Appendix B

On a class of $U(1) \times U(1)$ symmetric metrics found by V.Moncrief.

In this Appendix we shall prove that "strong cosmic censorship" holds in a six parameter family of *non–polarized* $U(1) \times U(1)$ symmetric metrics found by Moncrief[1] [93]. Apart from being interesting in their own right, these metrics provide a good testing ground for various *a priori* estimates one can obtain for general $U(1) \times U(1)$ symmetric metrics, *cf.* Chapter 3.

Throughout this Appendix the letter C denotes a constant the value of which may vary from line to line.

B.1 A harmonic map problem.

Let $x(t,\theta) = (\rho(t,\theta), \phi(t,\theta))$ be a map from two–dimensional Minkowski space to a two dimensional constant mean curvature hyperoboloid, set

$$X_t = \frac{\partial x^A}{\partial t}\frac{\partial}{\partial x^A} = \frac{\partial \rho}{\partial t}\frac{\partial}{\partial \rho} + \frac{\partial \phi}{\partial t}\frac{\partial}{\partial \phi}, \quad X_\theta = \frac{\partial x^A}{\partial \theta}\frac{\partial}{\partial x^A} = \frac{\partial \rho}{\partial \theta}\frac{\partial}{\partial \rho} + \frac{\partial \phi}{\partial \theta}\frac{\partial}{\partial \phi}\,.$$

On the hyperboloid one can introduce coordinates in which the metric takes the form

$$ds^2 = d\rho^2 + \sinh^2\rho\, d\phi^2\,.$$

[1]A similar class of harmonic maps has been considered independently by Shatah and Tahvildar-Zadeh in [118]; *cf.* also [63].

The Christoffel symbols are easily calculated to be

$$\Gamma^{\rho}_{\phi\phi} = -\sinh\rho\cosh\rho\,, \quad \Gamma^{\rho}_{\phi\phi} = \frac{\cosh\rho}{\sinh\rho}\,,$$

so that the Gowdy equations

$$\frac{DX_t}{Dt} - \frac{DX_\theta}{D\theta} = -\frac{X_t}{t}\,,$$

where D is the covariant derivative in the target space, $D_t \equiv D_{X_t}$, $D_\theta \equiv D_{X_\theta}$, take the form

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial\theta^2}\right)\phi = -\frac{1}{t}\frac{\partial\phi}{\partial t} - 2\coth\rho\left(\frac{\partial\rho}{\partial t}\frac{\partial\phi}{\partial t} - \frac{\partial\rho}{\partial\theta}\frac{\partial\phi}{\partial\theta}\right) \tag{B.1.1}$$

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial\theta^2}\right)\rho = -\frac{1}{t}\frac{\partial\rho}{\partial t} + \sinh\rho\cosh\rho\left(\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial\theta}\right)^2\right). \tag{B.1.2}$$

It has been observed by V. Moncrief [93] that the ansatz

$$\rho = \rho(t)\,, \quad \phi = n\theta \tag{B.1.3}$$

is compatible with the above equations, which then reduce to a single ordinary differential equation for ρ

$$\frac{d^2\rho}{d\tau^2} = -n^2\sinh\rho\cosh\rho\, e^{-2\tau}\,, \quad (t = e^{-\tau})\,. \tag{B.1.4}$$

For given θ_o the function $\rho(t)$ should be thought of as an affine parameter on the geodesic $\Gamma = \{\theta = \theta_o, \rho \geq 0\} \cup \{\theta = \pi + \theta_o, \rho \geq 0\}$ on the hyperboloid, rather than a radial coordinate constrained to satisfy $\rho \geq 0$, so that a change of sign of ρ means that $\rho(t)$ has crossed the origin along Γ. The following gives a complete description of the behaviour as $t \to 0$ of solutions of (B.1.4):

Proposition B.1.1 *1. For $\tau_o > -\infty$ and for every solution $\rho \in C_2([\tau_o, \infty))$ of (B.1.4) there exist constants $0 \leq |v_\infty| < 1$ and $\rho_\infty \in I\!R$ such that*

$$|\rho(\tau) - v_\infty\tau - \rho_\infty| + |\frac{d\rho}{d\tau}(\tau) - v_\infty| \leq C\, e^{-2(1-|v_\infty|)\tau} \tag{B.1.5}$$

for all $\tau_o \leq \tau < \infty$, for some constant C.

