# Chapter 2

# "Highly symmetric" space–times

In this chapter we shall describe the spaces of maximal globally hyperbolic (Hausdorff) spaces-times for which the groups $SO(3)\times U(1)$, $SO(3)$ (two-dimensional principal orbits) and $U(1) \times U(1) \times U(1)$ act by isometries on some compact, connected and orientable Cauchy surface. We start by proving the well known result, that symmetries of Cauchy data lead to symmetries of the space–time:

## 2.1 From symmetric Cauchy data to symmetric space–times.

In this Section we shall show that the existence of symmetries of Cauchy data implies the existence of symmetries of the space-time. Let us start with a "Killing vector approach" to this problem. The result that follows is essentially due to Moncrief [86] [Sections IV, V], we present the proof here for completeness (the proof below also seems to be somewhat simpler than the one in [86]; a similar proof can be found in [47]):

**Theorem 2.1.1** *Let $(M,\gamma)$ be a vacuum globally hyperbolic space-time, with a time function $t$, let $\Sigma_\tau = \{p \in M : t(p) = \tau\}$. Let $\mathring{X}^\mu$ be a vector field on $M$ defined in a neighbourhood of $\Sigma_0$ such that for $p \in \Sigma_o$ we have*

$$\left(\nabla_\alpha \mathring{X}_\beta + \nabla_\beta \mathring{X}_\alpha\right)(p) = 0\,, \tag{2.1.1}$$

$$\nabla_\sigma\left(\nabla_\alpha \overset{\circ}{X}_\beta + \nabla_\beta \overset{\circ}{X}_\alpha\right)(p) = 0\,, \tag{2.1.2}$$

*where $\nabla$ is the covariant derivative of the metric $\gamma$. There exists a vector field $X$ on $M$ satisfying*

$$\nabla_\alpha X_\beta + \nabla_\beta X_\alpha = 0\,, \tag{2.1.3}$$

$$p \in \Sigma_0 \qquad X_\alpha(p) = \overset{\circ}{X}_\alpha(p)\,, \qquad \nabla_\alpha X_\beta(p) = \nabla_\alpha \overset{\circ}{X}_\beta(p)\,. \tag{2.1.4}$$

**Proof.** Let $X^\alpha$ be the unique solution of

$$\Box X^\alpha = 0 \tag{2.1.5}$$

satisfying (2.1.4). (Because $(M,\gamma)$ is globally hyperbolic, a solution of (2.1.4)–(2.1.5) will exist on $M$, if *e.g.*, in local coordinates, we have $\partial_{\alpha_1}\dots\partial_{\alpha_i}\cdot\gamma_{\mu\nu} \in L^\infty_{\text{loc}}(\Sigma_t)$, $0 \le i \le k+2$, $k \in I\!N \cup \{0\}$, where $L^\infty_{\text{loc}}(\Sigma_t)$ denotes the space of functions defined almost everywhere on $\Sigma_t$ which are measurable and essentially bounded on every compact subset of $\Sigma_t$, and if $\partial_{i_1}\dots\partial_{i_\ell}\overset{\circ}{X}_\alpha|_{\Sigma_0} \in L^2_{\text{loc}}(\Sigma_0)$, $0 \le \ell \le k+1$, $\partial_{i_1}\dots\partial_{i_j}(\partial_t \overset{\circ}{X}_\alpha)|_{\Sigma_0} \in L^2_{\text{loc}}(\Sigma_0)$, $0 \le j \le k$; under these conditions we will have $(\partial_{\alpha_1}\dots\partial_{\alpha_i} X_\alpha)\,|_{\Sigma_t} \in L^2_{\text{loc}}(\Sigma_t)$ for all $t$, $0 \le i \le k+1$). Let us note that (as is well known) it follows from (2.1.3) that

$$\nabla_\alpha \nabla_\beta X_\gamma = R_{\lambda\alpha\beta\gamma} X^\lambda\,, \tag{2.1.6}$$

so that (2.1.5) is a necessary condition for (2.1.3) to hold in a vacuum space-time. We shall show that (2.1.1), (2.1.2), (2.1.4) and (2.1.5) and the fact that $(M,\gamma)$ in vacuum imply (2.1.3). Set

$$A_{\alpha\beta} = \nabla_\alpha X_\beta + \nabla_\beta X_\alpha\,.$$

From (2.1.5) and from

$$\nabla^\lambda R_{\alpha\beta\gamma\lambda} = 0$$

one obtains

$$\Box A_{\alpha\beta} + 2\,R_{\lambda\beta\gamma\alpha} A^{\gamma\lambda} = 0\,. \tag{2.1.7}$$

Under the conditions on the metric and on $\overset{\circ}{X}_\alpha$ outlined above with $k \ge 1$ (which will hold if *e.g.* $\gamma_{\mu\nu}$ and $\overset{\circ}{X}{}^\mu$ are smooth) it follows that every solution of (2.1.7) with zero initial data vanishes identically; and the vanishing of the initial data for $A_{\alpha\beta}$ follows from (2.1.1)–(2.1.2). □

**Corollary 2.1.1** *Under the hypotheses of Theorem 2.1.1, suppose that on $\Sigma_0$ there exists a smooth vector field $Y$ such that*

$$\mathcal{L}_Y\, g = \mathcal{L}_Y\, K = 0\,,$$

*where $\mathcal{L}_Y$ denotes a Lie derivative. There exists a smooth vector field $X$ on $M$ such that*

$$\mathcal{L}_X\, \gamma = 0\,,$$

$$p \in \Sigma_0 \qquad X(p) = i_{\Sigma_0 *}\, Y\,,$$

*where $i_{\Sigma_0}$ is the embedding of $\Sigma_0$ in $M$.*

**Proof.** Let us rewrite eqs. (2.1.1)–(2.1.2) in 3+1 notation; let $n^\alpha$ be the unit normal to the slicing $\Sigma_\tau$ ($n^\alpha\, n_\alpha = -1$), define

$$\begin{aligned}
&\beta_\sigma = n^\alpha\, \nabla_\alpha\, n_\sigma\,, \quad g_{\mu\nu} = \gamma_{\mu\nu} + n_\mu\, n_\nu\,, \quad K_{\mu\nu} = g_\mu{}^\alpha\, g_\nu{}^\beta\, \nabla_\alpha\, n_\beta\,,\\
&x = -X^\alpha\, n_\alpha\,, \quad Y^\alpha = g^\alpha{}_\beta\, X^\beta \quad (\Longrightarrow\ X^\alpha = x n^\alpha + Y^\alpha)\,,\\
&D_n x = n^\mu\, \nabla_\mu x\,, \quad D_n\, Y^\alpha = g^\alpha{}_\beta\, n^\mu\, \nabla_\mu\, Y^\beta\,.
\end{aligned}$$

