Appendix C

Maximal developments.

In this Appendix we shall discuss the existence of maximal developments, and we shall also prove some criteria which allow one to decide whether or not a given development is maximal.

C.1 Existence of maximal space–times.

In this section we shall prove the existence of maximal space–times; the reader should note that we are not making any global hyperbolicity hypotheses. The arguments here follow essentially those of [21]. Throughout this section "W manifold" stands for a connected, paracompact, Hausdorff n–dimensional manifold of differentiability class W such that $W \subset C^1$, where W stands for *e.g.* $C^{k,\alpha}$ or some Sobolev class, etc. The manifold will be said Lorentzian if it is equipped with a metric tensor, perhaps defined only almost everywhere, of a differentiability class adapted to that of W. For example, if $W = C^{k,\alpha}$ then we should have $k \geq 1$ and the metric, defined everywhere, will be of $C^{k-1,\alpha}$ differentiability class. It is useful to keep in mind that W can be a rather complicated space, *e.g.* for the purpose of the Cauchy problem in general relativity an appropriate space W is the set of maps which preserve the condition that the components of the metric tensor $\gamma_{\mu\nu}$ restricted to the hypersurfaces $\Sigma = \{t = \text{const}\}$ are of Sobolev class $H^k_{loc}(\Sigma)$, the time–derivatives of $\gamma_{\mu\nu}$ are in $H^{k-1}_{loc}(\Sigma)$, etc. A W Lorentzian manifold will be called vacuum if W is such that the equations $R_{\mu\nu} = 0$ can be defined, perhaps in a distributional sense, and if $R_{\mu\nu} = 0$ holds.

Theorem C.1.1 *Let (M, γ) be a W Lorentzian manifold, there exists a W Lorentzian manifold $(\tilde{M}, \tilde{\gamma})$ and an isometric embedding $\Phi : M \to \tilde{M}$ such that $\tilde{M}$ is inextendible in the class of W Lorentzian manifolds. The same is true if "Lorentzian W manifold" is replaced by "vacuum Lorentzian W manifold" everywhere above.*

Remarks:

1. The C^1 differentiability threshold for M cannot be weakened in the proof below. The author ignores whether or not the C^1 differentiability of M is necessary.

2. The maximal manifolds $(\tilde{M}, \tilde{\gamma})$ need not be unique, and may depend upon W. A non–trivial example of W dependence, with $W = C^{k,\alpha}$, is given by some Robinson–Trautman (RT) space–times, which for $k + \alpha \geq 123$ admit no non–trivial future extensions, while for $k + \alpha < 118$ they admit an infinite number of non–isometric vacuum RT extensions.

Proof: For $\ell \geq n$ let A_ℓ denote the set[1] of subsets of $\mathbb{R}^\ell$ which are n–dimensional manifolds, set $A_\infty = \cup_{\ell=0}^{\infty} A_\ell$. By a famous theorem of Whitney [128] every (C^1, connected, paracompact, Hausdorff) manifold can be embedded in $\mathbb{R}^\ell$ for some ℓ, which shows that every manifold has a representative which is an element of A_∞; it follows that without loss of generality a manifold can be defined as an element of A_∞; and we shall do so. With this definition the collection of all manifolds is A_∞, and therefore is a set. It follows from the axioms of set theory that the collection of all C^1 manifolds which are W manifolds forms a set; the same is true of the collection $\mathcal{M}_W$ of W Lorentzian manifolds (recall that a Lorentzian manifold can be identified with a subset of the bundle T_2M, where T_2M is the bundle of 2–covariant tensors on M), and of the collection $\mathcal{M}_{W,vac}$ of W vacuum Lorentzian manifolds. Let (M, γ) be a Lorentzian, respectively

[1] *cf. e.g.* [78][Appendix] for an overview of axiomatic set theory.