2. For every $0 \leq |v_\infty| < 1$ and every $\rho_\infty \in I\!R$ there exists a solution of (B.1.4) satisfying (B.1.5). If $\rho_\infty = v_\infty = 0$ then $\rho(\tau) \equiv 0$.

Proof: 1. Let

$$g(\tau) \equiv (\frac{d\rho}{d\tau})^2 + \frac{n^2}{2} \cosh^2 \rho\, e^{-2\tau} .$$

We have

$$\frac{dg(\tau)}{d\tau} = -n^2 \cosh^2 \rho\, e^{-2\tau} ,$$

which shows that g is monotonically decreasing, so that $g_\infty = \lim_{\tau\to\infty} g(\tau)$ exists and we have

$$g(\tau) = g_\infty + n^2 \int_\tau^\infty \cosh^2 \rho(s)\, e^{-2s} ds \quad \Longrightarrow$$

$$\int_{\tau_0}^\infty \cosh^2 \rho(s)\, e^{-2s} ds \ < \ \infty . \tag{B.1.6}$$

For $\tau_1 \geq \tau_2$ it follows from (B.1.4) that

$$\begin{aligned} |\frac{d\rho}{d\tau}(\tau_1) - \frac{d\rho}{d\tau}(\tau_2)| &= n^2 |\int_{\tau_2}^{\tau_1} \sinh \rho(s) \cosh \rho(s)\, e^{-2s}\, ds| \\ &\leq n^2 \int_{\tau_2}^{\tau_1} \cosh^2 \rho(s)\, e^{-2s} ds , \end{aligned}$$

which together with (B.1.6) implies that $\lim_{\tau\to\infty} \frac{d\rho}{d\tau}(\tau) = v_\infty$ exists.

Let us first assume $v_\infty = 0$, then for any $\epsilon > 0$ there exists τ_1 such for $\tau > \tau_1$ we have $|\frac{d\rho}{d\tau}| < \epsilon$, which implies that $|\rho| \leq \epsilon\tau + C$, and (B.1.4) gives

$$|\frac{d^2\rho}{d\tau^2}| \leq C_1\, e^{-2(1-\epsilon)\tau} \ \Rightarrow \ |\frac{d\rho}{d\tau}| \leq C_2\, e^{-2(1-\epsilon)\tau} ,$$

by integration, and one more integration shows that the limit $\lim_{\tau\to\infty} \rho(\tau) = \rho_\infty$ exists and we have

$$|\rho - \rho_\infty| \leq C_3\, e^{-2(1-\epsilon)\tau} \ \Rightarrow \ |\frac{d^2\rho}{d\tau^2}| \leq C_4\, e^{-2\tau} \ \Rightarrow \ |\rho - \rho_\infty| \leq C_5\, e^{-2\tau} ,$$

for some constants $C_1 - C_5$, which had to be established for $v_\infty = 0$.

If $v_\infty \neq 0$, it follows that $\frac{d\rho}{d\tau}$ has constant sign for $\tau \geq \tau_1$, τ_1 large enough, so that ρ has constant sign for large enough times, and multiplying ρ by -1 if necessary we may assume that for $\tau > \tau_1$ we have $\rho(\tau) > 0$, and also $v_\infty > 0$. Let us show that $v_\infty < 1$. Equation (B.1.4) implies that $\frac{d\rho}{d\tau}$ is non-increasing, so that if $0 < \frac{d\rho}{d\tau}(\tau_o) < 1$ we are done, let us therefore assume that $\frac{d\rho}{d\tau}(\tau_o) > 1$. Let $\tau_1 \leq \infty$ be such that for $\tau_o \leq \tau < \tau_1$ we have $\frac{d\rho}{d\tau}(\tau) > 1$, with $\frac{d\rho}{d\tau}(\tau_1) = 1$ if $\tau_1 < \infty$. For $\tau \in [\tau_o, \tau_1]$ we have

