10.44}$$

Then there is a harmonic function h such that $\log \lambda \leq h$.

Proof. Since D is a hyperbolic domain, there is a Green's function $G(\zeta, z)$ on D for any $\zeta \in D$ which is positive except at ζ and such that $G(\zeta, z) + \log|z - \zeta|$ is a harmonic function on D. Since $\triangle \log \lambda \in L^1(D)$,

$$u(\zeta) := \frac{1}{2\pi} \int_D G(\zeta, z) \, \triangle \log \lambda \, dx\, dy$$

solves the Poisson equation $\triangle u = - \triangle \log \lambda$. Note that $u \geq 0$, $h = u + \log \lambda \geq \log \lambda$ is the desired harmonic function. □

Now by Lemma 10.5 the induced metric by $X \circ \phi^{-1}$ on $\tilde{A}_R$ is $\Lambda^2 g_{ij}$. Then the Gauss curvature is

$$K = -\frac{\triangle \log \Lambda}{\Lambda^2}.$$

Since $K \leq 0$, we know that $\triangle \log \Lambda \geq 0$ is non-negative. Moreover, since

$$K(\tilde{A}_R) = \int_{\tilde{A}_R} K dA = -\int_{\tilde{A}_R} \frac{\triangle \log \Lambda}{\Lambda^2} \Lambda^2 \, dx \, dy = -\int_{\tilde{A}_R} \triangle \log \Lambda \, dx \, dy,$$

condition (10.44) is equivalent to $X \circ \phi^{-1}$ having finite total curvature. So we have a harmonic function $h \geq \log \Lambda$. Thus

$$\log \Lambda(z) + \log \Lambda(1/z) \leq H(z) := h(z) + h(1/z).$$

Obviously $e^{2H}\delta_{ij}$ is a complete Riemannian metric on $\hat{A}_R$. Since $H(z) = h(z) + h(1/z)$ is harmonic, by Proposition 10.6 we have $R = \infty$, a contradiction. This contradiction proves the first part of the following theorem due to Osserman:

Theorem 10.8 (Osserman, [66]) *Let M be a Riemann surface without boundary and $X : M \hookrightarrow \mathbf{R}^3$ a complete minimal surface such that the total curvature $K(M)$ is finite. Then*

1. *There exists a closed Riemann surface S_k and a finite number of points $p_1, \ldots, p_r$ on S_k such that M is conformally $S_k - \{p_1, \ldots, p_r\}$;*

2. *The Gauss map $g : M \to \Sigma$ can be extended to S_k such that the extension $\tilde{g} : S_k \to \Sigma$ is a holomorphic function. Moreover,*

$$K(M) = -4\pi \deg g. \tag{10.45}$$

Recall that if $g : S_k \to \mathbf{C}$ is a meromorphic function, where S_k is a closed Riemann surface, then there is a positive integer n such that for all but finitely many $p \in \mathbf{C}$, $g^{-1}(p) \subset S_k$ consists of n points. We say that g has *degree* n, and denote this by $\deg g = n$.

Proof. Since we have proved the first part, we only need prove the second part.

Recall that

$$K = -\frac{16|g'|^2}{|f|^2(1 + |g|^2)^4} \quad \text{and} \quad \Lambda^2 = \frac{1}{4}|f|^2(1 + |g|^2)^2.$$

Recall that $\tau^{-1} : \mathbf{C} \to S^2$ is a complex chart of S^2. In this chart the volume form of S^2 is

$$dS(w) = \frac{4}{(1 + |w|^2)^2} du \wedge dv$$

where $w = u + iv \in \mathbf{C}$. Obviously

$$\int_{S^2} dS = \int_{\mathbf{C}} dS(w) = 4\pi.$$

Let U be a coordinate neighbourhood in M on which g has no pole. Since g is holomorphic, $|g'|^2 = \det Dg$, where we interpret g as $g : U \to \mathbf{R}^2$. The induced metric by $g : U \to (\mathbf{C}, dS)$ has the volume form

$$g^*(dS) = \frac{4 \det Dg}{(1+|g|^2)^2} dx\, dy.$$

Since $g^*(dS)$ is well defined on $M - g^{-1}(\infty)$ (in fact on M), by

$$\int_U K dA = \int_U K\Lambda^2 dx \wedge dy = -\int_U \frac{4|g'|^2}{(1+|g|^2)^2} dx \wedge dy$$

$$= -\int_U \frac{4 \det Dg}{(1+|g|^2)^2} dx \wedge dy = -\int_U g^*(dS)$$

we have

$$\int_M K dA = -\int_M g^*(dS).$$

Thus by the area formula we finally get

$$\int_M g^*(dS) = \int_{\mathbf{C}} \sharp\{g^{-1}(w)\} dS(w) = 4\pi \deg g,$$

where $\sharp\{g^{-1}(w)\}$ is the number of points in $g^{-1}(w)$. Since g is meromorphic, for almost every $w \in \mathbf{C}$, $\sharp\{g^{-1}(w)\} = \deg g$. The proof is complete. □

Corollary 10.9 *If $X : M \hookrightarrow \mathbf{R}^3$ is a non-planar complete minimal surface of finite total curvature, then the Gauss map $g : M \to \mathbf{C}$ can miss at most a finite number of points of $\mathbf{C}$.*

Proof. Since g can be extended to a closed Riemann surface S_k and g is not a constant, (otherwise X will be contained in a plane) we know that $g(S_k) = \mathbf{C} \cup \{\infty\}$. Now $M = S_k - \{p_1, \cdots, p_r\}$, so $\mathbf{C} - g(M)$ has at most a finite number of points. □

Corollary 10.10 (Bernstein's Theorem) *Let $u : \mathbf{R}^2 \to \mathbf{R}$ be a solution to the minimal surface equation. Then u is an affine function, i.e., $u(x, y) = ax + by + c$ where a, b, and c are constants.*

Proof. If u is not affine, then the graph of u is a non-planar complete minimal surface S of conformal type $\mathbf{C} = S^2 - \{(0,0,1)\}$. In fact, the special isothermal coordinate in Section 5, $(\xi, \eta) : \mathbf{R}^2 \to \mathbf{R}^2$, is one to one and onto $\mathbf{R}^2$. If S has infinite total curvature, then the Gauss map g of S has an essential singularity at ∞, and hence g misses at most one point in $\mathbf{C}$. If S has finite total curvature and is non-planar, then Corollary 10.9 tells us that g misses at most a finite number of points of $\mathbf{C}$. But since S is a graph,

$$N = \frac{1}{(1 + u_x^2 + u_y^2)^{1/2}} (-u_x, -u_y, 1)$$

misses the lower hemisphere of S^2. This contradiction shows that S must be planar and forces $g \equiv$ constant, and hence u_x and u_y must be constant and u must be affine. □