COMPENSATED COMPACTNESS, COMMUTATORS AND HARDY SPACES

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ABSTRACT. Since Coifman, Lions, Meyer and Semmes discovered the link between Hardy space theory and compensated compactness theory, there has been much progresses in developing further connections between them. In this survey we show how the boundedness and compactness of commutators can be used to get the weak convergence and higher integrability of certain bilinear forms which are related to compensated compactness.

§1. Introduction.

Recent discoveries tie the weak continuity of various nonlinear quantities in compensated compactness with the theory of harmonic analysis, showing that many of these quantities are in fact in well-known Hardy spaces. We refer readers to the papers of Coifman, Lions, Meyer and Semmes [CLMS] , Müller [Mül] for more details.

The problem we are concerned with is set up as follows. Let

$$B_1 : \mathbb{R}^n \times \mathbb{R}^{N_1} \longrightarrow \mathbb{R}^{m_1}$$

$$B_2 : \mathbb{R}^n \times \mathbb{R}^{N_2} \longrightarrow \mathbb{R}^{m_2}$$

be two vector-valued bilinear forms. Therefore for every non-zero $\xi \in \mathbb{R}^n$, $B_j(\xi, \cdot)$ are linear maps from $\mathbb{R}^{N_j}$ to $\mathbb{R}^{m_j}$, $j = 1,2$. Let $Q : \mathbb{R}^{N_2} \longrightarrow \mathbb{R}^{N_1}$ be a linear map which satisfies $\mu^\tau Q\nu = 0$ whenever $B_1(\xi, \mu) = 0, B_2(\xi, \nu) = 0$ for some $\xi \neq 0$. Denote $q(\mu, \nu) = \mu^\tau Q\nu$, the bilinear form on $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ related to Q.

Tartar [T] proved that if $u_k \rightarrow u$ in L^2_{loc}, $v_k \rightarrow v$ in L^2_{loc}, and $B_1(D, u_k)$, $B_2(D, v_k)$ are compact in $W^{-1,2}_{loc}$, then $q(u_k, v_k)$ has a subsequence converges to $q(u, v)$ in the sense of distributions.

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Zhou [Z] generalised the result to L^p spaces, under the condition that $rankB_1(\xi,\cdot)$ and $rankB_2(\xi,\cdot)$ are constant. More precisely, suppose $1 < p,q < \infty$, $1/p + 1/q = 1$. Then if $u_k \to u$ in L^p_{loc} , $v_k \to v$ in L^q_{loc}, and $B_1(D,u_k), B_2(D,v_k)$ are compact in $W^{-1,p}_{loc}$ and $W^{-1,p}_{loc}$ respectively, $q(u_k,v_k)$ has a subsequence converges to $q(u,v)$ in the sense of distributions.

The well known 'div-curl lemma' of Murat's is a special case of this result. [M]

Later, the following result has been proved by using techniques in harmonic analysis. Suppose $rankB_1(\xi,\cdot)$ and $rankB_2(\xi,\cdot)$ are constant, and $1 < p,q < \infty$, $1/p + 1/q = 1$. Then for $u \in L^p$, $v \in L^q$, $B_1(D,u) = 0$, $B_2(D,v) = 0$, we have $q(u,v) \in H^1$, and $\|q(u,v)\|_{H^1} \leq C\|u\|_p\|v\|_q$, where H^1 is the Hardy space. See [CLMS] for the case when $p = q = 2$ and $B_1 = B_2$, and see [LMWZ] for the other cases. In fact, in the paper [LMWZ] by Li, McIntosh, Wu and Zhang, the condition that $rankB_1(\xi,\cdot)$ and $rankB_2(\xi,\cdot)$ are constant is dropped when $p = q = 2$.

The same result is also true in the local case, i.e., we can replace L^p, L^q and H^1 by L^p_{loc}, L^q_{loc} and H^1_{loc}, though we have not remove the constant rank condition in this case.

By using this result,together with Jones and Journe's result [JJ] and the 'biting lemma', Coifman, Lions, Meyers and Semmes [CLMS] proved the following result on weak convergence.