Using this notation, (2.1.1) can be rewritten in the form

$$\mathcal{L}_Y\, g_{\alpha\beta} \;=\; -2x\, K_{\alpha\beta}\,, \tag{2.1.8}$$

$$D_n x \;=\; -\beta_\sigma\, Y^\sigma\,, \tag{2.1.9}$$

$$D_n Y^\alpha \;=\; K^\alpha{}_\beta\, Y^\beta + D^\alpha x - x\beta^\alpha\,, \tag{2.1.10}$$

where $D$ is the covariant derivative operator of the Riemannian metric $g_{\alpha\beta}$ induced by $\gamma_{\alpha\beta}$ on $\Sigma_\tau$. When (2.1.8)–(2.1.10) hold, one finds that 1) the equations obtained by projecting all indices in (2.1.2) on $\Sigma_\tau$ are identically satisfied, 2) the equations obtained from (2.1.2) by projecting one of the indices along $n$ and the remaining on $\Sigma_\tau$ are equivalent to

$$\mathcal{L}_Y\, K_{\alpha\beta} + D_\alpha\, D_\beta x = \left({}^3R_{\alpha\beta} + K K_{\alpha\beta} - K_{\alpha\mu}\, K_\beta{}^\mu\right) x\,, \tag{2.1.11}$$

where $K = g^{\alpha\beta} K_{\alpha\beta}$, and ${}^3R_{\alpha\beta}$ is the Ricci curvature of the metric $g_{\alpha\beta}$, 3) and finally the equations obtained from (2.1.2) by projecting more than one index along $n^\mu$ involve second time derivatives of $x, Y^\alpha$ in a way consistent with eq. (2.1.5) if (2.1.8)–(2.1.11) hold. If $(g, K)$ are invariant under the flow of a vector field $Y$, then we can set $X^\mu(0,\cdot) = Y^\mu(\cdot)$, $(x(0,\cdot) = -X^\mu n_\mu(0,\cdot) = 0)$ and use (2.1.9)–(2.1.10) to determine $n^\mu \nabla_\mu X^\alpha(0,\cdot)$, obtaining thus Cauchy data for equation (2.1.5), which satisfy those of the equations (2.1.1)–(2.1.2) which do not contain second time derivatives of $X^\alpha$. Solving (2.1.5) we will obtain a Killing vector field on any globally hyperbolic development of $({}^3\Sigma, g, K)$.□

When considering symmetric data sets, it is natural to ask the following:

1. are discrete symmetries of $({}^3\Sigma, g, K)$ preserved under evolution?

2. suppose that we have a group $G$ acting on ${}^3\Sigma$, which leaves both $g$ and $K$ invariant; can we define an action of $G$ on some development of $({}^3\Sigma, g, K)$?

The argument that follows answers both of these questions, at least when ${}^3\Sigma$ is compact:

**Theorem 2.1.2** *Let $({}^3\Sigma, g, K)$ be a smooth Cauchy data set, with ${}^3\Sigma$ — compact, suppose that a group $G$ acts smoothly on ${}^3\Sigma$*

$$G \times {}^3\Sigma \ni (g,p) \longrightarrow \phi_g(p) \in {}^3\Sigma\,,$$

*and we have*

$$({\phi_g}^* g_{ij},\ {\phi_g}^* K_{ij}) = (g_{ij}, K_{ij})\ .$$

*For any development $(M,\gamma)$ of $({}^3\Sigma, g, K)$ there exists a (globally hyperbolic) neighbourhood $\mathcal{O} \subset \mathcal{M}$ of ${}^3\Sigma$, and an action $\Psi$ of the group $G$ on $\mathcal{O}$,*

$$G \times \mathcal{O} \ni (g,p) \longrightarrow \Psi_g(p) \in \mathcal{O}\,,$$

*such that*

$${\Psi_g}^*\gamma = \gamma\,.$$

*Moreover, there exists a diffeomorphism $\psi : \mathcal{O} \leftrightarrow (-\tau, \tau) \times^3 \Sigma$ such that we have*

$$\Psi_g = \Psi \circ \Phi_g \circ \Psi^{-1} ,$$

*where $\Phi_g$ is the following action of $G$ on $(-\tau, \tau) \times^3 \Sigma$:*

$$(-\tau, \tau) \times^3 \Sigma \ni (t, p) \leftrightarrow \Phi_g(t, p) = (t, \phi_g(p)) , \tag{2.1.12}$$

*and the hypersurfaces $\{t\} \times^3 \Sigma$, $t \in (-\tau, \tau)$, are space-like.*

**Remark.** Theorem 2.1.2 will still hold if $(g_{ij}\, K_{ij})(0, \cdot) \in H_k(\Sigma_0) \oplus H_{k-1}(\Sigma_0)$, $k \geq 4$. One would expect the result to be true under the condition $k > 5/2$, or, say $k \geq 3$, the proof presented here, however, fails if $k = 3$. This is due to the fact that the differentiability of the map $\Psi$ constructed below is not better than this of $\gamma$, which in turn leads to a differentiability class of $\tilde{\gamma} \equiv (\Psi^{-1})^* \gamma$ worse by one as compared to $\gamma$. If $k = 3$ the differentiability of $\tilde{\gamma}$ is not high enough to guarantee uniqueness of solutions, and the argument breaks down.

**Proof.** Let $\mathcal{D}(^3\Sigma)$ be the domain of dependence of $^3\Sigma$ in $(\mathcal{M}, \gamma)$; replacing $\mathcal{M}$ by $\mathcal{D}(^3\Sigma)$ if necessary we may assume that $\mathcal{M} = \mathcal{D}(^3\Sigma)$, and thus $\mathcal{M}$ is globally hyperbolic. Let $t$ be the unique solution of the problem

$$\Box_\gamma t \;=\; 0 , \tag{2.1.13}$$

$$t|_{^3\Sigma} \;=\; 0 , \qquad n^\mu\, \partial_\mu t|_{^3\Sigma} = 1 , \tag{2.1.14}$$

where $\Box_\gamma$ is the scalar wave operator of the metric $\gamma$, $\Box_\gamma = \nabla^\alpha \nabla_\alpha$. Let $\mathcal{O}$ be any neighbourhood of $^3\Sigma$ on which $\gamma_{\mu\nu}\, \nabla^\mu t \nabla^\nu t < 0$, by compactness of $^3\Sigma$ there exists $\sigma > 0$ such that $M_\sigma = \{p \in M : t(p) < \sigma\} \subset \mathcal{O}$. Let $\Sigma_t$ denote the level sets of $t$, we can use the integral curves of $\nabla t$ to identify $M_\sigma$ with $(-\sigma, \sigma) \times \Sigma_0$. Let $\mathring{g}$ be any smooth Riemannian $G$–invariant metric on $^3\Sigma$, on $M_\sigma$ we can define the Lorentzian metric

$$\mathring{\gamma} = \mathring{\gamma}_{\mu\nu}\, dx^\mu\, dx^\nu = -dt^2 + \mathring{g} \ .$$