$$\rho(\tau) - \rho(\tau_o) = \int_{\tau_o}^{\tau} \frac{d\rho}{d\tau}(s)\, ds \geq \tau - \tau_o\,,$$

so that

$$\begin{aligned}
\frac{d\rho}{d\tau}(\tau) &= \frac{d\rho}{d\tau}(\tau_o) - \int_{\tau_o}^{\tau} \frac{n^2}{4}\left(e^{2(\rho(\tau)-s)} - e^{-2\rho(s)-2s}\right) ds \\
&\leq \frac{d\rho}{d\tau}(\tau_o) - \int_{\tau_o}^{\tau} \frac{n^2}{4}\, e^{2(\rho(\tau_o)-\tau_o)} ds + \frac{n^2}{4}\int_{\tau_o}^{\tau} e^{-2(\rho(\tau_o)+s)}\, ds \\
&\leq \frac{d\rho}{d\tau}(\tau_o) - \frac{n^2}{4}(\tau - \tau_o)\, e^{2(\rho(\tau_o)-\tau_o)} + \frac{n^2}{8} e^{-2(\tau_o+\rho(\tau_o))}\,,
\end{aligned}$$

which is smaller than 1 for sufficiently large τ so that $\tau_1 < \infty$, and our claim follows. Define

$$r = \rho(\tau) - v_\infty \tau\,.$$

The function r satisfies $\lim_{\tau\to\infty} \frac{dr}{d\tau} = 0$, setting $\lambda = 2(1 - v_\infty) > 0$ one obtains

$$\frac{d^2 r}{d\tau^2} = -\frac{n^2}{4}\left(e^{2r-\lambda\tau} - e^{-2r-2(1+v_\infty)\tau}\right). \tag{B.1.7}$$

For $\tau > \tau_1(\epsilon)$, with $\tau_1(\epsilon) > 0$ sufficiently large, we have $|\frac{dr}{d\tau}| \leq \epsilon$ for any $\epsilon > 0$, therefore $|r(\tau)| \leq |r(\tau_1(\epsilon))| + \epsilon\tau$ and for $2\epsilon < \lambda/2$ one gets $\frac{d^2 r}{d\tau^2} = O(e^{-\lambda\tau/2})$; by integration one obtains $|\frac{dr}{d\tau}(\tau_2) - \frac{dr}{d\tau}(\tau)| = O(e^{-\lambda\tau/2})$, for $\tau_2 > \tau$. Passing with τ_1 to ∞ one gets $\frac{dr}{d\tau} = O(e^{-\lambda\tau/2})$, which integrating again yields, for $\tau_2 > \tau$

$$|r(\tau_2) - r(\tau)| = O(e^{-\lambda\tau/2})\,,$$

it ensues in a simple way that there exists a constant ρ_∞ such that

$$|r(\tau) - \rho_\infty| = O(e^{-\lambda\tau/2})\,.$$

Coming back to the equation (B.1.7) satisfied by r we have in fact

$$\left|\frac{d^2 r}{d\tau^2}\right| = O(e^{-\lambda\tau})\,,$$

which gives by a similar argument

$$|\frac{dr}{d\tau}(\tau)| + |r(\tau) - \rho_\infty(\tau)| \le C\, e^{-2(1-v_\infty)\tau} \ ,$$

so that (B.1.5) follows.

2. Multiplying ρ by -1 if necessary we may assume $v_\infty \ge 0$. Equations (B.1.4) and (B.1.5) are equivalent to the following integral equation for $p(\tau) = \rho(\tau) - v_\infty \tau - \rho_\infty$:

$$p(\tau) = T(p)(\tau)$$

$$T(p)(\tau) = -\frac{n^2}{4}\int_\tau^\infty (s-\tau)(e^{2p(s)+2\rho_\infty} - e^{-2p(s)+2\rho_\infty - 4v_\infty s})\, e^{-\lambda s}\, ds \ ,$$

with $\lambda = 2(1 - v_\infty)$. Let $H = \{p \in C([\tau_1, \infty))\,, \|p(\tau)\| = \sup_\tau |e^{\lambda\tau/2} p(\tau)| < \infty\}$. One checks without difficulty that there exists $\tau_1(v_\infty, \rho_\infty, n) < \infty$ such that T takes the unit ball of H into itself, and that T is a contraction — the claim follows by the contraction mapping principle. Finally if $v_\infty = \rho_\infty = 0$ then there are no "driving terms" in T so that the contraction property implies $p \equiv 0$. □

