VOLTAGE GRAPHS

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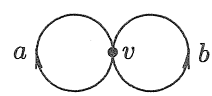
Abstract. An important theorem of Gross and Tucker characterises a directed graph E which admits a free action of a group as a skew product. Here we extend this to directed graphs admitting free actions of semigroups, S, under certain hypotheses. Indeed, the main criterion we employ can be completely characterised by properties of the quotient map $E \to E/S$.

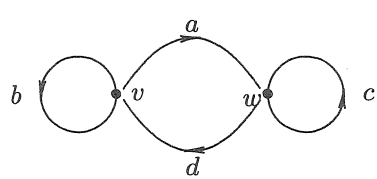
AMS subject classification. 05C28.

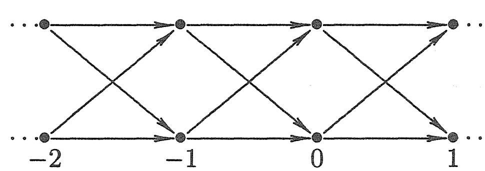
1. Introduction. This paper aims to study certain types of semigroup actions on directed graphs and extend properties already known of the case where S is a group. Thus, we begin by defining some basic notions used in previous literature that we will be using to develop our theory.

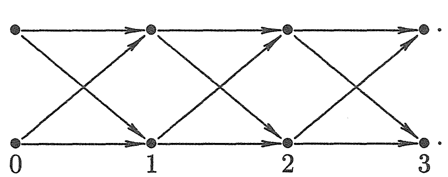
DEFINITION 1.1. A *directed graph* E is a quadruple $E = (E^0, E^1, r, s)$ consisting of countable sets E^0 of vertices and E^1 of edges, together with maps $r, s : E^1 \to E^0$ describing the range and source of each edge.

EXAMPLES 1.1.

$A :=$ 

$B :=$ 

$C :=$ 

$D :=$ 

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Definitions 1.2. If E and F are directed graphs, then a *graph morphism* ϕ is a pair $\phi = (\phi^0, \phi^1)$ of maps where $\phi^i : E^i \to F^i$ for $i = 0, 1$, which preserves connectivity; that is,

$$\phi^0(r(e)) = r(\phi^1(e)), \quad \phi^0(s(e)) = s(\phi^1(e))$$

for all $e \in E^1$. If $\phi^i : E^i \to F^i$ for $i = 0, 1$ are bijective, then ϕ is called a *graph isomorphism* and we write $E \cong F$; moreover, if $F = E$, then ϕ is called a *graph automorphism.*

The collection of all automorphisms of a graph E forms a group, $\mathrm{Aut}(E)$, under composition.

Example 1.2. By placing the graph D at the level marked $-n$ on C we may see that for every natural number n there is a graph morphism $\alpha_n : D \to C$.

Definition 1.3. A graph morphism $\phi : E \to F$ is said to have the *unique path lifting property* if, given a vertex $v \in E^0$ and an edge $e \in F^1$ with $s(e) = \phi^0(v)$, there is a unique edge $e' \in E^1$ such that $s(e') = v$ and $\phi^1(e') = e$.

In previous work, the first author has been interested in the actions of countable groups on directed graphs [4]. We describe some notions associated to this situation:

Definitions 1.4. Let Γ be a countable group and $c : E^1 \to \Gamma$ a function, then the pair (E, c) is called a *voltage graph.* From this we may form the *skew product graph* $E(c) = (\Gamma \times E^0, \Gamma \times E^1, r, s)$ or $E \times_c \Gamma$ where

$$s(g, e) = (g, s(e)) \text{ and } r(g, e) = (gc(e), r(e)).$$

for all $(g, e) \in E(c)^1$.

In the literature (see [3, Section 2.2.1]), $E(c)$ is sometimes referred to as a "derived graph". Voltage graphs have many applications, for instance, they are used in the theory of branched covering of surfaces [2].

Example 1.3. Suppose we define $c : B^1 \to \mathbb{Z}$ by $c(e) = 1$ for all $e \in E^1$ then $B(c) \cong C$. Suppose we define $c : A^1 \to \mathbb{Z}_2 = \{0, 1\}$ by $c(a) = 0$ and $c(b) = 1$, then $A(c) \cong B$.

Definition 1.5. An *action* of a group Γ on a directed graph E is a group homomorphism $\alpha : \Gamma \to \mathrm{Aut}(E)$.

