DARBOUX TRANSFORMATION AND INVERSE SCATTERING

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1. INTRODUCTION

This paper is a self–contained account of the application of the Darboux transformation to the one–dimensional Schrödinger inverse problem, giving results that provide solutions to the Korteweg–de Vries equation. No originality is claimed by the author; the quoted references have been used for the material presented.

2. FACTORIZATION OF SCHRÖDINGER EQUATION

Consider $(-\infty < x < \infty)$

$$-\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi \tag{1}$$

For $E = E_0$, let f be a solution with no zero. Define operators

$$B = \frac{d}{dx} - \frac{f\,'}{f} \ , \quad C = -\frac{d}{dx} - \frac{f\,'}{f} \ . \tag{2}$$

Then (1) is

$$(CB + E_0)\psi = E\psi \ . \tag{3}$$

The solution f contains two arbitrary parameters, one being E_0 , the other fixing a definite solution in the two–dimensional space of solutions to (1). The values of the parameters are restricted by the condition $f(x) \neq 0$, which for example requires $E_0 \leq E_1$, where E_1 is the least eigenvalue.

Example: $V(x) = 0$, $f(x) = \cosh(\mu x + \omega)$, $E_0 = -\mu^2$.

3. THE DARBOUX TRANSFORMATION

From (3)

$$(BC + E_0)B\psi = E(B\psi) ,$$

which is

$$-\frac{d^2\tilde{\psi}}{dx^2} + \tilde{V}\tilde{\psi} = E\tilde{\psi} \tag{4}$$

where
$$\tilde{\psi} = B\psi \tag{5}$$

and
$$\tilde{V}(x) = V(x) - 2[f'(x)/f(x)]' . \tag{6}$$

The new potential function $\tilde{V}$ contains the aforementioned two parameters, that may be assigned a continuous range of values. In quantum mechanics two systems with potentials V and $\tilde{V}$ are said to be related by a supersymmetry.

Example: $V(x) = 0$, $f(x) = \cosh(\mu x + \omega)$, $\tilde{V}(x) = -2\mu^2 \operatorname{sech}^2(\mu x + \omega)$

If
$$\int_{-\infty}^{\infty} (1 + |x|)|V(x)|\,dx < \infty \tag{7}$$

then $E_0 < 0$, say $E_0 = -\mu^2$ with $\mu > 0$.

Using $f(\pm\infty) \to e^{\mp\mu x}$ gives

$$\tilde{\psi}(\pm\infty) = \left[\frac{d}{dx} \pm \mu\right] \psi(\pm\infty) . \tag{8}$$

4. STATEMENT OF THE DIRECT SCATTERING PROBLEM

Assuming $\int_{-\infty}^{\infty} (1 + |x|)/V(x)|\,dx < \infty$ and putting $E = k^2$, calculate from (1) the following scattering data:

(a) functions $L(k)$, $R(k)$, $T(k)$ defined by $(k > 0\ ,\ E > 0)$

(i) $\psi(-\infty) \to e^{ikx} + L(k)e^{-ikx}$ if $\psi(\infty) \to T(k)e^{ikx}$.

(ii) $\psi(\infty) \to e^{-ikx} + R(k)e^{ikx}$ if $\psi(-\infty) \to T(k)e^{-ikx}$.

(b) numbers k_j , r_j , ℓ_j , M

$(j = 1,2,\cdots, M)$ such that the normalizable solutions ϕ_j exist for $E_j = -k_j^2\ (k = ik_j)$, and

$$\phi(-\infty) = \ell_j e^{k_j x}\ ,\ \phi(\infty) = r_j e^{-k_j x} \tag{9}$$

when $(\phi_j\ ,\ \phi_j) = \int_{-\infty}^{\infty} |\phi_j|^2\, dx = 1$. The numbers E_j are the eigenvalues, and the ℓ_j and r_j are called the asymptotic normalization constants.

5. COMPARISON OF THE EIGENVALUES SUPPORTED BY V AND BY $\tilde{V}$

Solutions of (4) are given by (5), and although in all cases $Bf = 0$ but $\tilde{f} = 1/f$ satisfies (4) with $E = E_0$.

From (8) , $\phi(\pm\infty) \to 0 \Longrightarrow \tilde{\phi}(\pm\infty) \to 0$.

Hence there are just three possibilities for the spectra:

(I) $E_0 = E_M = -k_M^2\ ,\ \ f = \phi_M\ ,\ \ f(\pm\infty) \to 0\ ,\ \ \tilde{f}(\pm\infty) \to \pm\infty$.

The spectrum for $\tilde{V}$ is the spectrum for V with E_M omitted (this is the least eigenvalue for V).

(II) Either $f(-\infty) \to 0$ and $f(\infty) \to \infty$, or $f(-\infty) \to \infty$ and $f(\infty) \to 0$.