Let us analyze the behaviour of solutions of (B.1.4) as $t \to \infty$ ($\tau \to -\infty$):

Proposition B.1.2 *For $t_o > 0$ let $\rho \in C^2([t_o, \infty))$ be a solution of (B.1.4).*

1. *There exists a constant C such that*

$$|\rho| + |\frac{d\rho}{dt}| \le Ct^{-1/2} \ . \qquad \text{(B.1.8)}$$

2. *There exist constants e_∞, C_1 such that*

$$\left| t[(\frac{d\rho}{dt})^2 + n^2 \sinh^2 \rho] - e_\infty \right| \le Ct^{-1} \ . \qquad \text{(B.1.9)}$$

If $e_\infty = 0$, then $\rho \equiv 0$.

Remark: The proof below suggests very strongly that we have the expansion

$$\rho = A\cos(nt + \delta)\, t^{-1/2} + B_1(t)\, t^{-3/2} + B_2(t)\, t^{-5/2} + \dots$$

for some constants A, δ and some functions $B_i(t)$ which are polynomials in $\sin(nt)$ and $\cos(nt)$.

Proof: Define

$$\psi(t) = t^{1/2}\rho(t) ;$$

ψ satisfies the equation

$$\frac{d^2\psi}{dt^2} = -\frac{\psi}{4t^2} - \frac{n^2t^{1/2}}{2}\sinh(2\rho) . \tag{B.1.10}$$

Let us set

$$e(t) = (\frac{d\psi}{dt})^2 + \frac{\psi^2}{4t^2} + \frac{n^2t}{2}(\cosh(2\rho) - 1) ; \tag{B.1.11}$$

we have

$$\frac{de}{dt} = -\frac{\psi^2}{2t^3} - n^2V(\rho) , \tag{B.1.12}$$

$$V(\rho) = \rho\sinh(2\rho) + 1 - \cosh(2\rho) .$$

We have $V(0) = V'(0) = 0$ and $V''(\rho) = 4\rho\sinh(2\rho) \geq 0$, thus $V(\rho) \geq 0$, so that

$$\frac{de}{dt} \leq 0 , \tag{B.1.13}$$

which shows that

$$(\frac{d\psi}{dt})^2 \leq C, \quad t\left(\cosh(2t^{-1/2}\psi) - 1\right) \leq C \tag{B.1.14}$$

for some constant C. The inequality $\cosh(2\rho) - 1 \geq 2\rho^2$ and the second inequality in (B.1.14) give

$$\psi^2 \leq C , \tag{B.1.15}$$

and (B.1.8) follows. Taylor expanding $\cosh\rho$ to fourth order in (B.1.11) and making use of (B.1.15) one obtains

$$e(t) = (\frac{d\psi}{dt})^2 + n^2\psi^2 + O(t^{-1}) . \tag{B.1.16}$$

From (B.1.13) it follows that e is monotone, therefore the limit $e_\infty = \lim_{t\to\infty} e(t)$ exists. There exists a constant C_V such that for $\rho \leq 1$ we have $|V(\rho)| \leq C_V\rho^4$, and since ρ tends to zero as t goes to infinity there exists a T such that for $t \geq T$ we have $|V(\rho)| \leq C_V\psi^4t^{-2} \leq C't^{-2}$, and integrating (B.1.12) one obtains

$$|e(t) - e_\infty| \leq Ct^{-1} , \tag{B.1.17}$$

so that (B.1.16) leads to

$$(\frac{d\psi}{dt})^2 + n^2\psi^2 - e_\infty = O(t^{-1}) \; ;$$

a simple calculation gives

$$t\,[(\frac{d\rho}{dt})^2 + n^2\sinh^2\rho] = e_\infty + O(t^{-1}) \; ,$$

which proves (B.1.9). Suppose finally that $e_\infty = 0$. The inequality (B.1.17) together with $e_\infty = 0$ implies