Example 1.4. $\mathbb{Z}_2$ acts on A by interchanging the edges a and b; it also acts on B by interchanging v with w, a with d, and b with c. $\mathbb{Z}_2$ also acts on C and D by interchanging the top and bottom rows. There is an action of $\mathbb{Z}$ on C where $1 \in \mathbb{Z}$ is mapped to the automorphism which shifts the graph one step to the right.

In this paper we shall focus our attention on actions of a particular type, and characterise those graphs which admit such an action.

DEFINITION 1.6. An action α of a group Γ on E is said to be *free* if α acts freely on the vertex set; that is, given $g \in \Gamma$, for any $v \in E^0$, if $\alpha_g^0 v = v$, then $g = 1$.

The terminology comes from the term "fixed–point free". Note that we don't insist that $\alpha_g^1 e = e \Rightarrow g = 1$ for all $e \in E^1$ in the definition 1.6 since this follows from the graph morphism properties of α.

EXAMPLE 1.5. Suppose that $c : E^1 \to \Gamma$ is a function, where Γ is a countable group. Then there is a natural free Γ–action $\beta : \Gamma \to \mathrm{Aut}(E(c))$ defined by

$$\beta_g^0(h, v) = (gh, v), \qquad \beta_g^1(h, e) = (gh, e),$$

for $g, h \in \Gamma$, $v \in E^0$ and $e \in E^1$. Hence we may see that $\mathbf{Z}$ acts freely on C and $\mathbf{Z}_2$ acts freely on B.

If a countable group Γ acts on a graph E, then there is a natural graph E/Γ called the *quotient graph* associated to this action. The edge and vertex sets of E/Γ consist of the edge and vertex orbits under the action of Γ; that is, $(E/\Gamma)^i = E^i/\Gamma$ for $i = 0, 1$. The range and source maps are defined for each $[e] \in (E/\Gamma)^1$ by $r([e]) = [r(e)]$ and $s([e]) = [s(e)]$. These maps are well–defined because the Γ–action preserves the connectivity of E. The *quotient map* $q : E \to E/\Gamma$ which maps each edge and vertex to their orbits is a surjective graph morphism.

EXAMPLE 1.6. If $c : E^1 \to \Gamma$ is a function, where Γ is a countable group. Then Γ acts freely on $E(c)$ as discussed in 1.6 above, and more importantly, $E(c)/\Gamma \cong E$. Hence, in relation to the examples of 1.1, $C/\mathbf{Z} \cong B$ and $B/\mathbf{Z}_2 \cong A$.

The following result of Gross and Tucker [3, Theorem 2.2.2] shows that any graph which admits a free Γ–action is in fact, a skew product graph of its quotient graph:

Theorem 1.1. *Let E be a directed graph and suppose that Γ acts freely on E, then there is a function $c : (E/\Gamma)^1 \to \Gamma$ such that*

$$(E/\Gamma)(c) \cong E.$$

A key element in the proof of Theorem 1.1 is the unique path lifting property of the quotient map $q : E \to E/\Gamma$. The purpose of the rest of this paper is to extend these notions and give a proof of the semigroup case.

2. The Main Theorem. We start this section with the definition of a particular type of countable semigroup. These semigroups will then be used extensively throughout the remainder of the paper. In particular, their properties are necessary for the main theorem.

DEFINITIONS 2.1. An *Ore semigroup* S is a cancellative semigroup which is *right–reversible*; that is, for any $s, t \in S$ we have $Ss \cap St \neq \emptyset$.

Ore semigroups have the distinguishing property that they may be embedded in a group Γ with $S^{-1}S = \Gamma$ where Γ is determined up to canonical isomorphism. Moreover, Ore semigroups are the only semigroups with this property (see [5, Theorem 1.1.2] and [1, §1.10], for example).

EXAMPLE 2.1. The natural numbers $\mathbf{N}$ and the direct product $\mathbf{Z}_2 \times \mathbf{N}$ are examples of Ore semigroups. Note also that every group is an Ore semigroup.

DEFINITION 2.2. Let E be a directed graph and $c : E^1 \to S$ a function, then as before we may form the *skew product graph* $E(c)$ by setting $E(c)^0 = S \times E^0$, $E(c)^1 = S \times E^1$, $s(t,e) = (t, s(e))$ and $r(t,e) = (tc(e), r(e))$.

