7 The Geometry of the Enneper-Weierstrass Representation

Let $X : M \hookrightarrow \mathbf{R}^3$ be a minimal surface. We will give the geometric data, such as the Gauss map, the first and second fundamental forms, the principal and Gauss curvatures, etc., of a minimal surface via its Enneper-Weierstrass representation.

One important fact is that the meromorphic function g in the Enneper-Weierstrass representation corresponds to the *Gauss map* N. For this we first recall that the Gauss map $N : M \to \Sigma = S^2$ of an immersion $X : M \hookrightarrow \mathbf{R}^3$ is defined as

$$N = |X_u \wedge X_v|^{-1}(X_u \wedge X_v) : M \to \Sigma.$$

Let $\tau : S^2 - \{\mathcal{N}\} \to \mathbf{C}$ be stereographic projection, where $\mathcal{N}$ is the north pole. Then

$$\tau(x,y,z) = \frac{x+iy}{1-z}, \qquad \tau^{-1}(w) = \frac{1}{1+|w|^2}(2\Re w, 2\Im w, |w|^2 - 1),$$

where $\Re$ and $\Im$ are the real and imaginary parts. We claim that

$$g = \tau \circ N : M \to \mathbf{C}.$$

In fact,

$$\tau^{-1} \circ g = \frac{1}{1+|g|^2}(2\Re g, \ 2\Im g, \ |g|^2 - 1).$$

By (6.15), (6.18), and (6.26)

$$\begin{aligned} X_u &= \Re\left(\frac{1}{2}f(1-g^2), \ \frac{i}{2}f(1+g^2), \ fg\right), \\ X_v &= -\Im\left(\frac{1}{2}f(1-g^2), \ \frac{i}{2}f(1+g^2), \ fg\right), \end{aligned}$$

thus

$$X_u \wedge X_v = \begin{pmatrix} -\Re\frac{i}{2}f(1+g^2)\Im fg + \Re fg \Im \frac{i}{2}f(1+g^2) \\ \Re\frac{1}{2}f(1-g^2)\Im fg - \Re fg\Im\frac{1}{2}f(1-g^2) \\ -\Re f(1-g^2)\Im\frac{i}{4}f(1+g^2) + \Re\frac{i}{4}f(1+g^2)\Im f(1-g^2) \end{pmatrix}$$

$$= \begin{pmatrix} \Im[\frac{i}{2}f(1+g^2)\overline{fg}] \\ \Im[\frac{1}{2}\overline{f(1-g^2)}fg] \\ \Im\left[\frac{-i}{4}\overline{f(1+g^2)}f(1-g^2)\right] \end{pmatrix} = \begin{pmatrix} \frac{1}{2}|f|^2\Re(\overline{g} + |g|^2 g) \\ \frac{1}{2}|f|^2\Im(g - |g|^2\overline{g}) \\ \frac{1}{4}|f|^2\Re(|g|^4 - 1 - \overline{g}^2 + g^2) \end{pmatrix}$$

$$= \frac{|f|^2(1+|g|^2)^2}{4(1+|g|^2)} \begin{pmatrix} 2\Re g \\ 2\Im g \\ |g|^2 - 1 \end{pmatrix} = \frac{1}{4}|f|^2(1+|g|^2)^2 \tau^{-1} \circ g.$$

Since $\tau^{-1} \circ g \in S^2$, $|\tau^{-1} \circ g| = 1$. Since X is conformal, the first fundamental form is given by $g_{12} = 0$ and

$$g_{11} = g_{22} = \Lambda^2 = |X_u||X_v| = |X_u \wedge X_v| = \frac{1}{4}|f|^2(1+|g|^2)^2. \tag{7.28}$$

Thus

$$N = |X_u \wedge X_v|^{-1}(X_u \wedge X_v) = \frac{1}{1+|g|^2}\left(2\Re g, 2\Im g, |g|^2 - 1\right) = \tau^{-1} \circ g, \tag{7.29}$$

as we claimed.

Later we will *also* call the function $g = \tau \circ N$ the *Gauss map of the immersion* $X : M \hookrightarrow \mathbf{R}^3$. We have seen that if X is a minimal surface then g is a meromorphic function. The converse is also true, i.e., X is minimal if and only if $g = \tau \circ N$ is meromorphic. We give a sketch of the proof of the converse direction; the reader can fill in the details or see [34], pages 7 to 14.