$\tilde{f} = 1/f$ is not normalizable. Eigenvalues are the same.

(III) $\quad f(\pm\infty) \to \infty\ ,\ \tilde{f}(\pm\infty) \to 0$

$\tilde{f}$ is the eigenfunction for a new eigenvalue. Spectrum for $\tilde{V}$ is that of V plus the new eigenvalue $E_{M+1} = E_0$ (which is the least eigenvalue for $\tilde{V}$).

Example: $V(x) = 0$ supports no eigenvalues; $\tilde{V}(x) = -2\mu^2 \operatorname{sech}^2(\mu x + \omega)$ supports one eigenvalue $-\mu^2$, the eigenfunction being $\tilde{f}(x) = \operatorname{sech}(\mu x + \omega)$.

Note that the type I Darboux transformation is essentially the inverse of type III.

6. NORMALIZATION OF THE EIGENFUNCTIONS FOR $\tilde{V}$

(i) Since C and B are adjoint,

$$(\tilde{\phi}, \tilde{\phi}) = (B\phi, B\phi) = (\phi, CB\phi)$$

$$= (E - E_0)(\phi, \phi),\ \text{using (3)}$$

$$= (\mu^2 - k_j^2)(\phi,\phi)\ . \tag{10}$$

(ii) For the extra eigenfunction in type III , let χ be a solution of (3) independent of f, and put $W = \chi' f - f'\chi$. (This Wronskian is constant.)

Then $(\chi/f)' = W/f^2 = W\tilde{f}^2$ so

$$(\tilde{f}, \tilde{f}) = W^{-1}[\chi/f]_{-\infty}^{\infty}\ . \tag{11}$$

Example: $V(x) = 0\ ,\ f(x) = 2\cosh(\mu x + \omega)\ ,\ \chi(x) = e^{-\mu x}$.

Then W can be evaluated as $W(\infty) = -2\mu e^{\omega}$, and $W^{-1}[\chi/f]_{-\infty}^{\infty} = \frac{1}{2\mu}$.

From (ii) we have (without any integration!)

$$\frac{1}{4}\int_{-\infty}^{\infty} \operatorname{sech}^2(\mu x + \omega)\,dx = \frac{1}{2\mu}\,.$$

The normalized eigenfunction is

$$\tilde{\phi} = (2\mu)^{\frac{1}{2}}\,\tilde{f} = \frac{(2\mu)^{\frac{1}{2}}}{2\,\cosh(\mu x + \omega)}$$

$$\to (2\mu)^{\frac{1}{2}}\,e^{-\omega}\,e^{-\mu x} \quad \text{as } x \to \infty \tag{12}$$

7. COMPARISON OF SCATTERING DATA

Consider the Darboux transformation of type III. Using (8) gives ($E_0 = -\mu^2$)

(a)

$$\begin{aligned} \tilde{T}(k) &= T(k)(k+i\mu)/(k-i\mu) \\ \tilde{L}(k) &= -L(k)(k+i\mu)/(k-i\mu) \\ \tilde{R}(k) &= R(k)(k+i\mu)/(k-i\mu) \end{aligned} \tag{13}$$

The directly observable quantities $|T|^2$, $|L|^2$, $|R|^2$ are unchanged by the transformation.

(b) Using (9) and (10) gives the asymptotic normalization constants for the M retained eigenvalues $-k_j^2$:

$$\tilde{\ell}_j = \ell_j\sqrt{\frac{\mu + k_j}{\mu - k_j}}\,, \quad \tilde{r}_j = -r_j\sqrt{\frac{\mu + k_j}{\mu - k_j}}\,. \tag{14}$$

For the new eigenfunction $\tilde{f} = 1/f\ (E_{M+1} = E_0)$ the asymptotic normalization constants ℓ and r follow from (11).

Similar formulas are easily written down for the Darboux transformations of types I and II.

8. STATEMENT OF THE INVERSE SCATTERING PROBLEM

Given the scattering data, find $V(x)$. The given data can be either $R(k)$ and k_j, r_j^2 $(j = 1,\cdots, M)$; or $L(k)$ and k_j, ℓ_j^2 $(j = 1,\cdots, M)$. The implied relations between these sets, and also $T(k)$, follow by considering the scattering problem for complex k. In particular, the k_j correspond to the poles of $T(k)$ in the upper half of the complex k–plane, at $k = ik_j$. The usual problem has $R(0) = -1$, $R(\infty) = 0$, $k_j \neq 0$, and zero is not an eigenvalue.

9. REFLECTIONLESS POTENTIALS

If $R(k) = 0$ is given, then the inverse problem is solved by using M successive Darboux Transformations of type III, starting from $V(x) = 0$. Each transformation introduces one of the required eigenvalues, and the second parameter in the transformation is chosen to fit the corresponding given asymptotic normalization constant (ℓ_j or r_j).