If $u_k \to u$ in L^p , $B_1(D,u_k) = 0$, $v_k \to v$ in L^q , $B_2(D,v_k) = 0$, then $q(u_k,v_k)$ has a subsequence converges to $q(u,v)$ weak-$*$ in H^1 .

In this paper, we give a direct proof of the weak convergence by using the compactness of commutators.

In the local case, by using the fact that $f \in H^1_{loc}$, $f \geq 0$ implies $f \in L(\log L)_{loc}$ [S], we conclude that $q(u,v) \in L(\log L)_{loc}$ if $q(u,v) \geq 0$ and u,v satisfy the above conditions.

S. Müller's [Mül] following result is therefore a special example of this case: suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ $(n \geq 2)$, and satisfies $\nabla f \in L^n_{loc}$. Let $J(f) = \det \nabla f$. If f is an orientation-preserving mapping, i.e. $J(f) \geq 0$, then $J(f) \in L(\log L)_{loc}$. This follows from our results because

$$J(f) = \frac{\partial f_1}{\partial x_1}\sigma_1 + \cdots + \frac{\partial f_1}{\partial x_n}\sigma_n$$

with $div\, \sigma = 0$.

In the Jacobian case, the following more general result has been proved: For $\alpha \in \mathbb{R}$, if $\nabla f \in L^n(\log L)^{\alpha-1}_{loc}$, and $J(f) \geq 0$, then $J(f) \in L(\log L)^{\alpha}_{loc}$. See [IS] for $\alpha = 0$, [GI] for $\alpha = 2$, [BFS] for $0 < \alpha < 1$ and [Mi] for all α.

In [LMZ], Li, McIntosh and Zhang discussed more general bilinear forms, and obtained the following result. If for $1 < p, q < \infty$, $1/p + 1/q = 1$, $\alpha \in \mathbb{R}$, $u \in L^p(\log L)^{\alpha-1}_{loc}$, $v \in L^q(\log L)^{\alpha-1}_{loc}$ $B_1(D, u) = 0$, $B_2(D, v) = 0$, $q(u, v) \geq 0$, we have $q(u, v) \in L(\log L)^{\alpha}_{loc}$.

Recently, Li and Zhang [LZ] considered the integrability of such bilinear forms on Orlicz spaces. Suppose ϕ is an Orlicz function on $\mathbb{R}^+$. Let

$$\psi(t) = \begin{cases} \displaystyle\int_t^\infty \frac{\phi(s)}{s} ds & \text{if } \displaystyle\int_1^\infty \frac{\phi(s)}{s} ds < \infty \\ \displaystyle\int_1^t \frac{\phi(s)}{s} ds & \text{if } \displaystyle\int_1^\infty \frac{\phi(s)}{s} ds = \infty \end{cases}$$

Then for $u \in L^p\phi(L^p)$, $v \in L^q\phi(L^q)$, $B_1(D, u) = 0$, $B_2(D, v) = 0$, $q(u, v) \geq 0$, we have $q(u, v) \in L\psi(L)$. Moscariello [Mo] proved the case when $\phi(t) \geq (\log t)^{-1}$ for Jacobians.

A side-product of our proof is that, if $\nabla f \in L^n\phi(L^n)$ with $\displaystyle\int_1^\infty \frac{\phi(s)}{s} ds = \infty$, then

$$\int_B J(f) \leq C\Big(\int_{2B} |\nabla f|^{\frac{n^2}{n+1}}\Big)^{\frac{n+1}{n}},$$

which gives a positive answer to the conjecture made by Greco, Iwaniec and Moscariello in [GIM].

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§2. On commutators.

First we define some notation.

A function $b \in L_{loc}$ is said to be in BMO of

$$\sup_{Q\subset\mathbb{R}^n} \frac{1}{|Q|}\int_Q |b-b_Q| < \infty$$

where b_Q is the mean value of b on the cube Q. If furthermore, it satisfies

$$\lim_{|Q|\to 0} \frac{1}{|Q|}\int_Q |b-b_Q| = 0,$$

then we say $b \in VMO$.