EXAMPLE 2.2. Let $c : B^1 \to \mathbf{N}$ be defined by $c(e) = 1$ for all $e \in B^1$, then $B(c) \cong D$. Let $c : A \to \mathbf{Z}_2 \times \mathbf{N}$ be defined by $c(a) = (0,1)$ and $c(b) = (1,1)$ where $\mathbf{Z}_2 = \{0,1\}$ with addition modulo two, then $A(c) \cong D$.

DEFINITION 2.3. An *action* of a semigroup S on a directed graph E is a semigroup homomorphism $\alpha : S \to \mathrm{End}(E)$, where $\mathrm{End}(E)$ is the semigroup of endomorphisms of E; that is, the set of injective graph morphisms $\phi : E \to E$ under composition.

EXAMPLE 2.3. Suppose that E is a directed graph and $c : E^1 \to S$ is a function, then S acts on $E(c)$ by defining for each $s \in S$, $\beta^0_s(t,v) = (st, v)$ and $\beta^1_s(t,e) = (st, e)$, giving a homomorphism $\beta : S \to \mathrm{End}((E/S)(c))$. Hence, $\mathbf{N}$ acts D by shifting the graph to the right and $\mathbf{Z}_2 \times \mathbf{N}$ acts on D by also swapping the upper and lower levels.

Again, we are only interested in a certain class of semigroup action. However, we must change the definitions given in Section 1 to account for the fact that there may be no inverses in the semigroup.

DEFINITION 2.4. Let $\alpha : S \to \mathrm{End}\,(E)$ be a semigroup action, then we say that the action of S is *free* if, for $v \in E^0$ and $s, t \in S$, we have $\alpha^0_s v = \alpha^0_t v$ only if $s = t$.

As before, it is not necessary to define freeness on the edge set, it follows because α preserves connectivity. Note also that if the semigroup S acts freely, then S is cancellative, this follows since α is a semigroup homomorphism.

EXAMPLE 2.4. The action of S on the skew product graph $E(c)$ described in example 2.3 is a free action, hence the actions of $\mathbf{N}$ and $\mathbf{Z}_2 \times \mathbf{N}$ on D are free.

Next, in analogy with the group case, we introduce the notion of a quotient graph for a semigroup action.

Proposition 2.1. *Let S be a right–reversible semigroup acting on a directed graph E. Then*
(i) the relation $\sim$ on E^0 (resp E^1) defined by $v \sim w$ if there exist $s, t \in S$ such that $\alpha_s v = \alpha_t w$ is an equivalence relation;
(ii) the equivalence classes of edges and vertices under $\sim$, denoted $(E/S)^1$ and $(E/S)^0$ respectively, together with maps $r, s : (E/S)^1 \to (E/S)^0$ defined by $r([e]) = [r(e)]$ and

$s([e]) = [s(e)]$, *form a directed graph.*

Proof. For the first part, symmetry and reflexivity of $\sim$ are obvious. In order for the relation $\sim$ to be transitive, it is necessary that S be right–reversible. The argument is as follows: suppose$v \sim w$ and $w \sim z$. Then there are $s, t, a, b \in S$ such that $\alpha_s v = \alpha_t w$ and $\alpha_a w = \alpha_b z$. Since S is right–reversible, $St \cap Sa \neq \emptyset$, so there are $m, n \in S$ such that $mt = na$. Thus, $\alpha_{ms} v = \alpha_{mt} w$ and $\alpha_{na} w = \alpha_{nb} z$, so $\alpha_{ms} v = \alpha_{nb} z$. Therefore, we have $\sim$ is transitive and, hence, an equivalence relation.

For the second part it suffices to show that the range and source maps are well–defined. So, suppose $e, f \in E^1$ are such that $[e] = [f]$. Then there exist $s, t \in S$ such that $\alpha_s e = \alpha_t f$, so we have $\alpha_s s(e) = s(\alpha_s e) = s(\alpha_t f) = \alpha_t s(f)$, and similarly, $\alpha_s r(e) = \alpha_t r(f)$. Therefore, $[s(e)] = [s(f)]$ and $[r(e)] = [r(f)]$, as required. □

DEFINITIONS 2.5. The equivalence relation defined in Proposition 2.1 part (i) is called *S–orbit equivalence*, and the directed graph defined in Proposition 2.1 part (ii) is called the *quotient graph* of E under the action of S and is denoted E/S. The map $q : E \to E/S$ which maps each element to its corresponding orbit is called the *quotient map.*

NOTE 2.1. By the definition of the range and source maps in Proposition 2.1 Part (*ii*), we see that q is a graph morphism. If S is a group, then the definition of S–orbit equivalence is the same as the usual orbit equivalence associated with the group action.