$$|\frac{d\psi}{dt}(t)| + |\psi(t)| \le Ct^{-1/2} \; . \tag{B.1.18}$$

Inserting (B.1.18) into (B.1.12) and repeating iteratively the above argument one shows that for any $\ell \in I\!N$ there exists $C(\ell)$ such that

$$|\psi(t)| + e(t) \le C(\ell)\, t^{-\ell} \; . \tag{B.1.19}$$

The inequalities (B.1.19) with $\ell = 2$ and $V(\rho) \le C_V\rho^4$ give

$$n^2V(\rho) \le C_V n^2\rho^4 = \frac{C_V n^2\psi^4}{t^2} \le \frac{C_V n^2 C(2)^2}{t^3}\frac{\psi^2}{t^3} \le \frac{\psi^2}{2t^3}$$

for $t \ge t_1 = [2C_V n^2 C(2)^2]^{1/3}$, which leads to

$$-\frac{e}{t} \le -\frac{\psi^2}{4t^3} = \frac{1}{4}\left[\frac{de}{dt} - \left(\frac{\psi^2}{2t^3} - n^2V(\rho)\right)\right] \le \frac{1}{4}\frac{de}{dt}$$

$$\Longrightarrow \quad \frac{de}{dt} \ge -\frac{4e}{t} \; ,$$

which implies

$$t_1 \le t_2 \le t_3 \qquad e(t_2) \le \frac{e(t_3)t_3^4}{t_2^4} \; ,$$

passing with t_3 to infinity one obtains $e(t) \equiv 0$, $\psi \equiv \rho \equiv 0$, which had to be established.

□

B.2 Moncrief's space–times.

Let $M = \{t \in (0,\infty),\ \theta,\ x^a \in [0,2\pi]_{|\mathrm{mod}\,2\pi}, a = 1,2\}$. Consider the following Gowdy–type metrics

$$ds^2 = \gamma_{\mu\nu}dx^\mu dx^\nu = e^{2B}(-dt^2 + d\theta^2) + \lambda t n_{ab}\,(dx^a + g^a d\theta)(dx^b + g^b d\theta)\ , \qquad \text{(B.2.1)}$$

$$n_{ab}\,dx^a dx^b = (\cosh\rho + \cos\phi\,\sinh\rho)\,(dx^1)^2 + 2\sinh\rho\sin\phi\,dx^1dx^2$$

$$+(\cosh\rho - \cos\phi\,\sinh\rho)\,(dx^2)^2\ ,$$

$$\rho = \rho(t,\theta), \quad \phi = \phi(t,\theta)\ ,$$

where λ and g^a are real constants, $\lambda > 0$. For a metric of the form (B.2.1) the dynamical part of Einstein equations reduces to the equations (B.1.1)–(B.1.2), and assuming Moncrief's ansatz (B.1.3) one finds that $B = B(t)$ (*cf. e.g.* eqs. (2.30) and (2.33) of [32]; the constants c_a appearing in these equations vanish for the metrics (B.2.1)), and

$$\frac{dB}{dt} = -\frac{1}{4t} + \frac{t}{4}[(\frac{d\rho}{dt})^2 + n^2\sinh^2\rho]\ . \qquad \text{(B.2.2)}$$

In this way we obtain a family of metrics parametrized by six parameters — λ, g^a, $a = 1,2$, an integration constant B_o for B and two real constants parametrizing solutions of the equation (B.1.4), *e.g.* v_∞ and ρ_∞ given by Proposition B.1.1. (Out of these parameters of course only v_∞ and ρ_∞ are dynamically interesting.) We shall refer to these space–times as *Moncrief's space–times.*

Proposition B.2.1 *All Moncrief's space–times are future causally geodesically complete.*