The Hardy space H^1 is defined, say, in [S] or [CLMS].

The relationship between these spaces is: $(VMO)^* = H^1$, $(H^1)^* = BMO$.

Let K be a Calderón-Zygmund operator on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Let $[b, K]$ denote the commutator of K and the operator of multiplication by the function b, i.e. $[b, K]f = bK(f) - K(bf)$. The following theorem is well known in harmonic analysis.

Theorem 2.1. *Suppose* $1 < p < \infty$.

(1) *If* $b \in BMO$, *then* $[b, K]$ *is bounded on* L^p, *and* $\|[b, K]\| \le C\|b\|_{BMO}$;
(2) *if* $b \in VMO$, *then* $[b, K]$ *is compact on* L^p;
(3) *if* $0 < \alpha < \min(1, p/n)$, $b \in \Lambda_\alpha$, *then* $[b, K]$ *is bounded from* L^p *to* L^q, *with* $1/q = 1/p - \alpha/n$, *and* $\|[b, K]\| \le C\|b\|_{\Lambda_\alpha}$.

For the proof we refer to [JP] and [L].

The other commutator is a nonlinear one. Let

$$S^{-\varepsilon}(f) = \left(\frac{|f|}{\|f\|_p}\right)^{-\varepsilon} f.$$

Theorem 2.2. *For* $0 < \varepsilon < 1$,

$$\|[S^{-\varepsilon}, K]f\|_{\frac{p}{1-\varepsilon}} \le C_p\varepsilon\|f\|_p$$

for each $f \in L^p$.

The proof can be found in [IS].

Now let us turn our attention to the bilinear forms. Zhou [Z] proved that if Q, B_1, B_2 satisfy the conditions listed in §1, then there exist $A_1(\xi)$, $A_2(\xi)$, which are smooth, homogeneous of degree -1, such that

$$Q = B_1^T(\xi)A_1^T(\xi) + A_2(\xi)B_2(\xi)$$

for any $\xi \neq 0$.

Thus, for a test function b, we have, if $B_1(D,u) = 0$, $B_2(D,v) = 0$, then

$$\int_{\mathbb{R}^n} bq(u,v)dx = \int_{\mathbb{R}^n} bu^T Qvdx = < [b,K]u, v >$$

where K is the Calderón-Zygmund operator defined by

$$\widehat{Ku}(\xi) = B_1^T(\xi)A_1^T(\xi)\hat{u}(\xi).$$

Please see [LMWZ] for the details.

As proved in [LZ], by properly choosing b on different occasions, and by using the results on commutators, we can get the following estimate.

Proposition 2.3. *Suppose* $1 < p, q < \infty$, $1/p + 1/q = 1$. *Suppose for* δ *small enough,* $u \in L_{loc}^{p-\delta}$, $v \in L_{loc}^{q-\delta}$, $B_1(D,u) = 0$, $B_2(D,v) = 0$, $q(u,v) \geq 0$, *then for* $p_1 < p - \delta$, $q_1 < q - \delta$, *with* $1/p_1 + 1/q_1 \leq 1 + 1/n$, *we have*

$$\int_B |q(u,v)|^{1-\varepsilon}dx$$

(*)

$$\leq C|B|^{1-\frac{1-\varepsilon}{p_1}-\frac{1-\varepsilon}{q_1}}(\int_{2B}|u|^{p_1})^{\frac{1-\varepsilon}{p_1}}(\int_{2B}|v|^{q_1})^{\frac{1-\varepsilon}{q_1}} + C\varepsilon(\int_{2B}|u|^{p-p\varepsilon})^{\frac{1}{p}}(\int_{2B}|v|^{q-q\varepsilon})^{\frac{1}{q}}$$

for $\varepsilon > \delta$, $\varepsilon(1/p_1 + 1/q_1) < 1/p_1 + 1/q_1 - 1$, *and for ball* $B \subset \mathbb{R}^n$.

Later on we shall see that this estimate is the critical point for proving the higher integrability of bilinear forms.