EXAMPLE 2.5. The free action of S on $E(c)$ is such that $E(c)/S \cong E$, hence $D/\mathbb{N} \cong B$, and $D/(\mathbb{Z}_2 \times \mathbb{N}) \cong A$. Note also that $C/\mathbb{N} = B$ and $C/(\mathbb{Z}_2 \times \mathbb{N}) \cong A$.

DEFINITIONS 2.6. Let $\alpha : S \to \mathrm{End}(E)$ be a free action of a right-reversible semigroup S on a directed graph E and let $x \in (E/S)^0$. Then a vertex $v_x \in E^0$ is said to be a *base vertex* of x if $q(v_x) = x$ and for any $w \in E^0$ with $q(w) = x$ there is a $t \in S$ such that $\alpha_t v_x = w$. Similarly, we may define a *base edge* $e_y \in E^1$ of an edge $y \in (E/S)^1$. The semigroup action is said to be *saturated* if for every $x \in (E/S)^0$ and $y \in (E/S)^1$ there is a base vertex v_x of x and a base edge e_y of y.

EXAMPLE 2.6. The free action of S on $E(c)$ described in Example 2.3 is saturated, since for each $(s, x) \in E(c)^0$, every element in the orbit of (s, x) can be written as $\beta_t(1, x)$ for some $t \in S$. Hence, the actions of $\mathbb{N}$ and $\mathbb{Z}_2 \times \mathbb{N}$ on D are saturated.

All group actions are saturated and so the results obtained in this section are generalisations that can be restricted to the group case. We now state and prove a result that enables us to parametrise base vertices (an equivalent result holds for edges).

Lemma 2.1. *Let $\alpha : S \to End(E)$ be a free and saturated action of an Ore semigroup S on a directed graph E and let P denote the group of invertibles in S. Then, for each $x \in (E/S)^0$ with base vertex v_x, the set $Pv_x = \{\alpha_p v_x \mid p \in P\}$ is precisely the set of base vertices of x.*

Proof. Suppose w_x is a base vertex of x, then for all $z \in E^0$ with $q(z) = x$, there exist $a, b \in S$ such that $z = \alpha_a v_x$ and $z = \alpha_b w_x$. In particular, $w_x = \alpha_p v_x$ and $v_x = \alpha_q w_x$ for some $p, q \in S$. Thus,

$$v_x = \alpha_q \alpha_p v_x = \alpha_{qp} v_x$$

and so $qp = 1$ since the action is free, and

$$w_x = \alpha_p \alpha_q w_x = \alpha_{pq} w_x$$

and so, again by freeness, $pq = 1$. Therefore, p is invertible with $p^{-1} = q$. Hence, the set of base vertices of x is a subset of Pv_x.

Now suppose $p \in P$. Then for all $z \in E^0$ with $q(z) = x$,

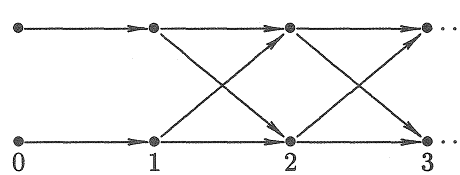
$$\begin{aligned} z &= \alpha_q v_x \quad \text{for some } q \in S \\ &= \alpha_{q(p^{-1}p)} v_x \quad \text{since } p^{-1} \in S \\ &= \alpha_{(qp^{-1})p} v_x \\ &= \alpha_t \alpha_p v_x \quad \text{where } t = qp^{-1} \end{aligned}$$

Hence, for all $p \in P$, $\alpha_p v_x$ is a base vertex of x, so Pv_x is a subset of the set of base vertices of x. Therefore, the set Pv_x is precisely the set of base vertices of x. □

DEFINITION 2.7. A saturated semigroup action $\alpha : S \to \text{End}(E)$ is said to be *co-saturated* if for each $y \in (E/S)^1$ there is a base edge e_y of y such that $s(e_y) = v_x$ where v_x is a base vertex of $x = s(y) \in (E/S)^0$; that is, every edge orbit has a base edge whose source is a base vertex.

