model atmospheres the normal modes must be calculated numerically. A finite difference solution of the eigenvalue problem for the differential equations leads to the matrix eigenvalue problem $A\mathbf{x} = \lambda\mathbf{x}$ ([16], [17]).

For nonadibatic perturbations the details will be determined by the equations adopted to model the nonideal behaviour of the background equilibrium atmosphere. In this paper we consider the case of a thermally conducting gas, in the absence of viscosity, and with infinite electrical conductivity. Previous studies have modelled the radiative conductivity by means of the diffusion approximation which is a valid approximation for the case of optically thick atmospheres ([1], [2]). In the absence of a magnetic field the differential equations are again fourth order however the system is now nonlinear in the eigenvalue and leads to a matrix eigenvalue problem of the form $A(\lambda)\mathbf{x} = \mathbf{0}$ [2]. The inclusion of a vertical magnetic field extends the order of the system to six with the eigenvalue parameter again appearing nonlinearly [1].

For plane-parallel atmospheres one of the simplest models which incorporates vertical stratification is the polytropic model in which the temperature variation is a linear function of height. Such models are good first approximations over regions of the solar atmosphere where the temperature gradient is approximately constant. Even for these simple models the normal modes must be evaluated numerically.

Previous numerical calculations to evaluate the eigenvalues for these models have invariably used some form of root finding technique to locate the eigenvalues. In the work of Antia, Chitre and Kale [2] the acoustic modes for adiabatic motions, where the eigenvalues are real, were located by looking for changes in the sign of det $\mathrm{A}(\lambda)$. In the case of nonadiabatic oscillations the complex eigenvalues were evaluated using Muller's method on a matrix system derived from first order differencing of the original differential equations. In searching for the overstable modes in the presence of radiative diffusion and a vertical magnetic field Antia and Chitre [1] used the root finding technique of Delves and Lyness [5] to locate the complex eigenvalues. In a stability study of nonadibatic oscillations in background polytropes both with and without the presence of a vertical magnetic field Lou ([12], [13]) has used a shooting method coupled with continuous variation of the ei

The complete spectrum for adiabatic oscillations in a vertical magnetic field was evaluated by Wood ([16], [17]) using a matrix method. This calculation showed that the previous evaluations using root finding methods had not located all of the possible modes. This technique was extended by Gore [7] to nonadiabatic oscillations when the coefficient of thermal conductivity was taken to be constant.

In this paper we consider a number of models which describe nonadiabatic oscillations in polytropic atmospheres and use a matrix method to evaluate normal modes. The results are then compared with the results found using root finding methods.

2. The Governing Equations

The basic atmosphere is taken to be an ideally conducting, invisid, compressible gas in the presence of a uniform and vertical magnetic field $\mathbf{B_0} = (0, 0, \mathrm{B_0})$ and a uniform gravitational acceleration $\mathbf{g} = (0, 0, -\mathrm{g})$. We use a standard cartesian coordinate system with z measuring the height through the atmosphere from $z = 0$ to $z = z_T$,the top of the atmosphere. The governing equations are:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{v}) = 0, \tag{2.1}$$

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla \mathbf{p} + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{\mu} + \mathbf{g}\rho, \tag{2.2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \tag{2.3}$$

$$\rho C_v \frac{dT}{dt} - RT\frac{d\rho}{dt} = \nabla.(\mathrm{K}\nabla \mathrm{T}), \tag{2.4}$$

$$p = R\rho T. \tag{2.5}$$

Here R is the gas constant, C_v is the specific heat and K is the coefficient of thermal conductivity.

The form of the thermal coefficient, K, in the energy equation (Equation (4)) places various constraints on the parameters used in modelling the small amplitude oscillations. If K = constant, then in equilibrium we obtain from Equation (4) that the temperature profile,T_0, is linear where

$$T_0 = az + T_B, \tag{2.6}$$

and $a = \frac{T_T - T_B}{L}$. Here T_T and T_B are the temperatures at the top and bottom of the atmosphere, respectively and L is the vertical extent of the slab.