Let $T_{X(p)}M \subset \mathbf{R}^3$ be the tangent space at $X(p)$, $p \in M$. $T_{X(p)}M$ is oriented by the basis (X_1, X_2). The orientation determined by (X_1, X_2) will be called the *positive orientation*. Thus we can regard $T_{X(p)}$ as a point in $G^+_{3,2}$, the Grasmann manifold of oriented two dimensional subspaces in $\mathbf{R}^3$. We want to embed $G^+_{3,2}$ in $\mathbf{CP}^2$, the two (complex) dimensional complex projective space.

One way to express $P \in G^+_{3,2}$ is to select a positive orthogonal basis (e_1, e_2). But if (e_1, e_2) is a positive orthogonal basis of P and A is a rotation in P, then $A(e_1, e_2)$ is also a positive orthogonal basis of P. If we consider $e_1 + ie_2$ as a vector in $\mathbf{C}^3$, then A corresponds to a unit complex number $e^{i\theta}$, and $(e_1, e_2)A$ corresponds to $e^{i\theta}(e_1 + ie_2) \in \mathbf{C}^3$. Moreover, $e^{i\theta}(e_1 + ie_2)/|e_1 + ie_2|$ corresponds to a positive orthonormal basis of P. Thus we find that given a positive orthogonal basis (e_1, e_2), all positive orthonormal bases can be written as $\Theta(e_1 + ie_2) \in \mathbf{C}^3$, where Θ is an nonzero complex number. Fixing a positive orthogonal basis (e_1, e_2) of P and identifying $\Theta(e_1 + ie_2) \in \mathbf{C}^3$ for all $\Theta \in \mathbf{C} - \{0\}$ gives us a point $[e_1 + ie_2] \in \mathbf{CP}^2$. Thus P corresponds to a unique point in $\mathbf{CP}^2$. This is our embedding $E : G^+_{3,2} \to \mathbf{CP}^2$. By local coordinates it is easy to verify that E is C^∞.

Now remember that for any conformal immersion $X : M \hookrightarrow \mathbf{R}^3$, the 1-forms $\overline{\phi} = X_1 + iX_2$ are well defined in a coordinate neighbourhood U. Since (X_1, X_2) is a positive orthogonal basis of $T_{X(p)}M \subset \mathbf{R}^3$, we can define $\overline{\phi} : U \to \mathbf{CP}^2$ by $\overline{\phi}(p) = \underline{E(T_{X(p)})} = [(X_1 + iX_2)(p)]$. X is conformal implies that (6.19) is true, thus the image of $\overline{\phi}$ is contained in the submanifold $Q_1 := \{[z_1, z_2, z_3] \in \mathbf{CP}^2 \,|\, z_1^2 + z_2^2 + z_3^2 = 0\}$. We claim that Q_1 is conformally homeomorphic to S^2. In fact, let (z_1, z_2, z_3) be a representative of $[z_1, z_2, z_3] \in Q_1$ and write $(z_1, z_2, z_3) = e_1 + ie_2$, where the e_i's are real vectors. Then $[z_1, z_2, z_3] \in Q_1$ implies that (e_1, e_2) is orthogonal, therefore there is a unique

$e_3 \in S^2$ such that (e_1, e_2, e_3) is a orthogonal basis of $\mathbf{R}^3$ with positive orientation. Define $\sigma([z_1, z_2, z_3]) = e_3$; clearly σ is a homeomorphism from Q_1 to S^2. A little calculation shows that σ is conformal. Clearly, $\sigma \circ \overline{\phi}(p) = N(p)$, where N is the Gauss map. Now $g(p) = \tau \circ \sigma \circ \overline{\phi}(p)$, or $\overline{\phi} = \sigma^{-1} \circ \tau^{-1} \circ g$. Since τ reverses orientation, it is anti-conformal. If g is holomorphic, then g is conformal and thus $\overline{\phi}$ is anti-conformal or anti-holomorphic. This implies that $\phi = \overline{\overline{\phi}}$ is holomorphic. Thus

$$\frac{1}{2}\left(\frac{\partial^2 X}{\partial x^2} + \frac{\partial^2 X}{\partial y^2}\right) = \frac{1}{2}\left[\frac{\partial^2 X}{\partial x^2} + i\frac{\partial^2 X}{\partial x \partial y} - i\left(\frac{\partial^2 X}{\partial x \partial y} + i\frac{\partial^2 X}{\partial y^2}\right)\right] = \frac{\partial \phi}{\partial \overline{z}} = 0.$$

Hence X is harmonic and therefore minimal. This ends the sketch of the proof.