The potential change due to each transformation is $-2(f'/f)' = -2(\ell n\ f)''$, and we construct a sequence of potentials

$$V_1 = -2(\ell n\ f_1)'' , \tag{15a}$$

$$V_2 = -2(\ell n\ f_1 + \ell n\ f_2)'' = -2\{\ell n(f_1 f_2)\}'' \tag{15b}$$

$$V_M = -2\{\ell n\ (f_1 f_2 \cdots f_M)\}'' \tag{15c}$$

where each f_j is a solution in the previous potential V_{j-1}. The eigenvalues for the successive potentials are $-k_1^2$; $\left\{-k_2^2, -k_1^2\right\}$; $\left\{-k_3^2, -k_2^2, -k_1^2\right\}$ $\cdots$ where $-k_M^2 < -k_{M-1}^2 < \cdots < -k_1^2$.

The example given previously solves the reflectionless potential problem for $M = 1$, with $\mu = k_1$. From (12), ω is chosen so that $e^{-\omega}\sqrt{2\mu} = r_1$, i.e. $\omega = \frac{1}{2}\log(2\mu/r_1^2)$.

In practice the functions f_i and χ_i are also generated by Darboux transformations. To illustrate the method consider the case $M = 3$ and suppose eigenvalues $E_i = -k_i^2$ and asymptotic normalization constants r_i are given. We start with three pairs of solutions for $V(x) = 0$, and indicate asymptotic forms as $x \to \infty$ $(x \to -\infty)$ to the right (left) of the function.

$$\tfrac{1}{2}\exp(-k_1x - \omega_1) \leftarrow f_1 = \cosh(k_1x + \omega_1) \rightarrow \tfrac{1}{2}\exp(k_1x + \omega_1)\,, \quad \chi_1 = e^{-k_1x} \tag{16a}$$

$$-\tfrac{1}{2}\exp(-k_2x - \omega_2) \leftarrow \sinh(k_2x + \omega_2) \rightarrow \tfrac{1}{2}\exp(k_2x + \omega_2)\,, \quad e^{-k_2x} \tag{16b}$$

$$\tfrac{1}{2}\exp(-k_3x - \omega_3) \leftarrow \cosh(k_3x + \omega_3) \rightarrow \tfrac{1}{2}\exp(k_3x + \omega_3)\,, \quad e^{-k_3x} \tag{16c}$$

The parameters ω_i will eventually be determined by the given r_i ; the alternation of cosh and sinh solutions ensures that the successive f_i will have no zeros.

The asymptotic forms in (16a) are used to obtain $W_1 = -k_1e^{\omega_1}$ for (11), and hence construct, as in (12), the normalized function $N_1\tilde{f}_1 = N_1/f_1$ where $N_1^2 = \tfrac{1}{2}k_1$. The operator for the first Darboux transformation, $B_1 = (d/dx) - (f_1'/f_1) \rightarrow (d/dx) \pm k_1$, is applied to the functions and asymptotic forms in (16b) and (16c). This gives the following solutions for the potential (15a):

$$N_1/f_1 \rightarrow \{2k_1\}^{\frac{1}{2}}\exp(-k_1x - \omega_1) \tag{17a}$$

$$\tfrac{1}{2}(k_2 + k_1)\exp(-k_2x - \omega_2) \leftarrow f_2 = B_1\sinh(k_2x + \omega_2) \rightarrow \tfrac{1}{2}(k_2 + k_1)\exp(k_2x + \omega_2) \tag{17b}$$

$$-(k_1 + k_i)\exp(-k_ix) \leftarrow B_1\,e^{-k_ix} \rightarrow (k_1 - k_i)\exp(-k_ix) \quad (i = 2,3) \tag{17c}$$

$$-\tfrac{1}{2}(k_3 + k_1)\exp(-k_3x - \omega_3) \leftarrow B_1\cosh(k_3x + \omega_3) \rightarrow \tfrac{1}{2}(k_3 + k_1)\exp(k_3x + \omega_3) \tag{17d}$$

In (17a) only the right asymptotic form is required since r_1 is given, and the positive asymptotic forms in (17b) show that $f_2(x) > 0$. Subsequent Darboux transformations need not be applied to (17a) since the changes in the asymptotic normalization constant follow from (14).