§3. Higher integrability of certain bilinear forms on Orlicz spaces.

Suppose $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous function, which satisfies the property that $\lim_{\lambda\to\infty} \lambda^{-\varepsilon}\phi(\lambda) = 0$ and $\lim_{\lambda\to\infty} \lambda^{\varepsilon}\phi(\lambda) = \infty$ for any $\varepsilon > 0$. Furthermore, suppose there exists $n \geq 0$ such that $\lambda^n\phi(e^\lambda)$ is non-decreasing.

For $p \geq 1$, the Orlicz space $L^p\phi(L^p)(\Omega)$ is defined as

$$\{f : \int_\Omega |f|^p \phi(\frac{|f|^p}{|f|^p_{\Omega,r}}) < +\infty\}$$

where $0 < r < p$ is fixed, and

$$|f|_{\Omega,r} = (\frac{1}{|\Omega|}\int_\Omega |f|^r)^{\frac{1}{r}}.$$

We denote

$$||f||_{L^p\phi(L^p)} = \{\int_\Omega |f|^p \phi(\frac{|f|^p}{|f|^p_{\Omega,r}})\}^{1/p}.$$

Different choices of r give equivalent norms.

The following type of Hölder inequality holds.

Proposition 3.1. *Suppose $\lambda\phi(\lambda)$ is log-convex, i.e. $\log(e^t\phi(e^t))$ is convex in t. Then for $1 < p, q < \infty$, $1/p + 1/q = 1$, we have*

$$||fg||_{L\phi(L)} \leq ||f||_{L^p\phi(L^p)}||g||_{L^q\phi(L^q)}.$$

We first look at ϕ such that

$$\int_1^\infty \frac{\phi(\lambda)}{\lambda} d\lambda = \infty.$$

In this case, let

$$\psi(\lambda) = \int_1^\lambda \frac{\phi(t)}{t} dt.$$

We need to use the following results on the boundedness of maximal functions on Orlicz spaces.

Proposition 3.2. *Suppose Ω is a bounded open subset of $\mathbb{R}^n$, then*

(1) *M_Ω, the maximal function on Ω, is bounded on $L^p\phi(L^p)(\Omega)$ for $p > 1$, and*

$$||M_\Omega f||_{L^p\phi(L^p)} \leq C||f||_{L^p\phi(L^p)};$$

(2) *if $f \in L_{loc}(\Omega)$ such that $M_\Omega f \in L\phi(L)(\Omega)$, then $f \in L\psi(L)(\Omega)$ and*

$$||f||_{L\psi(L)} \leq C||M_\Omega f||_{L\phi(L)}.$$

For the proof, see [Mo], [GIM]. Now we are ready to prove the following theorem.

Theorem 3.3. *Suppose ϕ is as above. Then for $u \in L^p\phi(L^p)(\Omega)$, $v \in L^q\phi(L^q)(\Omega)$ $B_1(D,u) = 0$, $B_2(D,v) = 0$, $q(u,v) \geq 0$, for ball $B \subset \Omega$, such that $2B \subset \Omega$,*

$$\frac{1}{|B|}\int_B q(u,v)dx \leq C\Big(\frac{1}{|2B|}\int_{2B}|u|^{p_1}\Big)^{1/p_1}\Big(\frac{1}{|2B|}\int_{2B}|v|^{q_1}\Big)^{1/q_1}$$

where $1 < p_1 < p$, $1 < q_1 < q$, $1/p_1 + 1/q_1 \leq 1 + 1/n$.

