Appendix E

On a wave equation with a singular source.

In this Appendix we shall show that the solutions of the problem (1.8.3)–(1.8.4), with $0 \notin \operatorname{supp} f$, $f \in C^\infty(\mathbb{R})$, $\varphi, \psi \in C^\infty(\mathbb{R}^2)$, are smooth on $\mathbb{R}^3 \setminus \operatorname{supp} \rho$. Consider thus the equation

$$\Box U = f(t)\,\delta_0\,,$$

recall that

$$\Delta \ln r = 2\pi\delta_0\,, \tag{E.0.1}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, and we also have

$$0 < \alpha \in \mathbb{R}, \quad \Delta\, r^\alpha \ln r = \alpha(\alpha \ln r + 2)\, r^{\alpha-2}\,. \tag{E.0.2}$$

From (E.0.1) one finds that the function $U_1 = U - f(t) \ln r / 2\pi$ satisfies

$$\Box U_1 = -\frac{1}{2\pi}\frac{\partial^2 f}{\partial t^2} \ln r\,,$$

and using (E.0.2) one shows by induction that there exist functions $\varphi_i \in C^\infty(\mathbb{R})$, $0 \notin \operatorname{supp} \varphi_i$, such that for any $k \in \mathbb{N}$ we have

$$\begin{aligned} \Box U_k &\equiv \Box \left(U - \sum_{i=0}^{k} \varphi_i(t)\, r^{2i} \ln r \right) \\ &= \chi_k(t)\, r^{2k} \ln r + \psi_k \equiv \rho_k\,, \end{aligned}$$

for some functions $\chi_k \in C^\infty(I\!R)$, $\psi_k \in C^\infty(I\!R^3)$. For any $\ell \in I\!N$ we can find k such that $\rho_k \in H_{\ell+2}(I\!R^3)$; we also have $U_k|_{t=0} = U|_{t=0}$, thus the Cauchy data for U_k are smooth, and by standard theory $U_k \in H_{\ell+2}(I\!R^3) \subset C_\ell(I\!R^3)$. This shows that for any ℓ we have $U \in C_\ell(I\!R^3 \backslash \mathrm{supp}\,\rho)$, thus $U \in C_\infty(I\!R^3 \backslash \mathrm{supp}\ \rho)$, which had to be established. Let us note that the argument presented above provides also an asymptotic expansion for U_ρ in a neighbourhood of supp ρ, which can be used to analyze in detail the nature of the singularities occuring in (M_ρ, γ_ρ).