The algorithm used on equations (16) is now applied to equations (17b), (17c) and (17d). The asymptotic forms in (17b) and (17c) are used to obtain $W_2 = k_2(k_2^2 - k_1^2)e^{\omega_2}$ from $\chi_2 = B_1 \exp(-k_2 x)$. Then using (II) gives the normalized function

$$N_2/f_2 \to \{2k_2(k_2 - k_1)/\{k_2 + k_1)\}^{\frac{1}{2}} \exp(-k_2 x - \omega_2) \ . \tag{18}$$

The operator for the second Darboux transformation, $B_2 = (d/dx) - (f_2'/f_2) \to (d/dx) \pm k_2$, is applied to the functions and asymptotic forms given in (17d) and (17c), giving

$$\tfrac{1}{2}(k_1 + k_3)(k_2 + k_3) \exp(-k_3 x - \omega_3) \leftarrow f_3 \to \tfrac{1}{2}(k_1 + k_3)(k_2 + k_3) \exp(k_3 x + \omega_3) \tag{19a}$$

$$\text{where } f_3 = B_2 B_1 \cosh(k_3 x + \omega_3)$$

$$(k_1 + k_3)(k_2 + k_3)\, e^{-k_3 x} \leftarrow \chi_3 = B_2 B_1\, e^{-k_3 x} \to (k_1 - k_3)(k_2 - k_3)\, e^{-k_3 x} \tag{19b}$$

These are solutions for the potential (15b).

Finally, the asymptotic forms in (19) are used to obtain

$$W_3 = -k_3(k_3^2 - k_1^2)\,(k_3^2 - k_2^2) \ ,$$

and then using (II) gives the normalized function

$$N_3/f_3 \to [2k_3(k_3 - k_1)(k_3 - k_2)/\{(k_3 + k_1)(k_3 + k_2)\}]^{\frac{1}{2}} \exp(-k_3 x - \omega_3) \ . \tag{20}$$

The values of the ω_i can now be expressed in terms of the given r_i.

From (20),

$$\omega_3 = \tfrac{1}{2} \log \left[\frac{2k_3(k_3 - k_1)(k_3 - k_2)}{r_3^2\,(k_3 + k_1)(k_3 + k_2)} \right] .$$

The effect on (18) of the last Darboux transformation is given by (14) with $\mu = k_3$, leading to $(r_2 < 0)$

$$\omega_2 = \frac{1}{2}\log\left[\frac{2k_2(k_2 - k_1)(k_2 + k_3)}{r_2^2(k_2 + k_1)(k_3 - k_2)}\right] .$$

The effect on (17a) of the last two Darboux transformations is given by a double application of (14) leading to

$$\omega_1 = \frac{1}{2}\log\left[\frac{2k_1(k_2 + k_1)(k_3 + k_1)}{r_1^2(k_2 - k_1)(k_3 - k_1)}\right] .$$

The required potential (15c) can be obtained by evaluating the f_i from (16a), (17b) and (19a). It is well–known that the result can be expressed in terms of the Wronskian of the functions in (16).

10. THE GENERAL INVERSE PROBLEM

If the given $R(k) \neq 0$ then Darboux transformations can still be used to fit the rest of the scattering data. Thus we assume some method is available for obtaining an initial potential $V_0(x)$ which has no eigenvalues $(M = 0)$ but fits a given reflection coefficient. However in view of (13), we fit $V_0(x)$ not to the given reflection coefficient $R(k)$, but to

$$R(k)\prod_{j=1}^{M}\frac{(k - ik_j)}{(k + ik_j)} .$$

Then (13) shows we shall reach the required $R(k)$ after making the M Darboux transformations as in the previous section. The initial potential $V_0(x)$ is added to each potential in the sequence (15).

The main difference from the previous reflectionless case is that the solutions for $V_0(x)$, corresponding to the sinh and cosh solutions in (16), may only be known numerically. This is a severe limitation because these solutions have to be differentiated in order to apply the transformations and obtain the potential.

However the asymptotic forms given in (16), (17) and (18) do remain valid, except that ω_i

should be replaced by ω_i^1 in the asymptotic forms as $x \to -\infty$. The value of ω_i^1 is fixed once the value of ω_i is chosen, and may always be obtained numerically. The alternation of cosh and sinh solutions in (16) is simply replaced by solutions with alternately the same or opposite signs for the two asymptotic forms as $x \to \pm\infty$.

The method can be extended to include a normalizable eigenfunction at $k = 0$. This requires extending the class of potentials so that $\int_{-\infty}^{\infty} |V(x)|\, dx < \infty$, allowing an asymptotic form

$$V(x) \to \frac{\ell(\ell + 1)}{(x - a)^2} \quad \text{as} \quad x \to \infty \quad \text{or} \quad x \to -\infty \quad \text{or both} .$$

Then if f_i $(i = 1,2)$ satisfy $f_i'' = Vf_i$, we have $f(x) = f_1(x) + cf_2(x) > 0$ for some continuous set of values of c. In the resulting Darboux transformation

$$\tilde{R}(k) = -R(k) \quad \text{and} \quad \tilde{T}(k) = T(k) \quad ,$$

and a value for c is fixed by a given asymptotic normalization constant r_0 for the zero–energy eigenfunction $\frac{1}{f}$.

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