Remark 7.1 Note that if $p \in M$ is a branch point of a branched minimal surface and (U, z) is an isothermal neighbourhood of p such that $z(p) = 0$, then we can write $\phi = z^m \psi$, where ψ is a holomorphic vector function and $\psi(0) \neq 0$. Since ϕ satisfies (6.19), ψ also satisfies (6.19). We can use $[\psi] \in \mathbf{CP}^2$ to define the tangent space $T_{X(p)}M$. Thus for a branched minimal surface, the tangent space is well defined even at branch points.

We next give a Gauss curvature formula of the minimal surface $X : M \hookrightarrow \mathbf{R}^3$ via the Enneper-Weierstrass representation, namely

$$K = -\left[\frac{4|g'|}{|f|(1+|g|^2)^2}\right]^2. \tag{7.30}$$

To prove this, remember that for a surface with conformal metric $ds^2 = \Lambda^2 |dz|^2$, where $dz = dx + i\,dy$ and $|dz|^2 = (dx)^2 + (dy)^2$, the Gauss curvature is given by

$$K = -\frac{1}{2\Lambda^2} \triangle \log \Lambda^2 = -\frac{2}{\Lambda^2} \frac{\partial}{\partial \overline{z}} \frac{\partial}{\partial z} \log \Lambda^2.$$

By (7.28), since $\log |f|$ is harmonic, we have

$$\begin{aligned}
\frac{2}{\Lambda^2} \frac{\partial}{\partial \overline{z}} \frac{\partial}{\partial z} \log \Lambda^2 &= \frac{4}{\Lambda^2} \frac{\partial}{\partial \overline{z}} \frac{\partial}{\partial z} \log |f| + \frac{4}{\Lambda^2} \frac{\partial}{\partial \overline{z}} \frac{\partial}{\partial z} \log(1+|g|^2) \\
&= \frac{4}{\Lambda^2} \frac{\partial}{\partial \overline{z}} \frac{g'\overline{g}}{1+|g|^2} = \frac{4}{\Lambda^2} \frac{g'\overline{g'}(1+|g|^2) - g'\overline{g}g\overline{g'}}{(1+|g|^2)^2} \\
&= \frac{4}{\Lambda^2} \frac{|g'|^2}{(1+|g|^2)^2} = \frac{16|g'|^2}{|f|^2(1+|g|^2)^4}.
\end{aligned}$$

We can also calculate the second fundamental form of X via the Enneper-Weierstrass representation. Recall that

$$X_1 - iX_2 = X_x - iX_y = (\phi_1, \phi_2, \phi_3)$$

are holomorphic functions of $z = x + iy$. Hence

$$X_{11} - iX_{12} = X_{xx} - iX_{xy} = (\phi_1', \phi_2', \phi_3').$$

Because X is harmonic, the data of the second fundamental form then must be

$$h_{11} = X_{11} \bullet N = \Re(\phi_1', \phi_2', \phi_3') \bullet N, \quad h_{22} = -h_{11},$$