Proof. For our choice of ϕ, $u \in L^p\phi(L^p)$ implies $u \in L^{p-\delta}$ for any $\delta > 0$. Thus from Proposition 2.3, we have

$$\begin{aligned} &\int_B |q(u,v)|^{1-\varepsilon}dx \\ (*)\quad &\leq C|B|^{1-\frac{1}{p_1}-\frac{1}{q_1}}\Big(\int_{2B}|u|^{p_1}\Big)^{1/p_1}\Big(\int_{2B}|v|^{q_1}\Big)^{1/q_1} + C\varepsilon\Big(\int_{2B}|u|^{p-p\varepsilon}\Big)^{\frac{1}{p}}\Big(\int_{2B}|v|^{q-q\varepsilon}\Big)^{\frac{1}{q}}\end{aligned}$$

for $0 < \varepsilon < 1$. If $\phi(\lambda) \geq C > 0$, then for $f \in L\phi(L)(B)$,

$$\varepsilon\int_B |f|^{(1-\varepsilon)}dx \to 0 \qquad \text{as } \varepsilon \to 0.$$

Therefore for $u \in L^p\phi(L^p)(2B)$, $v \in L^q\phi(L^q)(2B)$,

$$\varepsilon\Big(\int_{2B}|u|^{p-p\varepsilon}\Big)^{\frac{1}{p}}\Big(\int_{2B}|v|^{q-q\varepsilon}\Big)^{\frac{1}{q}} \to 0$$

as $\varepsilon \to 0$. Thus by letting $\varepsilon \to 0$ in $(*)$ we prove the lemma.

So we can suppose $\phi(\lambda) \to 0$ as $\lambda \to \infty$. Let

$$h(\lambda) = \int_\lambda^{2\lambda}\phi(e^s)ds,$$

then $h(\lambda)$ is decreasing, and $\lambda^{n+1}h(\lambda)$ is increasing, $\displaystyle\int_1^\infty \frac{h(\lambda)}{\lambda}d\lambda = \infty$, $\displaystyle\frac{h(\log\lambda)}{\log\lambda} \leq C\phi(\lambda)$.

Let $L(\mu) = \displaystyle\int_{1/a}^{1/\mu}\frac{h(s)}{s}ds = \int_\mu^a \frac{h(\varepsilon^{-1})}{\varepsilon}d\varepsilon$. Then $L(\mu) \to \infty$ as $\mu \to 0$. Multiplying both side of $(*)$ by $\dfrac{h(\varepsilon^{-1})}{\varepsilon}$ and integrating from μ to a, and dividing both side by $L(\mu)$, we have

$$\frac{1}{L(\mu)}\int_\mu^a\int_B\frac{h(\varepsilon^{-1})}{\varepsilon}|q(u,v)|^{1-\varepsilon}dxd\varepsilon\le\frac{C|B|}{L(\mu)}\int_\mu^a\frac{h(\varepsilon^{-1})}{\varepsilon}(|u|_{2B,p_1}|v|_{2B,q_1})^{1-\varepsilon}d\varepsilon$$
$$+C\frac{1}{L(\mu)}(\int_\mu^a\int_{2B}h(\varepsilon^{-1})|u|^{p-p\varepsilon}dxd\varepsilon)^{\frac{1}{p}}(\int_\mu^a\int_{2B}h(\varepsilon^{-1})|v|^{q-q\varepsilon}dxd\varepsilon)^{\frac{1}{q}}.$$

We first prove that

$$\int_0^a h(\varepsilon^{-1})\lambda^{-\varepsilon}d\varepsilon\le C\frac{h(\log\lambda)}{\log\lambda}$$

for $\lambda\ge e$.

$$\begin{aligned}
&\int_0^a h(\varepsilon^{-1})\lambda^{-\varepsilon}d\varepsilon\\
&=\int_{1/a}^\infty h(s)\lambda^{-\frac{1}{s}}\frac{ds}{s^2}\\
&=\int_{1/a}^{\log\lambda} h(s)\lambda^{-\frac{1}{s}}\frac{ds}{s^2}+\int_{\log\lambda}^\infty h(s)\lambda^{-\frac{1}{s}}\frac{ds}{s^2}\\
&\le(\log\lambda)^{n+1}h(\log\lambda)\int_{1/a}^{\log\lambda}s^{-n-1}\lambda^{-\frac{1}{s}}\frac{ds}{s^2}+h(\log\lambda)\int_{\log\lambda}^\infty\lambda^{-\frac{1}{s}}\frac{ds}{s^2}\\
&\le(\log\lambda)^{n+1}h(\log\lambda)\int_{1/a}^{\log\lambda}s^{-n-1}e^{-\frac{1}{s}\log\lambda}\frac{ds}{s^2}+\frac{h(\log\lambda)}{\log\lambda}\\
&\le\frac{h(\log\lambda)}{\log\lambda}\int_1^{a\log\lambda}t^{n+1}e^{-t}dt+\frac{h(\log\lambda)}{\log\lambda}\\
&\le C\frac{h(\log\lambda)}{\log\lambda}.
\end{aligned}$$