a vacuum Lorentzian, W manifold, consider the subset $\mathcal{M}_W(M,\gamma)$ of $\mathcal{M}_W$, respectively $\mathcal{M}_{W,vac}(M,\gamma)$ of $\mathcal{M}_{W,vac}$, defined as the set of those Lorentzian manifolds $(\tilde{M},\tilde{\gamma})$ for which there exists an isometric C^1 embedding $\Phi : M \to \tilde{M}$. On both $\mathcal{M}_W(M,\gamma)$ and $\mathcal{M}_{W,vac}(M,\gamma)$ we can define a partial order $\prec$ as follows: $(\tilde{M},\tilde{\gamma}) \prec (\tilde{M}_1,\tilde{\gamma}_1)$ if there exists an isometric C^1 embedding $\Phi : \tilde{M} \to \tilde{M}_1$. If $A \subset \mathcal{M}_W(M,\gamma)$ or $A \subset \mathcal{M}_{W,vac}(M,\gamma)$ is a chain, define $\bar{M} = \left(\cup_{(\tilde{M},\tilde{\gamma})\in A}\tilde{M}\right) / \sim$, where for $p \in \tilde{M}$ and $q \in \tilde{M}_1$ we set $p \sim q$ iff $q = \Phi(p)$, where $\Phi : \tilde{M} \to \tilde{M}_1$ is an isometric C^1 embedding. It is not too difficult to show that $\bar{M}$ is a W manifold (Hausdorff, paracompact, connected), a Lorentzian metric $\bar{\gamma}$ can be defined on $\bar{M}$ in an obvious way. Since every element $(\tilde{M},\tilde{\gamma})$ of A can be embedded in $\bar{M}$ ($\tilde{M} \ni p \to [p]_\sim \in \bar{M}$), it follows that $\bar{M}$ is an upper bound for A. Zorn's Lemma (*cf. e.g.* [78]) shows that both $\mathcal{M}_W(M,\gamma)$ and $\mathcal{M}_{W,vac}(M,\gamma)$ have maximal elements, which had to be established. □

C.2 Some maximality criteria.

In this Appendix we will discuss some inextendability criteria. Let us start with some terminology. Recall that we have defined $(\tilde{M},\tilde{\gamma})$ to be an extension of (M,γ) if there exists an isometric embedding Φ of M into $\tilde{M}$, and $M \neq \tilde{M}$. We shall often identify M with $\Phi(M)$. If M is a subset of $\tilde{M}$, by ∂M we always mean the topological boundary of M in $\tilde{M}$: $\partial M \equiv \{p \in \tilde{M} \setminus M$ such that $\exists p_i \in M$ with the property that $p_i \to p\}$. When considering inextendability criteria, it is useful to keep in mind the possibility of allowing extensions in which a weak form of violation of Hausdorffness occurs. Following Clarke [40] we shall say that a (possibly[2] non–Hausdorff) space–time is a *Hajiček space–time* if there exists no bifurcating causal curves in (M,γ) (more precisely, let $\Gamma_\alpha : [0,b) \to M$, $\alpha = 1,2$ be two continuous causal curves such that $\Gamma_1|[0,a) = \Gamma_2|[0,a)$ for some $a < b$, then $\Gamma_1(a) = \Gamma_2(a)$. In this section all geodesics are affinely parametrized; unless specified otherwise we do not impose any particular orientation on the affine parameter. In all the results presented in this section we make essential use of both existence and uniqueness

[2]If M is Hausdorff, then the Hajiček condition trivially holds.

of solutions of the initial value problem for geodesics; those are guaranteed by $C^{1,1}_{loc}$ differentiability of the metric and by the requirement that the space–times considered satisfy the Hajiček condition. The results here need not to hold in an arbitrary non–Hausdorff space–time, or in space–times with metrics the derivatives of which are not Lipschitz continuous.

Let us recall a criterion which has been used by Misner and Taub [85] to prove inextendability of the Taub–NUT space–time:

Proposition C.2.1 *Let (M, γ) be a Hausdorff space–time with a $C^{1,1}_{loc}$ metric, suppose that for every geodesic segment $\Gamma : [s_0, s_1) \to M$ which cannot be extended beyond s_1 there exists a compact set K such that $\Gamma \subset K$. Then (M, γ) is inextendible in the class of Hausdorff space–times with $C^{1,1}_{loc}$ metrics.*

Remark: It should be noted that in Taub–NUT space–time there exist inextendible geodesic segments which remain in a compact set.