EXAMPLE 2.7. The action of $\mathbf{N}$ on D is co-saturated. However, the action of $\mathbf{N}$ on the following graph F is saturated but not co-saturated:

$F :=$



The reason we restrict our attention to co-saturated actions is given in the following result.

Proposition 2.2. *Let $\alpha : S \to End(E)$ be a free and saturated action of an Ore semigroup S on a directed graph E. Then the quotient map $q : E \to E/S$ has the unique path lifting property if and only if the S–action is co-saturated.*

Proof. Suppose the S–action is free and co-saturated. Let $x \in (E/S)^0$ and suppose there is a $y \in (E/S)^1$ such that $s(y) = x$. Then, since the action is co-saturated, there is a base edge e_y of y such that $s(e_y) = v_x$ for some base vertex v_x of x (note that the base vertex with this property must be associated to x, since $q(s(e_y)) = s(q(e_y)) = s(y) = x$ by the definition of the quotient graph source map).

For each $w \in E^0$ with $q(w) = x$ there exists a $t \in S$ such that $w = \alpha_t v_x$. We claim that the edge $\alpha_t e_y \in E^1$ is the unique edge such that $q(\alpha_t e_y) = y$ and $s(\alpha_t e_y) = w$. It is obvious that $q(\alpha_t e_y) = y$, and $s(\alpha_t e_y) = \alpha_t s(e_y) = \alpha_t v_x = w$. Also, supposing that we have another $f \in E^1$ with $q(f) = y$ and $s(f) = w$, then $f = \alpha_s e_y$ for some $s \in S$ and so $w = s(f) = s(\alpha_s e_y) = \alpha_s s(e_y)$. But $\alpha_t s(e_y) = s(\alpha_t e_y) = w$, so $\alpha_s s(e_y) = \alpha_t s(e_y)$. Hence, by freeness, $s = t$, making $f = \alpha_t e_y$. Therefore, q has the unique path lifting property.

Now suppose the quotient map q has the unique path lifting property. For each $y \in (E/S)^1$ there is an $x \in (E/S)^0$ such that $s(y) = x$. Since the S–action on E is saturated, there exists a base vertex v_x of x, thus, by the unique path lifting property, there is a unique $f \in E^1$ such that $s(f) = v_x$ and $q(f) = y$. We claim that f is a base edge of y. To see this let $f' \in E^1$ be such that $q(f') = y$, then $s(f') = \alpha_t v_x$ for some $t \in S$ since $q(s(f')) = s(q(f')) = s(y) = x$. Now, $s(\alpha_t f) = \alpha_t s(f) = \alpha_t v_x = s(f')$ and so, by the unique path lifting property, $f' = \alpha_t f$. Therefore, f is a base edge of y whose source is a base vertex of x, hence the S–action is co-saturated. □

The unique path lifting property plays an important part in the proof our main theorem.

Theorem 2.1. *Let $\alpha : S \to End(E)$ be a free action of an Ore semigroup S on a directed graph E. Then there exists a function $c : (E/S)^1 \to S$ such that $(E/S)(c) \cong E$ in an S-equivariant way if and only if the S-action is co-saturated.*

Proof. First, we suppose the S–action on E is co-saturated. For each $x \in (E/S)^0$ fix a base vertex v_x of x. By Proposition 2.2, the quotient map has the unique path lifting property, so for each $y \in (E/S)^1$ such that $s(y) = x$, there is a unique base edge e_y of y such that $s(e_y) = v_x$. In E we have $r(e_y) = \alpha_a v_z$ where $a \in S$ and v_z is the base vertex associated to $z = r(y)$, so we define $c(y) = a$.

Now we define two functions, $\phi^i : ((E/S)(c))^i \to E^i$ $(i = 0, 1)$ by $\phi^0(t, x) = \alpha_t v_x$ and $\phi^1(t, y) = \alpha_t e_y$. For $y \in (E/S)^1$, $x = s(y)$, $z = r(y)$ and $t \in S$, we compute

$$\begin{aligned}
\phi^0(s(t,y)) &= \phi^0((t,x)) \quad \text{since } s(y) = x \\
&= \alpha_t v_x \quad \text{by definition of } \phi \\
&= \alpha_t s(e_y) \quad \text{since } v_x = s(e_y) \\
&= s(\alpha_t e_y) \quad \text{since } \alpha \text{ preserves connectivity} \\
&= s(\phi^1(t,y)) \quad \text{by definition of } \phi
\end{aligned}$$

and

$$\begin{aligned}
\phi^0(r(t,y)) &= \phi^0((tc(y),z)) \quad \text{since } r(y) = z \\
&= \alpha_{tc(y)} v_z \quad \text{by definition of } \phi \\
&= \alpha_t \alpha_{c(y)} v_z \quad \text{since } \alpha \text{ is a homomorphism} \\
&= \alpha_t r(e_y) \quad \text{by definition of } c \\
&= r(\alpha_t e_y) \quad \text{since } \alpha \text{ preserves connectivity} \\
&= r(\phi^1(t,y)) \quad \text{by definition of } \phi.
\end{aligned}$$