Proof: If the constant e_∞ given by Proposition B.1.2 vanishes ($\Rightarrow \rho \equiv 0$, *cf.* Proposition B.1.2) one easily checks that the metric can be put in Kasner's form with exponents $(p_1, p_2, p_3) = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$ or permutation thereof (*cf.* Section 2.4 for a description of Kasner metrics), in which case it is easy to show future geodesic completeness, we shall thus consider the case $e_\infty \neq 0$ only. Let $\Gamma(s) = \{x^\mu(s)\}$ be a future inextendible future directed affinely parametrized causal geodesic. From $\gamma_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = -\epsilon$, $\epsilon \in \{0,1\}$, and from

$$\frac{dp_a}{ds} \equiv \frac{d}{ds}\left(\gamma_{\mu\nu}\frac{dx^\mu}{ds}X^\nu_a\right) = 0 \ ,$$

where $X^\nu_a\frac{\partial}{\partial x^\nu}$ are the Killing vectors $\frac{\partial}{\partial x^a}$, $a = 1,2$, one obtains the following equations,

$$e^{2B}[(\frac{dt}{ds})^2 - (\frac{d\theta}{ds})^2] = \epsilon + \bar{g}^{ab}p_a p_b \ ,$$

$$\frac{dx^a}{ds} = -g^a\frac{d\theta}{ds} + p^a, \quad p^a \equiv \bar{g}^{ab}p_b \ ,$$

where $\bar{g}^{ab} \equiv (g_{ab})^{-1}$, $g_{ab} = \lambda t n_{ab}$. The t part of the equations satisfied by a geodesic reads

$$\frac{d}{ds}(B^2\frac{dt}{ds}) = B^2e^{-2B}[(\epsilon + \bar{g}^{ab}p_a p_b)\frac{dB}{dt} - \frac{1}{2}\frac{\partial g_{ab}}{\partial t}p^a p^b] \ . \tag{B.2.3}$$

Non–spacelikeness of Γ implies that Γ can be parametrized by t, which allows us to rewrite (B.2.3) as

$$\frac{d}{dt}\left(B^4(\frac{dt}{ds})^2\right) = f \ , \tag{B.2.4}$$

$$f \equiv 2B^4e^{-2B}[(\epsilon + \bar{g}^{ab}p_a p_b)\frac{dB}{dt} - \frac{1}{2}\frac{\partial g_{ab}}{\partial t}p^a p^b] \ . \tag{B.2.5}$$

From (B.2.2) and Proposition B.1.2, point 2, we have

$$\frac{dB}{dt} = \frac{e_\infty}{4} + O(t^{-1}) \quad \Longrightarrow \quad B = \frac{e_\infty}{4}t + O(\ln t) \ ,$$

which together with (B.1.8) shows that f converges exponentially fast to zero as t goes to infinity, therefore there exists a constant C such that

$$B^4(\frac{dt}{ds})^2 \leq C \ ,$$

which for t large enough gives

$$t^4\left(\frac{dt}{ds}\right)^2 \leq \left(\frac{8}{e_\infty}\right)^4 C \ ,$$

and for $s_2 \geq s_1$ one obtains

$$s_2 \geq s_1 + [3\left(\frac{8}{e_\infty}\right)^2 C^{1/2}]^{-1}\left(t^3(s_2) - t^3(s_1)\right) \ ,$$

so that $s_2 \to \infty$ as $t(s_2) \to \infty$, which had to be established. □

Proposition B.2.2 *Let Γ be either a past inextendible timelike curve parametrized by proper time, or an affinely parametrized past inextendible null geodesic, in a Moncrief's space–time. Then*

1. *Γ reaches the boundary $t = 0$ in finite proper time or finite affine time, say s_o, and*

2. *we have*

$$\lim_{s \to s_0} \left(R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \right) |_{\Gamma}(s) = \infty \ .$$

Proof: From (B.2.2) and from Proposition B.1.1 it follows that there exists a constant B_o such that

$$e^B = e^{B_o} t^{(v_\infty^2 - 1)/4} \left(1 + O(t^{2(1-|v_\infty|)})\right) \ . \tag{B.2.6}$$

Consider first a timelike curve $\Gamma = \{x^\mu(s)\}$ parametrized by proper time s, with $t(s)$ decreasing as s increases,

$$e^{2B}[(\frac{dt}{ds})^2 - (\frac{d\theta}{ds})^2] - \lambda t n_{ab} \left(\frac{dx^a}{ds} + g^a \frac{d\theta}{ds}\right) \left(\frac{dx^b}{ds} + g^b \frac{d\theta}{ds}\right) = 1 \ . \tag{B.2.7}$$