Note that the action (2.1.12) preserves $\mathring{\gamma}$:

$$\Phi_g{}^* \, \mathring{\gamma} = \mathring{\gamma} \ . \tag{2.1.15}$$

Consider the following initial value problem for a map $\Psi : ((-\tau,\tau)\times^3\Sigma,\gamma) \to ((-\sigma,\sigma)\times {}^3\Sigma,\mathring{\gamma})$,

$$\Box(\gamma,\mathring{\gamma})\,\Psi = 0\,, \tag{2.1.16}$$

$$\Psi(t=0,\cdot) = id_{^3\Sigma}(\cdot)\,, \quad n^\mu \frac{\partial \Psi^\alpha}{\partial x^\mu}(t=0,\cdot) = \delta^\alpha_0\,, \tag{2.1.17}$$

where $\Box(\gamma,\mathring{\gamma})$ is the Lorentzian harmonic map operator; in local coordinates

$$\Box(\gamma,\mathring{\gamma})\,\Psi^\gamma = \gamma^{\alpha\beta}\Big(\partial_\alpha\,\partial_\beta\,\Psi^\gamma - \Gamma^\lambda_{\alpha\beta}(\gamma)\,\frac{\partial\Psi^\gamma}{\partial x^\lambda} + \Gamma^\lambda_{\mu\nu}(\mathring{\gamma})\,\frac{\partial\Psi^\mu}{\partial x^\alpha}\,\frac{\partial\Psi^\nu}{\partial x^\beta}\Big)\,, \tag{2.1.18}$$

where $\Gamma^\alpha_{\beta\gamma}(h)$ denotes the Christoffel symbols of a metric $h$. There exists $\tau > 0$ such that there exists a unique smooth[1] solution of the problem (2.1.16)–(2.1.17) defined on $(-\tau,\tau)\times^3\Sigma$. Note that from (2.1.13) and (2.1.14) we have

$$\gamma^{\alpha\beta}\,\Gamma^t_{\alpha\beta}(\gamma) = 0\,, \qquad \Gamma^t_{\alpha\beta}(\mathring{\gamma}) = \Gamma^\alpha_{t\beta}(\mathring{\gamma}) = 0\,,$$

and uniqueness of solutions of (2.1.16) implies $\psi^0 = t$. This shows that, decreasing $\tau$ if necessary, there exist diffeomorphisms $\psi_t :^3\Sigma \leftrightarrow {}^3\Sigma$ such that

$$(-\tau,\tau)\times^3\Sigma \ni (t,p) \leftrightarrow \Psi(t,p) = (t,\psi_t(p))\,,$$

and from (2.1.17) we have

$$\psi_0 = id_{^3\Sigma}\,, \qquad n^\mu\,\frac{\partial\psi_t}{\partial x^\mu}\bigg|_{t=0} = 0\,. \tag{2.1.19}$$

On $M_\tau$ we can define an action of the group $G$ as follows:

$$g \in G\,, \qquad \Psi_g = \Psi^{-1}\circ\Phi_g\circ\Psi\,.$$

The claim that the maps $\Psi_g$ are isometries of $\gamma$ is equivalent to the statement that the maps $\Phi_g$ are isometries of the metric $\tilde{\gamma} \equiv (\Psi^{-1})^*\gamma$. The covariance of the equation (2.1.18) under changes of coordinates in the source space implies that the identity map

$$id_{M_\tau} : ((-\tau,\tau)\times^3\Sigma,\tilde{\gamma}) \leftrightarrow ((-\tau,\tau)\times^3\Sigma,\mathring{\gamma})$$

[1] It is not too difficult to show, using *e.g.* the methods of [20], that if $\partial_{\alpha_1}\ldots\partial_{\alpha_j}\gamma_{\mu\nu}(t,\cdot) \in H^{\rm loc}_{k-j}({}^3\Sigma_t)$, $0 \le j \le k$, $k \ge 3$, then there will exist a solution of (2.1.16)–(2.1.17) satisfying $\partial_{\alpha_1}\ldots\partial_{\alpha_j}\Psi^\mu(t,\cdot) \in H^{\rm loc}_{k-j}({}^3\Sigma_t)$.

also satisfies (2.1.18) which in local coordinates reads

$$id^{\alpha}_{\mathcal{M}_\tau}(x^\mu) = x^\alpha \quad \Longrightarrow$$

$$A^\sigma \equiv \Box(\tilde\gamma, \mathring{\gamma})\, id^{\sigma}_{M_\tau} = \tilde\gamma^{\alpha\beta}\left(\Gamma^{\sigma}_{\alpha\beta}(\mathring{\gamma}) - \Gamma^{\sigma}_{\alpha\beta}(\tilde\gamma)\right) = 0\,. \tag{2.1.20}$$

As is well known (*cf. e.g.* [20]) Einstein equations with conditions (2.1.18) are a well posed hyperbolic system for the metric $\tilde\gamma$, the solutions being determined uniquely by the initial data $(\tilde g, \tilde K)$, with $(\tilde g, \tilde K)$ — obtained by appropriately transforming $(g, K)$. In our case, (2.1.19) implies $(g, K) = (\tilde g, \tilde K)$. Consider now the metric $\tilde\gamma_g = {\Phi_g}^*\tilde\gamma$. $\tilde\gamma_g$ satisfies vacuum Einstein equations, and by coordinate–invariance of (2.1.20) under a simultaneous change of coordinates for both the metric $\mathring{\gamma}$ and $\tilde\gamma$ it follows that $A^\gamma$ defined by (2.1.20) satisfies $A^\gamma = 0 = \tilde\gamma_g^{\alpha\beta}\left(\Gamma^{\gamma}_{\alpha\beta}(\mathring{\gamma}) - \Gamma^{\gamma}_{\alpha\beta}(\tilde\gamma_g)\right)$ (and we have used ${\Phi_g}^*\mathring{\gamma} = \mathring{\gamma}$). The initial data for $\tilde\gamma_g$ are given by

$$(\tilde g_g, \tilde K_g) = ({\phi_g}^*\tilde g, {\phi_g}^*\tilde K) = ({\phi_g}^* g, {\phi_g}^* K) = (g, K) = (\tilde g, \tilde K)\,,$$

and uniqueness implies

$${\Phi_g}^*\tilde\gamma = \tilde\gamma \Longrightarrow {\Psi_g}^*\gamma = \gamma\,.$$

