5 Isothermal Coordinates for Minimal Surfaces

There is a direct construction of isothermal coordinates for minimal surfaces. Let $X : M \hookrightarrow \mathbf{R}^3$ be a minimal surface and $p \in M$. Without of loss generality we can assume that $X(p) = (0,0,0)$ and $N(p) = (0,0,1)$, and there is a simply connected domain $(0,0) \in \Omega \subset \mathbf{R}^2$ such that near $(0,0,0)$, $X(M)$ can be written as a graph $(x, y, u(x,y))$, with $u : \Omega \to \mathbf{R}$ a solution to the minimal surface equation. Writing $p = u_x$, $q = u_y$ and $W = (1+p^2+q^2)^{1/2}$, we see that $pdx + qdy$ is a closed form, i.e., $d(pdx+qdy) = 0$ on Ω. Furthermore, it is also easy to check that the two 1-forms

$$\eta_1 := \frac{1}{W}\left((1+p^2)dx + pq\,dy\right), \quad \eta_2 := \frac{1}{W}\left(pq\,dx + (1+q^2)dy\right),$$

are closed. Since Ω is simply connected,

$$\xi(x,y) := x + \int_{(0,0)}^{(x,y)} \eta_1 = x + F(x,y), \quad \eta(x,y) := y + \int_{(0,0)}^{(x,y)} \eta_2 = y + G(x,y),$$

are well defined. Thus

$$\frac{\partial \xi}{\partial x} = 1 + \frac{1+p^2}{W}, \quad \frac{\partial \xi}{\partial y} = \frac{pq}{W},$$

$$\frac{\partial \eta}{\partial x} = \frac{pq}{W}, \quad \frac{\partial \eta}{\partial y} = 1 + \frac{1+q^2}{W},$$

and

$$J = \frac{\partial(\xi,\eta)}{\partial(x,y)} = 2 + \frac{2+p^2+q^2}{W} = \frac{(W+1)^2}{W} > 0.$$

Thus the transformation $(x,y) \to (\xi,\eta)$ has a local inverse $(\xi,\eta) \to (x,y)$ and setting $x = x(\xi,\eta)$, $y = y(\xi,\eta)$, $z(\xi,\eta) = u(x(\xi,\eta), y(\xi,\eta))$, we find

$$\frac{\partial x}{\partial \xi} = \frac{W+1+q^2}{(W+1)^2}, \quad \frac{\partial x}{\partial \eta} = -\frac{pq}{(W+1)^2},$$

$$\frac{\partial y}{\partial \xi} = -\frac{pq}{(W+1)^2}, \quad \frac{\partial x}{\partial \eta} = \frac{W+1+p^2}{(W+1)^2},$$

$$\frac{\partial z}{\partial \xi} = p\frac{\partial x}{\partial \xi} + q\frac{\partial y}{\partial \xi}, \quad \frac{\partial z}{\partial \eta} = p\frac{\partial x}{\partial \eta} + q\frac{\partial y}{\partial \eta}.$$

Calculation shows that

$$|X_\xi|^2 = |X_\eta|^2 = \frac{W}{J} = \frac{W^2}{(W+1)^2}, \quad X_\xi \bullet X_\eta = 0.$$

Thus (ξ,η) is an isothermal coordinate. Furthermore, (ξ,η) has the property that

$$|(\xi,\eta)|^2 > |(x,y)|^2. \tag{5.13}$$

To see this, note that

$$\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x};$$

thus there is a function E such that

$$\frac{\partial E}{\partial x} = F, \quad \frac{\partial E}{\partial y} = G,$$

and

$$\left(\frac{\partial^2 E}{\partial x \partial y}\right) = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{1+p^2}{W} & \frac{pq}{W} \\ \frac{pq}{W} & \frac{1+q^2}{W} \end{pmatrix}$$

is positive.

Lemma 5.1 *Let $E \in C^2$ such that the Hessian of E is positive. Then the mapping $x = (x_1, x_2) \to (u_1, u_2) = (E_{x_1}, E_{x_2}) = u(x)$ satisfies*

$$(v - u) \bullet (y - x) > 0, \tag{5.14}$$

for $y \neq x$ in Ω and $v = u(y)$, $u = u(x)$.

Proof. Let $G(t) = E(ty + (1-t)x)$, $0 \leq t \leq 1$. Then

$$G'(t) = \sum_{i=1}^{2} \left[\frac{\partial E}{\partial x_i}(ty + (1-t)x)\right](y_i - x_i),$$

$$G''(t) = \sum_{i,j=1}^{2} \left[\frac{\partial^2 E}{\partial x_i \partial x_j}(ty + (1-t)x)\right](y_i - x_i)(y_j - x_j) > 0,$$

for $0 \leq t \leq 1$. Hence $G'(1) > G'(0)$, or

$$\sum v_i(y_i - x_i) > \sum u_i(y_i - x_i),$$

which is (5.14). □

Lemma 5.2 *Under the hypotheses of Lemma 5.1, define a map*

$$(x_1, x_2) \to (\tau_1, \tau_2) = \tau,$$

where $\tau_i = x_i + u_i(x_1, x_2)$. Then for $x \neq y$,

$$(\tau(y) - \tau(x)) \bullet (y - x) > |y - x|^2.$$

Proof. Since $\tau(y) - \tau(x) = (y - x) + (v - u)$, this comes from (5.14). □

Now by the Cauchy-Schwarz inequality,

$$|\tau(y) - \tau(x)| > |y - x|.$$

Note that our transformation $(x, y) \to (\xi, \eta)$ is the form defined in Lemma 5.2. Taking $x = (0, 0)$ we have $|\tau(y)| > |y|$ since $\tau(0) = 0$. If $\Omega = \mathbf{R}^2$, then the map $(x, y) \to (\xi, \eta)$ is a diffeomorphism from $\mathbf{R}^2$ to $\mathbf{R}^2$.