Thus, ϕ is a graph morphism. The injective and surjective properties of ϕ follow from freeness and saturatedness, respectively. Moreover, for $s \in S$

$$\begin{aligned}
\alpha_s\phi^0(t,x) &= \alpha_s\alpha_t v_x \quad \text{by definition of } \phi \\
&= \alpha_{st}v_x \quad \text{since } \alpha \text{ is a homomorphism} \\
&= \phi^0(st,x) \quad \text{by definition of } \phi \\
&= \phi^0(\beta_s(t,x)) \quad \text{since } \beta \text{ is a homomorphism}
\end{aligned}$$

and

$$\begin{aligned}
\alpha_s\phi^1(t,y) &= \alpha_s\alpha_t e_y \quad \text{by definition of } \phi \\
&= \alpha_{st}e_y \quad \text{since } \alpha \text{ is a homomorphism} \\
&= \phi^1(st,y) \quad \text{by definition of } \phi \\
&= \phi^1(\beta_s(t,y)) \quad \text{since } \beta \text{ is a homomorphism.}
\end{aligned}$$

Therefore, $(E/S)(c) \cong E$ in an S–equivariant way. Now suppose that $(E/S)(c) \cong E$ in an S–equivariant way, so we have an isomorphism $\psi : (E/S)(c) \to E$. For each $x \in (E/S)^0$ we claim that elements of the form $\psi^0(1,x)$ are base vertices in E and, similarly, for each $y \in (E/S)^1$, elements of the form $\psi^1(1,y)$ are base edges in E. To see this, take any $z \in E^0$ with $q(z) = x$, then

$$\begin{aligned}
z &= \psi^0(s,w) \quad \text{for some } s \in S \text{ and } w \in (E/S)^0 \\
&= \psi^0(\beta_s(1,w)) \quad \text{since } \beta \text{ is a homomorphism} \\
&= \alpha_s\psi^0(1,w) \quad \text{since } \psi \text{ is } S\text{–equivariant}
\end{aligned}$$

Thus, all $z \in E^0$ with $q(z) = x$ are of the form $z = \alpha_s v_x$ for some $s \in S$, where $v_x = \psi^0(1,w)$. In the same way it can be shown that all $f \in E^1$ with $q(f) = y$ are of the form $f = \alpha_t e_y$ for some $t \in S$, where $e_y = \psi^1(1,u)$ for some $u \in (E/S)^1$. Therefore, the S–action is saturated. Furthermore, for each $y \in (E/S)^1$, there exists an $e_y \in E^1$ such that

$$\begin{aligned}
s(e_y) &= s(\psi^1(1,u)) \quad \text{for some } u \in (E/S)^1 \\
&= \psi^0(s(1,u)) \quad \text{since } \psi \text{ preserves connectivity} \\
&= \psi^0(1,s(u)) \quad \text{by definition of the skew product graph source map.}
\end{aligned}$$

But $\psi^0(1,s(u))$ is a base vertex. Therefore, the S–action is co-saturated. □

REMARKS 2.1. If S is a group, then we recover the original Gross and Tucker result. We have completely characterised those graphs which admit a co-saturated S–action – they are precisely the skew product graphs for functions $c : E^1 \to S$. If, in the proof of Theorem 2.1, we choose a different set of base vertices, then we obtain a new function $d : (E/S)^1 \to S$. Any two functions $c, d : (E/S)^1 \to S$ derived this way will be cohomologous in the sense that there exists a function $b : (E/S)^0 \to S$ such that, for all $y \in (E/S)^1$,

$$b(s(y))c(y) \;=\; d(y)b(r(y)).$$

In fact, from 2.1 the range of b lies in the invertibles of S, since b essentially assigns to each $x \in (E/S)^0$ the element of S which takes the base vertex of x used to define c to the base vertex of x used to define d.

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