□

It must be stressed that while the Killing vectors argument, Theorem 2.1.1, proves the existence of Killing vector fields defined on the whole of $(M, \gamma)$, provided $(M, \gamma)$ is globally hyperbolic, Theorem 2.1.2 shows only existence of a neighbourhood of ${}^3\Sigma$ on which the group $G$ acts. It is easily seen that the hypothesis of compactness of $\Sigma$ is necessary: consider the space–time $M = \{(t, x) \in I\!R^2 : |t| < f(x)\}$ with the metric $\gamma_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dx^2$, where $0 < f$ is a differentiable function such that $|df/dx| < 1$, and $\lim_{|x|\to\infty} f(x) = 0$. $(M, \gamma)$ is globally hyperbolic, the translations in $x$ are isometries of the Cauchy data at $t = 0$, but the action does not extend to $M$. Similarly it can be seen that for spatially compact space–times the action of the symmetry group given by Theorem 2.1.2 needs not to extend beyond a neighbourhood of the Cauchy surface: consider the space–time $M = \{(t, x) \in I\!R \times S^1 : |t| < 1 + (\sin x)/2\}$, with the metric

$-dt^2 + dx^2$, and we have identified $S^1$ with $[0, 2\pi]_{|\mathrm{mod}\, 2\pi}$. The translations in $x$ do not extend beyond the strip $\{|t| < 1/2\}$.

It is a remarkable fact, that isometries of Cauchy data always extend to the maximal globally hyperbolic developments, regardless of whether the Cauchy surface is compact or not. Before proving this, let us restate the Choquet–Bruhat — Geroch theorem 1.1.2 in a form more suitable for our further applications:

**Theorem 2.1.3** *Let $(\Sigma, g, K)$ be a Cauchy data set, where $\Sigma$ is a Hausdorff manifold and $g, K \in C^\infty(\Sigma)$. There exists a $C^\infty$, vacuum, Hausdorff, globally hyperbolic development $(M, \gamma, i)$ of $(\Sigma, g, K)$ such that for every smooth, Hausdorff, globally hyperbolic development $(\tilde{M}, \tilde{\gamma}, \tilde{\imath})$ of $(\Sigma, g, K)$ there exists an isometric embedding $\Psi : \tilde{M} \to M$ satisfying*

$$\Psi \circ \tilde{\imath} = i \ .$$

Any developement $(M, \gamma)$ satisfying the above will be called a *maximal globally hyperbolic* developement. It is clear from the above that maximal developements are unique, up to isometry, and inextendible in the class of smooth, Hausdorff, globally hyperbolic space–times. We shall also need the following:

**Lemma 2.1.1** *Let $(M, \gamma)$ be a smooth, Hausdorff, connected Lorentzian manifold, let $\Psi : M \to M$ be a smooth map such that*

$$\Psi^* \gamma = \gamma \ , \qquad \Psi|_S = id \quad (S \neq \emptyset) \ ,$$

*where $S$ is either an open set, or a non–everywhere–null submanifold of codimension 1; in this last case we moreover assume that $\Psi$ is orientation–preserving. Then*

$$\Psi = id \ .$$

**Remark:** Note that in the case when $S$ is a submanifold one does not need to assume any kind of completeness of $S$.

**Proof:** Suppose first that $S$ is an open set, let $\tilde{S}$ be the largest open set such that $\Psi|_{\tilde{S}} =$ id. Suppose that $\tilde{S}$ is not closed, thus there exists $p \in \partial\tilde{S}$, let $\mathcal{O}$ be any neighbourhood of $p$ with a local coordinate system such that $x^\mu(p) = 0$, smoothness of $\Psi$ implies, in local coordinates,

$$\Psi^\mu(0) = 0 \ , \quad \frac{\partial \Psi^\mu}{\partial x^\nu}(0) = \delta^\mu_\nu \ , \quad \partial_{\sigma_1} \cdots \partial_{\sigma_1} \Psi^\mu(0) = 0 \ . \tag{2.1.21}$$

From $\Psi^*\gamma = \gamma$ one has

$$\gamma_{\alpha\beta}(x) = \gamma_{\mu\nu}\left(\Psi(x)\right) \frac{\partial \Psi^\mu}{\partial x^\alpha} \frac{\partial \Psi^\nu}{\partial x^\beta} \ , \tag{2.1.22}$$

$$\frac{\partial^2 \Psi^\mu}{\partial x^\alpha \partial x^\beta} = \Gamma^\sigma_{\alpha\beta}(x) \frac{\partial \Psi^\mu}{\partial x^\sigma} - \Gamma^\mu_{\nu\rho}\left(\Psi(x)\right) \frac{\partial \Psi^\nu}{\partial x^\alpha} \frac{\partial \Psi^\rho}{\partial x^\beta} \ , \tag{2.1.23}$$

where $\Gamma$ denotes the Christoffel symbols of the metric $\gamma$. Setting $A^\alpha_\beta \equiv \frac{\partial \Psi^\alpha}{\partial x^\beta}$, from the equation (2.1.23) one obtains the following system of ODE's along rays emanating from the origin:

$$\frac{d\Psi^\mu}{dr} = A^\mu_\beta x^\beta / r \ , \qquad r = \left(\sum (x^\alpha)^2\right)^{1/2} \ ,$$

$$\frac{dA^\mu_\beta}{dr} = \left(\Gamma^\sigma_{\alpha\beta}(x) A^\mu_\sigma - \Gamma^\mu_{\nu\rho}\left(\Psi(x)\right) A^\nu_\alpha A^\rho_\beta\right) x^\alpha / r \ ,$$

and the initial conditions (2.1.21) together with uniqueness of solutions of systems of ODE's imply $\Psi^\mu = x^\mu$ in $\mathcal{O}$, which leads to a contradiction, and shows that $\partial S = \emptyset$, thus $S = M$.