Thus, since $\frac{h(\log\lambda)}{\log\lambda}\le\phi(\lambda)$, we have

$$\int_\mu^a\int_{2B}h(\varepsilon^{-1})|u|^{p-p\varepsilon}dxd\varepsilon\le C(|B|+\int_{2B}|u|^p\phi(|u|^p)dx)$$

and

$$\int_\mu^a\int_{2B}h(\varepsilon^{-1})|v|^{q-q\varepsilon}dxd\varepsilon\le C(|B|+\int_{2B}|v|^q\phi(|v|^q)dx).$$

Thus by letting $\mu\to0$ and using L'Hospital's rule, we get

$$\int_B q(u,v)dx\le C|B|^{1-\frac{1}{p_1}-\frac{1}{q_1}}(\int_{2B}|u|^{p_1})^{1/p_1}(\int_{2B}|v|^{q_1})^{1/q_1}.$$

In fact, this estimate gives us the inequality

$$M_B(q(u,v))(x) \leq CM_{2B}(u^{p_1})^{1/p_1}M_{2B}(v^{q_1})^{1/q_1},$$

where M_B is the Hardy-Littlewood maximal function on B.

Thus from the properties of maximal function on Orlicz spaces, we have the following conclusion.

Theorem 3.4. *If $u \in L^p\phi(L^p)(2B)$, $v \in L^q\phi(L^q)(2B)$, $B_1(D,u) = 0$, $B_2(D,v) = 0$, $q(u,v) \geq 0$, then $q(u,v) \in L\psi(L)(B)$, and*

$$||q(u,v)||_{L\psi(L)} \leq C||u||_{L^p\phi(L^p)}||v||_{L^q\phi(L^q)}.$$

□

Now we turn to the case when $\int_1^\infty \frac{\phi(\lambda)}{\lambda}d\lambda < \infty$. In this case, we let $\psi(\lambda) = \int_\lambda^\infty \frac{\phi(t)}{t}dt$.

Lemma 3.5. *Suppose ψ is concave, and $\psi(\lambda) \geq c\lambda^{-\varepsilon}$ for any $\varepsilon > 0$. Then there exists h, bounded, non-decreasing on $\mathbb{R}^+$, continuous at 0, such that*

$$\phi(\lambda) \sim \int_0^a \varepsilon\lambda^{-\varepsilon}dh(\varepsilon)$$

and

$$\psi(\lambda) \sim \int_0^a \lambda^{-\varepsilon}dh(\varepsilon)$$

for some $0 < a < 1$.

For the proof, see [GIM].

By using this lemma and estimate $(*)$, we can prove that Theorem 3.4 is still true, with the norm estimate replaced by

$$\int_B q(u,v)\psi(\frac{q(u,v)}{||u||_{p_1}||v||_{q_1}}) \leq C(\int_{2B} u^p\phi(\frac{|u|^p}{||u||^p_{p_1}}))^{1/p}(\int_{2B} v^q\phi(\frac{|v|^q}{||v||^q_{q_1}}))^{1/q}.$$

The details of the proof can be found in [LZ]. □

Examples

(1) When $\phi(\lambda) = (\log(\lambda+e))^{-1+\alpha}$ for $\alpha \in \mathbb{R}$, then we have $\psi(\lambda) = (\log(\lambda+e))^{\alpha}$. Thus we recover the result in [LMZ], especially the Jacobian case proved by Milman [Mi] and others.