Proof: Suppose that $(\tilde{M}, \tilde{\gamma})$ is an extension of (M, γ). For any $p \in \partial M$ there exist some geodesic segment $\Gamma_p : [0, a] \to \tilde{M}$ such that $\Gamma_p(0) = p$, $\Gamma_p(a) \in M$ (Γ_p — timelike, spacelike or null). Since $p \notin M$, it follows that $\Gamma_p \cap M$ is inextendible in M and is not contained in any compact set K, which leads to a contradiction. □

In exactly the same way one proves:

Proposition C.2.2 *Let (M, γ) be a Hajiček space–time with a $C^{1,1}_{loc}$ metric, suppose that for every geodesic segment $\Gamma : [s_0, s_1) \to M$ which cannot be extended beyond s_1 either*

1. *there exists a compact set K such that $\Gamma \subset K$, or*
2. *some polynomial scalar of the curvature tensor is unbounded on Γ as $s \to s_1$.*

Then (M, γ) is inextendible in the class of Hajiček space–times with $C^{1,1}_{loc}$ metrics.

Note that if γ and $\tilde{\gamma}$ are assumed to be twice differentiable, then point 2 above can be weakened to "some polynomial scalar of the curvature tensor has no finite limit on Γ as $s \to s_1$" (in other words, if the limit exists, it is infinite).

If we have an extension of (M, γ) and a point $p \in \partial M$ such that there exists a timelike curve from M to p, it is easily seen that there necessarily exists a *timelike geodesic* from M to p. Whenever (M, γ) is time–orientable, it is natural to divide extensions in the following classes:

1. there exists a future directed timelike geodesic from M to $\tilde{M}$,
2. there exists a past directed timelike geodesic from M to $\tilde{M}$,
3. there exist both future and past time directed timelike geodesics from M to $\tilde{M}$,
4. there exists no timelike geodesics from M to $\tilde{M}$.

Let us show that case 4 above cannot occur:

Proposition C.2.3 *Let (M, γ) be a Hajiček space–time with a $C^{1,1}_{loc}$ metric, suppose that $(\tilde{M}, \tilde{\gamma})$ is a Hajiček extension thereof with a $C^{1,1}_{loc}$ metric. Then there necessarily exists*

1. *a timelike geodesic from M to ∂M, and*
2. *a null geodesic from M to ∂M.*

Proof: Suppose there exists $p \in \partial M$ such that there exists no timelike geodesic from M to p, thus $I(p) \cap M = \emptyset$, where $I(p)$ is the union of the future and of the past of p in $\tilde{M}$. There exists a sequence $p_i \in M$ such that $p_i \to p$, choose any normal geodesic neighbourhood $\mathcal{O} \subset \tilde{M}$ of p with a local coordinate chart, for i large enough we have $p_i \in \mathcal{O}$. It is easily seen that for i large enough a maximally extended geodesic through p_i with tangent vector $\partial/\partial t$ at p must leave M and enter $\tilde{M}$, and the result follows because in a Hajiček space–time there exists no bifurcate timelike geodesics. The result for null geodesics is proved in a similar way. □

We shall say that a time–orientable space–time (M, γ) is *future inextendible* if there exists no extension $(\tilde{M}, \tilde{\gamma})$ of (M, γ) in which a future directed causal curve starting in M enters $\tilde{M}$; the notion of *past inextendibility* is defined similarly. Proposition C.2.3 shows that a space–time which is both future and past inextendible is inextendible. This is a useful result, because together with Proposition C.2.2 it reduces the task of proving inextendability of M to an analysis of the behaviour of either null or timelike geodesics in (M, γ):

Proposition C.2.4 *1. In Propositions C.2.1 and C.2.2 "every geodesic segment" can be replaced by "every timelike geodesic segment".*