Suppose now that $S$ is a hypersurface, let $p \in S$ be such that $S$ is not null in a neighbourhood of $p$, let $\mathcal{O}$ be a neighbourhood of $p$ with a coordinate system $(x, y) = (x, y^1, \ldots, y^n)$ such that $S \cap \mathcal{O} = \{x = 0\}$ and $\gamma_{xi}(0, y) = 0$; note that $\gamma_{xx}(0, y) \neq 0$. From (2.1.23) it follows that along the curves $y =$ const we have

$$\frac{d\Psi^\mu}{dx} = A^\mu_x \ ,$$

$$\frac{dA^\mu_\beta}{dx} = \Gamma^\sigma_{x\beta}(x, y) A^\mu_\sigma - \Gamma^\mu_{\nu\rho}\left(\Psi(x, y)\right) A^\nu_x A^\rho_\beta \ ,$$

and $\Psi|_S =$id implies

$$\Psi^i(0, y) = y^i, \quad \Psi^x(0, y) = 0, \quad \frac{\partial \Psi^i}{\partial y^j}(0, y) = \delta^i_j, \quad \frac{\partial \Psi^x}{\partial y^j}(0, y) = 0 \ ,$$

which together with (2.1.22) gives

$$\frac{\partial\Psi^\alpha}{\partial x^\beta}(0,y) = \delta^\alpha_\beta \ ,$$

and $\Psi|_{\mathcal{U}} = \text{id}$ for some neighbourhood $\mathcal{U}$ of $p$ follows. $\mathcal{U} = M$ follows by part 1 of this Lemma. □

**Corollary 2.1.2** *Let $(\Sigma, g, K)$ be smooth Cauchy data on a Hausdorff manifold $\Sigma$, let $\phi : \Sigma \to \Sigma$ be a smooth diffeomorphism, set $\tilde{g} = \phi^* g$, $\tilde{K} = \phi^* K$. Let $(M, \gamma, i)$, respectively $(\tilde{M}, \tilde{\gamma}, \tilde{i})$ be the maximal globally hyperbolic (vacuum, Hausdorff, smooth) development of $(\Sigma, g, K)$, respectively of $(\Sigma, \tilde{g}, \tilde{K})$. There exists a diffeomorphism $\Phi : \tilde{M} \to M$ such that $\Phi^*\gamma = \tilde{\gamma}$, and*

$$\Phi \circ \tilde{i} = i \circ \phi \ .$$

**Proof:** By definition we have $(\phi^{-1})^*\tilde{g} = g$, $(\phi^{-1})^*\tilde{K} = K$, thus $(\tilde{M}, \tilde{\gamma}, \tilde{i} \circ \phi^{-1})$ is a development of $(\Sigma, g, K)$. By maximality of $(M, \gamma, i)$ it follows from Theorem 2.1.3 that there exists an isometric embedding $\Phi : \tilde{M} \to M$ such that $\Phi \circ \tilde{i} \circ \phi^{-1} = i$. A similar argument using maximality of $(\tilde{M}, \tilde{\gamma}, \tilde{i})$ shows that there exists an isometric embedding $\Psi : M \to \tilde{M}$ such that $\Psi \circ i \circ \phi = \tilde{i}$. One thus has $\Psi \circ \Phi \circ \tilde{i} = \tilde{i}$, $(\Psi \circ \Phi)^*\tilde{\gamma} = \tilde{\gamma}$, so that $\Psi \circ \Phi$ is an isometry which is the identity on $\tilde{i}(\Sigma)$, therefore $\Psi \circ \Phi = \text{id}$ by Lemma 2.1.1. This shows that $\Phi$ is invertible, and the result follows. □

The main result of this Section is the following:

**Theorem 2.1.4** *Let $(\Sigma, g, K)$ be smooth Cauchy data on a Hausdorff manifold $\Sigma$, let $(M, \gamma, i)$ be the maximal globally hyperbolic (vacuum, Hausdorff, smooth) development of $(\Sigma, g, K)$, suppose that a group $G$ acts on $(\Sigma, g, K)$ by smooth isometries:*

$$G \times \Sigma \ni (g, p) \to \phi_g(p) \in \Sigma$$

*$(\phi^*_g g = g,\ \phi^*_g K = K)$. There exists an action of $G$ on $M$,*

$$G \times M \ni (g, p) \to \Phi_g(p) \in M \ ,$$

*such that*

$$\forall g \in G \qquad \Phi_g^*\gamma = \gamma \; , \quad \Phi_g \circ i = i \circ \phi_g \; .$$

**Proof:** Let $(M_g, \gamma_g, i_g)$ be a maximal globally hyperbolic development of $(\Sigma, \phi_g^* g, \phi_g^* K)$. By Corollary 2.1.2 there exists a diffeomorphism $\Psi_g : M_g \to M$ such that $\Psi_g \circ i_g = i \circ \phi_g$, $\Psi_g^*\gamma = \gamma_g$. Since $\phi_g^* g = g$, $\phi_g^* K = K$, Theorem 2.1.3 moreover implies the existence of a diffeomorphism $\Xi_g : M_g \to M$ such that $\Xi_g \circ i_g = i$, $\Xi_g^*\gamma = \gamma_g$. Consider the diffeomorphism $\Phi_g = \Psi_g \circ \Xi_g^{-1} : M \to M$. We have

$$\Phi_g^*\gamma = (\Xi_g^{-1})^* \Psi_g^* \gamma = (\Xi_g^{-1})^* \gamma_g = \gamma \; ,$$

thus the $\Phi_g$'s are isometries, moreover

$$\Phi_g \circ i = \Psi_g \circ \Xi_g^{-1} \circ i = \Psi_g \circ i_g = i \circ \phi_g \; .$$

Lemma 2.1.1 easily implies $\Phi_{gh} \circ \Phi_h^{-1} \circ \Phi_g^{-1} = \mathrm{id}$, thus $\Phi_{gh} = \Phi_g \circ \Phi_h$. Finally continuity of $\Phi_g$ in $g$ follows from the continuous dependence upon Cauchy data (on compact subsets) of the solutions of the initial value problem for Einstein equations. □

## 2.2 $SO(3) \times U(1)$ symmetric space-times, ${}^3\Sigma = L(p,1)$.

Consider the set of space-times with Cauchy-data on compact, connected, orientable manifolds ${}^3\Sigma$, invariant under an effective action of the group $G = SO(3) \times U(1)$ or $G = \Big(SU(2) \times U(1)\Big)/D$, $D = \{(-1,-1),(1,1)\}$, with three-dimensional principal orbits. In this Section we shall outline the proof of the claim, that this set consists of the Taub-NUT metrics. As discussed by Fischer [45], ${}^3\Sigma$ is necessarily $S^3$ or a lens space $L(p,1)$, and the action of $G$ is unique up to equivalence. Identifying $U(1)$ with the subgroup of $SU(2)$ consisting of matrices of the form $\mathrm{diag}(e^{i\alpha}, e^{-i\alpha})$, and identifying $SU(2)$ with $S^3$ in the standard way, it follows that up to homomorphism of $G$ and diffeomorphism of ${}^3\Sigma$ the action of $G$ on ${}^3\Sigma$ is given by

$$G \times {}^3\Sigma \ni (g = (g_1, g_2), p) \to gp = g_1 p g_2^{-1} \in {}^3\Sigma \, , \tag{2.2.1}$$