(2) When $\phi(\lambda) = (\log(\lambda + e))^{-1}(\log(\log(\lambda + e) + e))^{-1+\alpha}$, for $\alpha \in \mathbb{R}$, we have $\psi(\lambda) = (\log(\log(\lambda + e) + e))^{\alpha}$. In the Jacobian case, when $\alpha \geq 1$, the result can be deduced from [Mo] (in this case, $\phi(\lambda) \geq c(\log\lambda)^{-1}$); when $\alpha < 1$, the proof can be found in S. Wu's paper [Wu].

§4. Weak convergence of bilinear forms.

First of all, the following theorem is true.

Theorem 4.1. *Suppose $u_k \to u$ weakly in L^p , $v_k \to v$ weakly in L^q , $B_1(D, u_k) = 0$, $B_2(D, v_k) = 0$, then $q(u_k, v_k)$ has a subsequence converges to $q(u, v)$ weak-$*$ in H^1.*

Proof. In section 2 we have the expression

$$\int_{\mathbb{R}^n} bq(u,v)dx = \int_{\mathbb{R}^n} bu^{\tau}Qvdx =< [b,K]u, v >$$

if $B_1(D, u) = 0$, $B_2(D, v) = 0$. Now by the boundedness of $[b, K]$, we have

$$< q(u,v), b > \leq C||b||_{BMO}||u||_p||v||_q.$$

Then by duality, we obtain $q(u,v) \in H^1$ and $||q(u,v)||_{H^1} \leq C||u||_p||v||_q$. Thus for the sequences in the Theorem, we have

$$||q(u_k,v_k)||_{H^1} \leq C||u_k||_p||v_k||_q \leq C'.$$

From the fact that H^1 is weak-$*$ compact, we can choose a subsequence of $q(u_k, v_k)$ which converges weak-$*$ in H^1. What is left is to identify the limit. In [CLMS] it is proved that the subsequence converges to $q(u, v)$, by using the 'biting lemma ' and Jones–Journé approach. Here we give another proof. Since for $b \in VMO$, $[b, K]$ is compact, thus we can choose a subsequence of $[b,K]u_k$ which converges strongly to $[b,K]u$, therefore for this subsequence (we still denote as $[b,K]u_k$), we have

$$< q(u_k, v_k), b > = < [b,K]u_k, v_k > \longrightarrow < [b,K]u, v >$$

for any $b \in VMO$. Thus by duality, $q(u_k, v_k)$ converges to $q(u,v)$ weak-$*$ in H^1.

In §3 we have proved that if $u_k \in L^p\phi(L^p)(\Omega)$ is bounded, $v_k \in L^q\phi(L^q)(\Omega)$ is bounded, $B_1(D,u_k)=0$, $B_2(D,v_k)=0$, and $q(u_k,v_k)\geq 0$, then

$$||q(u_k,v_k)||_{L\psi(L)} \leq C||u_k||_{L^p\phi(L^p)}||v_k||_{L^q\phi(L^q)} \leq C'.$$

We recall the 'criteria of de La Vallée Poussin' which says that if $f_k \in L\psi(L)(\Omega)$ is bounded for some ψ such that $\psi(t) \to +\infty$ when $t \to \infty$, then f_k has a subsequence which converges weakly in $L(\Omega)$. Thus we have the following theorem.

Theorem 4.2. *Suppose ϕ is as before, and satisfies $\int_1^\infty \frac{\phi(t)}{t}dt = \infty$. If $u_k \in L^p\phi(L^p)(\Omega)$ bounded, $v_k \in L^q\phi(L^q)(\Omega)$ bounded, $B_1(D,u_k)=0$, $B_2(D,v_k)=0$, and $q(u_k,v_k)\geq 0$, then $q(u_k,v_k)$ has a subsequence which converges to $q(u,v)$ weakly in $L(\Omega)$, where u and v are the weak limits of u_k, and v_k in $L^p\phi(L^p)(\Omega)$ and $L^q\phi(L^q)(\Omega)$ respectively.*

Proof. The only thing left to be proved is that the weak limit is $q(u,v)$. It can be obtained by using the same method that S. Müller used in [Mü2]. The details can be found in [LZ].

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