Appendix D

Some flat metrics with Cauchy horizons.

In this Appendix we present a family of flat $n+1$ dimensional space–times which have at least two non–isometric extensions across a smooth Cauchy horizon; this example is a generalization of Misner's model for the Taub–NUT space–times [84]. The space–times considered here do not have compact spacelike hypersurfaces, as opposed to Misner's example which, in n+1 dimensions, can be given $\mathbb{R} \times S^1 \times T^{n-1}$ topology, where T^ℓ is an ℓ dimensional torus. In view of the renewed interest in flat Lorentzian manifolds [130] [92] [83] it would be useful to understand the global structure of all flat Lorentzian manifolds, at least in 4 dimensions.

For $n \geq 2$ let (t, x, y), $y = y^1$ if $n = 2$ or $y = (y^1, \ldots, y^{n-1})$ otherwise, be coordinates in $n+1$ dimensional Minkowski space,

$$ds^2 = -dt^2 + dx^2 + dy^2 \ , \tag{D.0.1}$$

with $dy^2 = (dy^1)^2$ if $n = 2$ and $dy^2 = d\vec{y}^{\,2} = (dy^1)^2 + \ldots + (dy^{n-1})^2$ otherwise. Let Λ_0 be a fixed Lorentz boost in the $t - x$ plane,

$$\Lambda_0 = \begin{bmatrix} \gamma_0 & \gamma_0\beta & 0 \\ \gamma_0\beta & \gamma_0 & 0 \\ 0 & 0 & id_{\mathbb{R}^{n-1}} \end{bmatrix} ,$$

$$\gamma_0 = \frac{1}{\sqrt{1-\beta_0^2}}\,, \qquad |\beta_0| < 1\,,$$

where $id_{\mathbb{R}^{n-1}}$ is the identity matrix in $\mathbb{R}^{n-1}$. Let $\mathcal{H}_\tau$ denote a hyperboloid in $\mathbb{R}^{n+1}$,

$$\mathcal{H}_\tau = \{\tau = t^2 - x^2 - \vec{y}^{\,2}\}\,,$$

with $\mathcal{H}_0$ — the light cone at the origin. Define $\mathcal{M}_{\Lambda_0}$ as the quotient of the interior $I^+(0)$ of the solid future light cone $J^+(0)$ from the origin by the group generated by Λ_0. In $I^+(0)$ we can introduce coordinates (τ, β, β_1) as follows:

$$\tau = t^2 - x^2 - y^2 - z^2 > 0\,,$$

$$t = ch\,\beta_1\, ch\,\beta + \frac{(\tau-1)e^{-\beta}}{2ch\,\beta_1}\,, \qquad x = ch\,\beta_1\, sh\,\beta - \frac{(\tau-1)e^{-\beta}}{2ch\,\beta_1}\,, \tag{D.0.2}$$

$$y = sh\,\beta_1 \ \text{ if } n=2\,, \quad \vec{y} = sh\,\beta_1\,\vec{\omega}\,,\ \vec{\omega} \in S^{n-1}(1) \ \text{ if } n>2\,, \tag{D.0.3}$$

where $S^k(1)$ is the k-dimensional unit sphere. In terms of (D.0.2) the metric (D.0.1) takes the form

$$\begin{aligned} ds^2 = \gamma_{\mu\nu} dx^\mu\, dx^\nu = {} & (\tau + sh^2\,\beta_1)\, d\beta^2 + \frac{1+\tau sh^2\,\beta_1}{ch^2\,\beta_1}\, d\beta_1^2 - d\tau\, d\beta \\ & -\frac{sh\,\beta_1}{ch\,\beta_1}\, d\tau\, d\beta_1 + \frac{2(\tau-1)\, sh\,\beta_1}{ch\,\beta_1}\, d\beta_1\, d\beta\,, \end{aligned} \tag{D.0.4}$$

and a term $sh^2\,\beta_1\, d\vec{\omega}^{\,2}$ has to be added to (D.0.4) if $n > 2$. One finds

$$\det \gamma_{\mu\nu} = -\tfrac{1}{4}\, ch^2\,\beta_1\, sh^{2(n-2)}\,\beta_1 \,\det h^0_{AB}\,, \tag{D.0.5}$$