$$g_1 \in SO(3) \quad \text{or} \quad SU(2)\,, \quad g_2 \in U(1)\,.$$

When $G = [SU(2)\times U(1)]/D$, and/or when ${}^3\Sigma = L(p,1)$, $p \neq 1$, appropriate equivalence relations in (2.2.1) should be taken into account[2]. For any $p \in {}^3\Sigma$ there exists a neighbourhood $\mathcal{O}_p$ of $p$ diffeomorphic with a neighbourhood $U_e$ of the identity of $SU(2) \approx S^3$, using this diffeomorphism any $G$-invariant metric on ${}^3\Sigma$ can be pulled–back to define a $G$-invariant metric on $U_e$. Let $X_i(e)$ be any basis of $T_e\,SU(2)$, let

$$X_i(g) = (R_g)_*\,X_i(e)\,,$$

where $R_g$ is the right action of $SU(2)$ on itself. Since the right and left actions commute, the vector fields $X_i$ are left-invariant. Also, by definition of the adjoint representation $ad$,

$$(R_{h^{-1}})_*\,X_i = ad_h\,X_i\,.$$

As is well-known, the $ad$ representation of $SU(2)$ acts on $T_e\,SU(2) \approx I\!\!R^3$ by rotations, and we can choose $X_3(e)$ so that $U(1) = \mathrm{diag}(e^{i\alpha}, e^{-i\alpha})$ acts as rotations around the $X_3(e)$-axis. Let $\langle\cdot,\cdot\rangle$ be a metric on $U_e \subset SU(2)$, invariant under the action (2.2.1). The $SU(2)$ invariance implies that the functions

$$g_{ij}(p) = \langle X_i, X_j\rangle_{|p}$$

are $p$-independent. The $U(1)$ invariance implies that $g_{ij}$ is a 2-covariant tensor on $I\!\!R^3$ invariant under rotations around the "$z$-axis", which by straightforward considerations leads to

$$g_{ij} \equiv \mathrm{diag}(e^a, e^a, e^b)\,,$$

for some $a, b \in I\!\!R$. It follows that any $G$–invariant metric on $S^3$ is of the form

$$g = e^a[(\omega^1)^2 + (\omega^2)^2] + e^b(\omega^2)^2\,, \tag{2.2.2}$$

where the $\omega^i$'s are left invariant forms on $S^3$, dual to the vectors $X_i$ as defined above. The argument presented in Ref. [12], Chapter II, Section 3, shows that the vacuum

[2]Equation (2.2.1) defines an action of $SU(2)\times U(1)$ on ${}^3\Sigma$, this action is however not effective.

dynamics does not lead to a rotation of the eigen-axes of the $U(1)$ action, thus in a vacuum $SO(3) \times U(1)$ or $[SU(2) \times U(1)]/D$ symmetric space-time an appropriate choice of time $\tau$ and of the $\omega^i$'s leads to

$$ds^2 = -d\tau^2 + e^{a(\tau)}[(\omega^1)^2 + (\omega^2)^2] + e^{b(\tau)}(\omega^3)^2 \,, \tag{2.2.3}$$

for some functions $a(\tau)$, $b(\tau)$. In Section 8.2 of Ref. [111] all solutions of Einstein equations with a metric of the form (2.2.3) have been found—these are the Taub-NUT metrics.

## 2.3 $SO(3)$ symmetry, 2-dimensional principal orbits.

In this Appendix we shall describe the family of vacuum, maximal globally hyperbolic space-times ${}^4M$ which admit compact, connected, orientable Cauchy surfaces ${}^3\Sigma$ with Cauchy data invariant under an $SO(3)$ action with two dimensional principal orbits[3]. The results presented below are a global version of the generalized Birkhoff theorem (*cf.* Appendix B to [66]).

As has been shown in Ref. [45], one has ${}^3\Sigma \approx S^1 \times S^2$ or $S^3$ or $I\!P^3$ or $I\!P^3 \# \, I\!P^3$ (connected sum of two projective spaces), and the action of $SO(3)$ is in each case unique up to homomorphism of $SO(3)$ and diffeomorphism of ${}^3\Sigma$. $S^1 \times S^2$ as well as $S^3$ or $I\!P^3$ or $I\!P^3 \# \, I\!P^3$ are of the form

$${}^3\Sigma = ([0,1] \times S^2)/\sim \,, \tag{2.3.1}$$

where

- in the $S^1 \times S^2$ case the relation " $\sim$ " identifies $\{0\} \times S^2$ with $\{1\} \times S^2$ via the identity map from $S^2$ to $S^2$,
- in the $I\!P^3 \# \, I\!P^3$ case " $\sim$ " identifies $\{0\} \times S^2$ with itself via the antipodal map from $S^2$ to $S^2$, similarly for $\{1\} \times S^2$,

[3]The author is grateful to dr G. Cieciura for useful discussions about the results presented here.

- in the $S^3$ case " $\sim$ " shrinks $\{0\} \times S^2$ to a point $p_0$ and $\{1\} \times S^2$ to a point $p_1$,

- in the $I\!P^3$ case " $\sim$ " shrinks $\{0\} \times S^2$ to a point $p_0$ and identifies $\{1\} \times S^2$ with itself via the antipodal map.

From [45] it follows that in all cases there exist coordinates $(\psi, (\theta, \phi))$ on $[0,1] \times S^2$ such that the action of $SO(3)$ consists of $\psi$-independent rotations of $S^2$, which implies that any $SO(3)$ invariant metric on ${}^3\Sigma$ in these coordinates takes the form

$$ds^2 = f^2(\psi)\, d\psi^2 + A^2(\psi)(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{2.3.2}$$

Let $({}^3\Sigma, g_{ij}, K_{ij})$ be any $SO(3)$ invariant Cauchy data, let $({}^4M, \gamma)$ be the maximal globally hyperbolic development thereof, let $t$ be a time function in a neighborhood $\mathcal{O}$ of ${}^3\Sigma$ such that for $\tau \in (t_1, t_2)$ the group $SO(3)$ acts on the level sets $\mathcal{I}_\tau \equiv \{p : t(p) = \tau\}$ of $t$ by isometries (*cf.* Section 2.1). By Lie dragging along the normals to $\mathcal{I}_\tau$ the above coordinates $(\psi, (\theta, \phi))$ can be extended to $\mathcal{O}$, and one finds that in this coordinate system the metric takes the form

$$ds^2 = -F^{-2}(t,\psi)\, dt^2 + X^2(t,\psi)\, d\psi^2 + Y^2(t,\psi)(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{2.3.3}$$

for some (strictly) positive functions $F, X$, and a non–negative function $Y$; $Y$ strictly positive for $\psi \in (0,1)$, and

- $Y$ strictly positive if ${}^3\Sigma \approx S^1 \times S^2$ or ${}^3\Sigma \approx I\!P^3 \# I\!P^3$,

- if ${}^3\Sigma \approx S^3$ the area function $Y$ vanishes at $\psi = 0$ and $\psi = 1$ only,

- if ${}^3\Sigma \approx I\!P^3$ the function $Y$ vanishes at $\psi = 0$ only.