Equation (B.2.7) implies

$$e^B \frac{dt}{ds} \leq -1$$

(recall that Γ is past–directed) which together with (B.2.6) gives, for $s_2 \geq s_1$, with $t(s_1)$ — small enough,

$$s_2 \leq s_1 + \frac{8e^{B_o}}{3 + v_\infty^2} \left(t^{(3+v_\infty^2)/4}(s_1) - t^{(3+v_\infty^2)/4}(s_2) \right) \ ,$$

so that any timelike curve reaches $t = 0$ in finite proper time. To prove the result for null geodesics, some more work is required. In what follows we shall write that

$$f \approx g$$

if

$$\lim_{t \to 0} \frac{f}{g} = 1 \ .$$

From (B.2.2) and from Proposition B.1.1 we have

$$\frac{dB}{dt}(t) \approx \frac{v_\infty^2 - 1}{4t}, \quad B(t) \approx \frac{v_\infty^2 - 1}{4} \ln t \,, \tag{B.2.8}$$

$$ds^2 \approx e^{2B_o} t^{(v_\infty^2-1)/2} \left(-dt^2 + d\theta^2\right) + \lambda t^{1-|v_\infty|} \{ \frac{1+\cos(n\theta)}{2} (dx^1 + g^1 d\theta)^2$$
$$+ \sin(n\theta)\,(dx^1 + g^1 d\theta)(dx^2 + g^2 d\theta) + \frac{1-\cos(n\theta)}{2}(dx^2 + g^2 d\theta)^2 \} \,, \tag{B.2.9}$$

$$\frac{\partial g_{ab}}{\partial t} \equiv \frac{\partial(\lambda t\, n_{ab})}{\partial t} \approx \frac{1-|v_\infty|}{t} g_{ab} \,, \tag{B.2.10}$$

with (B.2.9) holding for $|v_\infty| > 0$, and (B.2.8), (B.2.10) holding for $0 \leq |v_\infty| < 1$. Consider null geodesics such that $(p^1)^2 + (p^2)^2 \neq 0$; the case $p^1 = p^2 = 0$ is analyzed by a similar simpler argument. From (B.2.4)-(B.2.5) and (B.2.10) one obtains t small enough,

$$\frac{d}{dt}\left(B^4 (\frac{dt}{ds})^2\right) \approx -\frac{e^{-2B_o}}{2}\left(\frac{v_\infty^2 - 1}{4}\right)^4 (3 - 2|v_\infty| - v_\infty^2)\, t^{-(1+v_\infty^2)/2} \ln^4 t\, \bar{g}^{ab} p_a p_b \leq 0 \,,$$

so that $B^4(\frac{dt}{ds})^2$ increases as $t(s)$ goes to zero for $t(s_1) \leq t_1$, for some t_1 small enough, therefore for $s \geq s_1$

$$B^4 \left(\frac{dt}{ds}\right)^2 \geq C \,,$$

and an argument similar to the one for timelike curves shows that Γ must reach $t = 0$ in finite affine time.

To analyze the behaviour of the curvature near $t = 0$, with the help of a SHEEP calculation[2] one finds

$$R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \approx \frac{e^{-4B}}{4t^4} \left[(1 - v_\infty^2)^2 (3 + v_\infty^2)\right] \,,$$

and (B.2.6) gives

$$R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \geq C t^{-(v_\infty^2 + 3)} \,, \tag{B.2.11}$$

for some constant C, so that the curvature blows up uniformly along all curves as t approaches zero. □

From Propositions B.2.1, B.2.2, and C.2.4 one obtains

[2]The author is grateful to D. Singleton for performing this calculation.

Theorem B.2.1 *Let* $\Sigma = T^3$, *let* $X(\Sigma)$ *be the space of Cauchy data for Moncrief's metrics. The Theorem–To–Be–Proved holds in* $X(\Sigma)$, *with* $Y(\Sigma) = X(\Sigma)$; *more precisely, every maximal Hausdorff developement of a Cauchy data set for a Moncrief metric is globally hyperbolic, therefore unique, and inextendible, in vacuum or otherwise, in the class of Hausdorff Lorentzian manifolds with* $C^{1,1}_{loc}$ *metrics.*