$$h_{12} = X_{12} \bullet N = -\Im(\phi_1', \phi_2', \phi_3') \bullet N.$$

By (6.15), (6.18), and (6.26),

$$\begin{aligned}
X_{11} \bullet N &= \Re(\phi_1', \phi_2', \phi_3') \bullet N \\
&= \Re\left[\left(\frac{1}{2}f'(1-g^2), \frac{i}{2}f'(1+g^2), f'g\right) + (-fgg', ifgg', fg')\right] \bullet N \\
&= \frac{1}{1+|g|^2}\Big(\Re f'(1-g^2)\Re g - \Im f'(1+g^2)\Im g + \Re f'g(|g|^2-1) \\
&\quad -2\Re fgg'\Re g - 2\Im fgg'\Im g + \Re fg'(|g|^2-1)\Big) \\
&= \frac{1}{1+|g|^2}\Big(\Re f'\Re g - \Re f'g^2\Re g - \Im f'\Im g - \Im f'g^2\Im g \\
&\quad +\Re f'g(|g|^2-1) - 2\Re fgg'\overline{g} + \Re fg'(|g|^2-1)\Big) \\
&= \frac{1}{1+|g|^2}\Big(\Re f'g - \Re f'g^2\overline{g} + \Re f'g(|g|^2-1) - 2|g|^2\Re fg' + \Re fg'(|g|^2-1)\Big) \\
&= \frac{1}{1+|g|^2}\Big(-\Re fg'(|g|^2+1)\Big) = -\Re fg'.
\end{aligned}$$

Similarly, we have $h_{12} = \Im fg'$. From these we see that for a minimal surface,

$$h_{11} - ih_{12} = -fg' \tag{7.31}$$

is a holomorphic function.

Again let $dz = dx + i\,dy$ and $(dz)^2 = (dx)^2 - (dy)^2 + 2i\,dx\,dy$. The second fundamental form of X can be written as

$$h_{11}(dx)^2 + 2h_{12}\,dx\,dy + h_{22}(dy)^2 = -\Re(fg')((dx)^2 - (dy)^2) + 2\Im(fg')\,dx\,dy$$

$$= -\Re(fg')\Re(dz)^2 + \Im(fg')\Im(dz)^2 = -\Re(fg'(dz)^2) = -\Re(f\,dg\,dz).$$

Let $V \in T_pM$ be a unit tangent vector and write

$$V = \Lambda^{-1}\left(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial x}\right) = 2\Re\Lambda^{-1}e^{i\theta}\frac{\partial}{\partial z} = \Lambda^{-1}e^{i\theta}\frac{\partial}{\partial z}\Lambda^{-1}e^{-i\theta}\frac{\partial}{\partial \overline{z}}$$

in complex form; then

$$II(V,V) = -\Lambda^{-2}\Re(fg'e^{2i\theta})$$

by the previous formulae. Thus the two principal curvatures are

$$\kappa_1 = \max_{0\leq\theta\leq 2\pi} -\Lambda^{-2}\Re(fg'e^{2i\theta}) = \Lambda^{-2}|fg'| = \frac{4|g'|}{|f|(1+|g|^2)^2}, \tag{7.32}$$

$$\kappa_2 = \min_{0\leq\theta\leq 2\pi} -\Lambda^{-2}\Re(fg'e^{2i\theta}) = -\Lambda^{-2}|fg'| = -\frac{4|g'|}{|f|(1+|g|^2)^2}. \tag{7.33}$$

Then from $K = \kappa_1\kappa_2$ we recover formula (7.30).

Now let $r(t) = r_1(t) + ir_2(t)$ be a curve on M and $r'(t) = r_1'(t) + ir_2'(t)$; then

$$\begin{aligned} II(r'(t), r'(t)) &= -\Re\{f[r(t)]\, g'[(r(t)]\, [r'(t)]^2\}(dt)^2 \\ &= -\Re\{d[g(r(t)]f[r(t)]dr(t)\} \\ &= -\Re\{d[g(r(t)]\eta[r(t)]\}, \end{aligned} \tag{7.34}$$

since $\eta = fdz$. Remember that a regular curve r is an *asymptotic line* on a surface M if $II(r'(t), r'(t)) \equiv 0$; a curve r is a *curvature line* if and only if $r'(t)$ is in a principal direction, if and only if $|r'(t)|^{-2}II(r'(t), r'(t))$ takes either maximum or minimum value of $II(v, v)$ for all unit tangent vectors in $T_{r(t)}M$. We have the following criteria:

1. A regular curve r is an asymptotic line if and only if $f[r(t)]\, g'[r(t)]\, [r'(t)]^2 \in i\mathbf{R}$.

2. A regular curve r is a curvature line if and only if $f[r(t)]\, g'[r(t)]\, [r'(t)]^2 \in \mathbf{R}$.

The last assertion comes from the fact that $-\Re\{f[r(t)]g'([(t)][r'(t)]^2\}$ achieves its maximum or minimum for all unit vectors $r'(t)$ at $r(t)$ only if $f[r(t)]g'[r(t)][r'(t)]^2$ is real.