In order to analyze the constraint equations it is useful to introduce the following "null" derivatives:

$$Y_\pm \equiv \frac{1}{X}\frac{\partial Y}{\partial \psi} \pm F\frac{\partial Y}{\partial t}\,.$$

We have the following:

**Lemma 2.3.1** *Let Cauchy data for a metric of the form (2.3.3) satisfy the vacuum constraint equations, suppose that there exists $\psi_+$ such that*

$$Y_+(0, \psi_+) = 0 \ .$$

*Then*

$$\psi < \psi_+ \Rightarrow Y_+(0, \psi) < 0 \ ,$$

$$\psi > \psi_+ \Rightarrow Y_+(0, \psi) > 0 \ . \qquad (2.3.4)$$

*The same statement holds with the subscript "+" replaced by the subscript "−".*

**Proof:** From equations (A.2)-(A.3) on p. 370 in [66] we have

$$\frac{\partial Y_\pm}{\partial \psi} = h_\pm Y_\pm + \frac{X}{2Y} \ , \qquad (2.3.5)$$

$$h_\pm \equiv \pm F \frac{\partial X}{\partial t} - \frac{X Y_\mp}{2Y} \ ,$$

thus

$$Y_+(\psi) = \int_{\psi_+}^{\psi} \frac{X(s)}{2Y(s)} e^{\int_s^\psi h_+(u) du} ds \begin{cases} > 0, & \psi > \psi_+ \ , \\ < 0, & \psi < \psi_+ \ . \end{cases} \qquad (2.3.6)$$

□

Lemma 2.3.1 implies, that in the vacuum the topologies $S^3$ and $I\!P^3$ are excluded:

**Proposition 2.3.1** *Let $({}^3\Sigma, g_{ij}, K_{ij})$ be $SO(3)$ invariant Cauchy data, ${}^3\Sigma$ compact, connected, orientable. Then ${}^3\Sigma \approx S^1 \times S^2$ or $I\!P^3 \# I\!P^3$; moreover, $\nabla Y$ is timelike, where $Y$ is the area function* (cf. *(2.3.3)).*

**Proof:** Suppose first, that ${}^3\Sigma \approx S^3$. By construction of the coordinate system (2.3.3) we must have

$$Y(t=0, \psi=0) = 0 \ , \quad \frac{\partial Y}{\partial \psi}(t=0, \psi=0) > 0 \ , \quad \frac{\partial Y}{\partial t}(t=0, \psi=0) = 0 \ ,$$

$$Y(t=0, \psi=1) = 0 \ , \quad \frac{\partial Y}{\partial \psi}(t=0, \psi=1) < 0 \ , \quad \frac{\partial Y}{\partial t}(t=0, \psi=1) = 0 \qquad (2.3.7)$$

(*cf. e.g.* Appendix C to [32] for a detailed description of functions and tensors invariant under rotations; in that reference rotations around a single axis are considered, the results generalize to the full rotation group in a rather straightforward manner). (2.3.7) implies that

$$Y_{\pm}(0,0) > 0 \;, \quad Y_{\pm}(0,1) < 0 \;, \tag{2.3.8}$$

therefore there exist $\psi_{\pm}$ such that

$$Y_{\pm}(0,\psi_{\pm}) = 0 \;,$$

which makes (2.3.8) inconsistent with (2.3.4), thus on $S^3$ no solutions of the vacuum constraints exist. Since a solution of the vacuum constraints on $\mathbb{P}^3$ can be pulled–back to $S^3$ via the covering map, no such solutions exist on $\mathbb{P}^3$ either. It thus remains to show, that on $S^1 \times S^2$ and $\mathbb{P}^3 \# \mathbb{P}^3$ the vector field $\nabla Y$ must be timelike. Let us first note that the map $\Phi : [0,1] \times S^2 \to [0,1] \times S^2$ defined by

$$\Phi(\psi,\omega) = \begin{cases} (2\psi,\omega), & \psi \in [0,\frac{1}{2}] \;, \\ (2(1-\psi), R\omega) \;, & \psi \in (\frac{1}{2},1] \;, \end{cases}$$

where $R$ is the antipodal map, extends to a double covering map $\tilde{\Phi} : S^1 \times S^2 \to \mathbb{P}^3 \# \mathbb{P}^3$. Since $S^1 \times S^2$ is itself covered by $\mathbb{R} \times S^2$ it is sufficient to show timelikeness of $\nabla Y$ for periodic solutions (with period 1) of the constraint equations on $\mathbb{R} \times S^2$. Suppose that there exists a point $\psi_+$ such that $Y_+(\psi_+) = 0$, or there exists a point $\psi_-$ such that $Y_-(\psi_-) = 0$, or both, then from (2.3.6) it follows

$$Y_{\pm}(\psi_{\pm}+1) = \int_{\psi_{\pm}}^{\psi_{\pm}+1} \frac{X(s)}{2Y(s)}\, e^{\int_s^{\psi_{\pm}+1} h_{\pm}(u)du} ds > 0 \;, \tag{2.3.9}$$

which contradicts periodicity of $Y_{\pm}$, thus neither $Y_+$ nor $Y_-$ can change signs. Let $\psi_o$ be a local extremum of $Y$, then $\frac{\partial Y}{\partial \psi}(t=0,\psi_0) = 0$, and we have

$$(Y_+Y_-)(t=0,\psi_0) = -F^2 \left(\frac{\partial Y}{\partial t}\right)^2 \leq 0 \;,$$

which implies

$$(Y_+Y_-)(t=0,\psi) < 0 \quad \Longrightarrow \quad g^{\mu\nu} Y_{,\mu} Y_{,\nu} < 0 \;.$$

□

One can now proceed as in the Appendix B to [66] to conclude that ${}^3\Sigma$ can be deformed in ${}^4M$ in such a way that $\frac{\partial A}{\partial \psi} = 0$, a reparametrization of the $\psi$'s leads to $\frac{\partial f}{\partial \psi} = 0$, $A$ and $f$ as in (2.3.2), and thus in both $S^1 \times S^2$ and $I\!P^3 \# I\!P^3$ cases one obtains a two–parameter family of metrics on ${}^4M$ (parametrized by the area $A$ of the group orbits and the length $f$ of closed geodesics orthogonal to the orbits[4]) which are, locally, isometric to the metric under the horizon of the Schwarzschild–Kruskal–Szekeres manifold.