2. In Propositions C.2.1 and C.2.2 "every geodesic segment" can be replaced by "every null geodesic segment".

Given a globally hyperbolic space–time (M, γ), it might be useful in some situations to be able to determine whether (M, γ) is maximal in the class of globally hyperbolic space–times (note that this does not exclude the existence of non–globally–hyperbolic extensions of (M, γ)). We shall say that a globally hyperbolic Lorentzian manifold (M, γ) admits a *crushing future boundary* if there exists a foliation of (M, γ) by spacelike Cauchy hypersurfaces Σ_τ, $\tau \in (\tau_o, \tau_1)$, $-\infty \le \tau_o < \tau_1 \le \infty$, such that

$$lim_{\tau \to \tau_1} K^+_\tau = \infty \ ,$$

where

$$K^+_\tau \equiv \inf_{p \in \Sigma_\tau} (g^{ij} K_{ij})(p) \ ,$$

g^{ij} is the inverse of the metric g_{ij} induced by γ on Σ_τ, and K_{ij} is the extrinsic curvature of Σ_τ. Similarly a *crushing past boundary* is defined by the condition

$$lim_{\tau \to \tau_0} K^-_\tau = -\infty \ , \qquad K^-_\tau \equiv \sup_{p \in \Sigma_\tau} (g^{ij} K_{ij})(p) \ .$$

(Eardley and Smarr [43] have proposed the term "crushing singularity" for the above described behaviour: we find that terminology misleading, because the existence of a crushing boundary does not imply existence of a singularity in the geometry[3], there might exist perfectly smooth extensions of (M,γ) as is seen *e.g.* in some polarized Gowdy metrics [37].) Although it is irrelevant for further purposes, it might be of some interest to note that in spatially compact space–times (M,γ) with crushing boundaries there exist constant mean curvature (CMC) surfaces [7]; if moreover the timelike convergence condition holds ($R_{\mu\nu}X^\mu X^\nu \geq 0$ for all timelike vectors X), then (M,γ) can be foliated by CMC surfaces.

As has been noted by Moncrief [87], the existence of crushing boundaries implies maximality in the class of globally hyperbolic space–times:

Proposition C.2.5 *Let (M,γ) be a globally hyperbolic, spatially compact, Hajiček space–time with a $C^{1,1}_{loc}$ metric.*

1. *Suppose that (M,γ) has a future crushing boundary. Then (M,γ) is future inextendible in the class of globally hyperbolic, Hajiček space–times with $C^{1,1}_{loc}$ metrics.*
2. *The same is true if "future" is replaced by "past" everywhere above.*

Proof: Let Σ_τ, $\tau_0 < \tau < \tau_1$ be a foliation of (M,γ) as in the definition of future crushing boundary, suppose that $(\tilde{M},\tilde{\gamma})$ is a future extension of (M,γ), let $p \in \tilde{M} \setminus M$ be such that $M \cap I^-(p;\tilde{M}) \neq \emptyset$, where $I^-(p;\tilde{M})$ is the past of p in $\tilde{M}$. By global hyperbolicity of $(\tilde{M},\tilde{\gamma})$ for any $\tau < \tau_1$ there exists a future directed maximizing timelike geodesic $\Gamma_\tau : [0, s_1(\tau)] \to \tilde{M}$ parametrized by proper time such that $\Gamma_\tau(0) \in \Sigma_\tau$, $\Gamma_\tau(s_1(\tau)) = p$, for some $s_1(\tau) > 0$, where throughout this proof "maximizing" stands for "maximizing in the class of geodesics which start on Σ_τ and have p as endpoint" . Choose some $\tau_o < \tilde{\tau} < \tau_1$, there exists $\tilde{s} < s_1(\tilde{\tau})$ such that $\Gamma_{\tilde{\tau}}(\tilde{s}) \in \partial M$. Since the Σ_τ's are Cauchy surfaces it follows that for $\tau > \tilde{\tau}$ we have $\Sigma_\tau \cap \Gamma_{\tilde{\tau}} \neq \emptyset$, and since the Γ_τ's are maximizing geodesics from Σ_τ to p it follows that $s_1(\tau) \geq s_1(\tilde{\tau}) - \tilde{s} > 0$. But by [66][Corollary,

[3]If the space–time is spatially compact and the timelike convergence condition holds ($R_{\mu\nu}X^\mu X^\nu \geq 0$ for all timelike vectors X) (M,γ) will be, however, geodesically incomplete, *cf. e.g.* [66][Chapter 8, Theorem 4].

Section 6.7] and [66][Proposition 4.4.3] there are no future directed maximizing geodesics of length more then $3/K_\tau^+$ starting at Σ_τ, which gives a contradiction since $3/K_\tau^+ \to 0$ on Γ as $s \to s_0$. □