h^0_{AB} — the round metric on a $n-1$ dimensional sphere ($\det h^0_{AB} = 1$ if $n = 2$). Equation (D.0.4) and the definition of $\mathcal{M}_{\Lambda_0}$ show that (τ, β, β_1) can be used as coordinates on $\mathcal{M}_{\Lambda_0}$ if β is identified with $\beta + \beta_0$. By (D.0.5) it follows that (D.0.4) gives an analytic extension of the flat metric on $\mathcal{M}_{\Lambda_0}$ to the larger manifold M,

$$\tau \in (-\infty, \infty)\,, \quad \beta_1 \in (-\infty,\infty) \text{ for } n=2\,, \quad \sinh\beta_1\,\vec{\omega} \in \mathbb{R}^{n-1} \text{ for } n>2\,, \tag{D.0.6}$$

$$\beta \in [0, \beta_0]|_{\mathrm{mod}\,\beta_0}\,.$$

Another extension $(\tilde{M}, \tilde{\gamma})$ is obtained by defining coordinates $(\tau, \tilde{\beta}, \beta_1)$, with $\tilde{\beta} \in [0, \beta_0]|_{\mathrm{mod}\,\beta_0}$ and τ, β_1 — as in (D.0.6), by

$$t = ch\beta_1 ch\tilde{\beta} + \frac{(\tau-1)e^{\tilde{\beta}}}{2ch\beta_1}\,, \qquad x = ch\beta_1 sh\tilde{\beta} + \frac{(\tau-1)e^{\tilde{\beta}}}{2ch\beta_1}\,, \tag{D.0.7}$$

y as in (D.0.3), which yields

$$ds^2 \;=\; \tilde{\gamma}_{\mu\nu} dx^\mu\, dx^\nu \;=\; (\tau + sh^2\beta_1)\, d\tilde{\beta}^2 + \frac{(1+\tau sh^2\beta_1)}{ch^2\beta_1}\, d\beta_1^2 + d\tau d\tilde{\beta} \tag{D.0.8}$$

$$-\frac{sh\beta_1}{ch\beta_1}\, d\tau d\beta_1 - \frac{2(\tau-1)sh\beta_1}{ch\beta_1}\, d\tilde{\beta}\, d\beta_1\,, \tag{D.0.9}$$

$$\det \tilde{\gamma}_{\mu\nu} = -\frac{ch^2\beta_1}{4}\, sh^{2(n-2)}\,\beta_1\, \det h^0_{AB}\,.$$

There exists an isometry $\Phi : \{p \in M : \tau(p) > 0\} \to \tilde{M}$, where Φ is obtained by calculating $(\tau, \tilde{\beta}, \beta_1)$ as a function of (τ, β, β_1) from (D.0.2) and (D.0.7). From the equation

$$e^{\tilde{\beta}} = \frac{e^\beta ch^2\beta_1}{\tau + sh^2\beta_1}\,,$$

it follows that $\tilde{\beta}$ blows up at $\tau = \beta_1 = 0$, thus Φ cannot be extended beyond $\tau = 0$. It should be noted that the map $\Psi : M \to \tilde{M}$ defined by

$$\tau \to \tau, \quad \beta \to -\tilde{\beta}, \quad \beta_1 \to \beta_1,$$

is a globally defined isometry, thus (M, γ) and $(\tilde{M}, \tilde{\gamma})$ *are* isometric. (M, γ) and $(\tilde{M}, \tilde{\gamma})$ are, however, *inequivalent* space–times from a Cauchy problem point of view for the following reason: let $i : \Sigma \to M$, $\tilde{\imath} : \tilde{\Sigma} \to \tilde{M}$ be embeddings such that M and $\tilde{M}$ are developments of the Cauchy data set (Σ, g, K), then there exists no isometry $\Xi : M \to \tilde{M}$ such that $\Xi \circ i = \tilde{\imath}$ (*cf.* [36]).

It should be noted that the space–times described above, restricted to the t–x plane, coincide with the so–called "Misner model" for the Taub–NUT space–time.