It may be of some interest to describe a larger family of space–times, with $S^1 \times S^2$ spatial topology, the metric of which is *locally* isometric to the "$r < 2m$ Schwarzschild metric", but on which no *global* action of $SO(3)$ by isometries exists. Consider the metric (2.3.2) with $\frac{\partial A}{\partial \psi} = \frac{\partial f}{\partial \psi} = 0$ on the manifold (2.3.1) in which the relation " $\sim$ " identifies $\{0\} \times S^2$ with $\{1\} \times S^2$ via a rotation $\omega \in SO(3)$ of $S^2$:

$$(0, p) \sim (1, \omega p)\,, \qquad \omega \in SO(3)\,.$$

In this way one obtains a five–parameter family of smooth metrics, parametrized by[5] $(f, A, \omega) \in (0, \infty) \times (0, \infty) \times SO(3)$. The four–dimensional metric is, again, locally isometric to the "$r < 2m$ Schwarzschild" metric with an appropriately chosen $m$.

## 2.4 Spatially compact Bianchi I space-times.

In this Section we shall discuss the space of maximally developed, globally hyperbolic, smooth Hausdorff vacuum metrics with compact $U(1) \times U(1) \times U(1)$ symmetric Cauchy surfaces ${}^3\Sigma$. $G = U(1) \times U(1) \times U(1)$ symmetry implies that ${}^3\Sigma$ is a three dimensional torus $T^3$ (cf. e.g. [45]) and the action of $G$ is transitive on ${}^3\Sigma$, choosing group coordinates on ${}^3\Sigma$ and the geodesic distance on geodesics normal to ${}^3\Sigma$ as a time coordinate in ${}^4M$

[4]Strictly speaking, $f$ is the length of such geodesics when ${}^3\Sigma = S^1 \times S^2$, and half of the length when ${}^3\Sigma = I\!P^3 \# I\!P^3$.

[5]It is easily seen that two such metrics with $\omega_1 \neq \omega_2$ will not be isometric: a geodesic orthogonal to a sphere $S^2_{\psi_o} \equiv \{\psi = \psi_0\}$ and passing through a point $p \in S^2_{\psi_o}$ will again interesect $S^2_{\psi_o}$ at $\omega p$.

we have

$$ds^2 = -dt^2 + g_{ij}dx^i dx^j, \quad g_{ij} = g_{ij}(t). \tag{2.4.1}$$

Let us momentarily forget about the identifications $x^i \equiv x^i + 2\pi$ and consider the metric (2.4.1) on $I\!R^3$. At any chosen time $t = t_o$ there exists a matrix $L^i_j \in SL(3, I\!R)$ such that

$$g_{ij}(t_o)L^i_k L^j_l = \delta_{kl}. \tag{2.4.2}$$

Since the right hand side of (2.4.2) is invariant under $SO(3)$ the matrix $L^i_j$ is not uniquely defined, and we may choose $\omega^i_j \in SO(3)$ so that

$$P_{ij}(t_o)L^i_k L^j_l = diag(P_1, P_1, P_3); \tag{2.4.3}$$

by an abuse of notation we have used the symbol $L^i_j$ to denote $L^i_k\omega^k_j$. Let

$$y^i = (L^{-1})^i_j x^j; \tag{2.4.4}$$

at $t = t_o$ we have

$$ds^2 = -dt^2 + \sum dy^i dy^i,$$
$$P_{ij}dy^i dy^j = P_1(dy^1)^2 + P_2(dy^2)^2 + P_3(dy^3)^2. \tag{2.4.5}$$

From 1) the existence of diagonal solutions of the equations of motion, namely the flat metrics on $I\!R^4$ or the Kasner metrics [79] on $(0,\infty)\times I\!R^3$; 2) uniqueness up to coordinate transformation of solutions of vacuum Einstein equations; it follows that all solutions can be diagonalized for all t and are either Minkowski space-time or the Kasner metrics[6]

$$ds^2 = -dt^2 + t^{2p_1}(dy^1)^2 + t^{2p_2}(dy^2)^2 + t^{2p_3}(dy^3)^2,$$
$$t \in (0,\infty), \quad y^i \in I\!R^3, \quad \sum p_i^2 = \sum p_i = 1. \tag{2.4.6}$$

Returning to $T^3$, the equations (2.4.4), (2.4.6) and (2.4.1) give

$$ds^2 = -dt^2 + g_{ij}dx^i dx^j, \quad \partial_\mu g_{ij} = 0, \tag{2.4.7}$$

[6]This simple argument is due to V.Moncrief.

in the $T^3$ — Minkowski case, or

$$ds^2 = -dt^2 + \sum t^{2p_i}(\omega^i)^2,$$

$$\omega^i \equiv L^i_j dx^j, \quad x^i \in [0, 2\pi]_{|\mathrm{mod}2\pi}, \tag{2.4.8}$$

in the $T^3$ — Kasner case. All coordinate transformations preserving (2.4.7) can be shown to be of the form

$$t \to t + \tau, \quad x^i \to Z^i_j x^j + x^i_o, \quad Z^i_j \in SL(3, Z) \quad (\det Z^i_j = 1, Z^i_j \in Z),$$

$$\partial_\mu \tau = \partial_\mu x^i_o = 0, \tag{2.4.9}$$

while those preserving (2.4) are

$$t \to t, \quad x^i \to Z^i_j x^j + x^i_o, \quad Z^i_j \in SL(3, Z), \quad \partial_\mu x^i_o = 0. \tag{2.4.10}$$

Let $Diff_o(^4M)$ be the path-connected component of the identity of the diffeomorphism group of $^4M$. Elementary singular homology considerations show that the diffeomorphisms (2.4.9) — (2.4.10) are in $Diff_o(^4M)$ if and only if $Z^i_j = \delta^i_j$. It follows that the set of maximally developed globally hyperbolic Hausdorff Bianchi I space-times with spatially compact Cauchy surfaces divided by $Diff_o$ can be given the structure of a manifold which is the union of three disconnected pieces:

- a six dimensional manifold of metrics (2.4.7) (parametrized by $g_{ij}$),
- a ten dimensional manifold of metrics (2.4) (one parameter for the $p_i$'s, nine for the $L^i_j$'s), and
- a ten dimensional manifold of metrics which differ from (2.4) by a change of time orientation, $t \to -t$.

The subset of extendible metrics of the above type consists of six disconnected nine dimensional submanifolds, which consist of Kasner metrics with one of the parameters $p_i$ equal to 1, thus a generic maximal Bianchi I space-time is inextendible.