In equilibrium, Equation (2) gives

$$\frac{dp_0}{dz} = -\rho_0 g$$

and using Equation (5) we obtain

$$\rho_0 = \rho_B T_B^m T_0^{-m} \tag{2.7}$$

and

$$p_0 = R\rho_B T_B^m T_0^{1-m} \tag{2.8}$$

where $m = \frac{Ra+g}{Ra}$. The polytropic index

$$\Gamma = \frac{d(\ln p_0)}{d(\ln \rho_0)} = \frac{m-1}{m}. \tag{2.9}$$

If on the other hand we model radiative effects by means of the diffusion approximation with an opacity that varies with height, then K is of the form

$$K = K_0 T_0^{3+\nu+m(\lambda+1)},$$

where K_0, ν and λ are constants. To ensure that the atmosphere is polytropic in the equilibrium state then it follows that we must put

$$3 + \nu + m(\lambda + 1) = 0. \tag{2.10}$$

For a particular atmospheric model with T_B *and* T_T specified then m is fixed while λ *and* ν are free adjustable parameters provided Equation (10) is satisfied. Note that for $\lambda = -1$ *and* $\nu = -3$ the conductivity is constant.

It is convenient to nondimensionalize the equations by expressing distance in units of scale height $H = RT_B/g$ and time in units of $\sqrt{RT_B}/g$, while the pressure and density are expressed in units of p_B and ρ_B respectively. In these units the equilibrium temperature is

$$T_0 = 1 - (\frac{\Gamma - 1}{\Gamma})z,$$

while the equilibrium density is $\rho_0 = T_0^{-m}$, where z is now dimensionless.

Equations (1)—(5) are linearized by assuming each of the variables is of the form

$$f = f_0(z) + f_1(z)e^{\omega t + i\alpha x},$$

where ω is the dimensionless eigenvalue and $\alpha = Hk$ is the dimensionless horizontal wave number.

The resulting equations are:

$$E' + \alpha\rho_o A = \frac{\omega}{T_o}(\rho_o D - p), \tag{2.11}$$

$$p' + T_o^{-1}(p - \rho_o D) = -\omega E, \tag{2.12}$$

$$G_K(D'' - \alpha^2 D) - \frac{(\lambda+1)G_K D'}{T_o} + (1 - \frac{\gamma}{\Gamma})E = \omega(\gamma\rho_o D - (\gamma - 1)p + \frac{G_K(\Gamma - 1)(\lambda + 1)E}{\Gamma T_o \rho_o}), \tag{2.13}$$

$$G_B(A'' - \alpha^2 A) = \omega^2\rho_o A - \omega\alpha p. \tag{2.14}$$

Here E, p, D and A are the perturbations in the vertical component of momentum, pressure, temperature and the $x-$ component of velocity, respectively. The ratio of specific heats is denoted by γ while

$$G_K = \frac{K_o g}{\rho_B C_v (RT_B)^{\frac{3}{2}}}$$

and

$$G_B = \frac{B_o^2}{\mu R T_B \rho_B}$$

are dimensionless parameters.

If we let $\mathbf{X} = [\mathrm{E}, \mathrm{p}, \mathrm{D}, \mathrm{A}]^{\mathrm{T}}$ then Equations (11) - (14) may be written formally as

$$L_1\mathbf{X} = \mathrm{M}_1(\omega)\mathbf{X}\ , \tag{2.15}$$

where L_1 is a differential operator. Introducing an additional variable $Q = \omega A$, Equation (15) may be written as

$$L_2\mathbf{Y} = \omega\mathrm{M}_2\mathbf{Y}\ , \tag{2.16}$$

where M_2 is independent of ω, $\mathbf{Y} = [\mathrm{E}, \mathrm{p}, \mathrm{D}, \mathrm{A}, \mathrm{Q}]^{\mathrm{T}}$ and L_2 is an algebraic-differential operator.

The system of equations must be supplemented by appropriate boundary conditions. In order to compare the results from the matrix computations presented in this paper with the results of previous authors we choose $E = D = A' = 0$ *at* $z = 0$ *and* $z = z_T$.

Equation (16) is converted to a matrix eigenvalue problem by second order central differencing. The eigenvalues are evaluated by means of the QZ algorithm using commonly available numerical routines.

Setting either of the parameters G_K *or* G_B equal to zero leads to reduced systems of equations which give rise to various sets of normal modes. These modes have been evaluated previously using the various root finding techniques discussed above. Here we reproduce some of these results together with some new results by means of the companion matrix method in Equation (16).

3. Non Magnetic Atmospheres

In the absence of a magnetic field, $G_B = 0$ and the normal modes arise due to the combined effects of compression and gravity. These are the acoustic - gravity waves in an atmosphere which is optically thick. We restrict our attention to the case of a constant coefficient of thermal conductivity ($\lambda = -1$) to obtain:

$$\begin{aligned} \alpha^2 p &= \frac{\omega^2}{T_o}(p - \rho_o D) + \omega E' , && (3.1) \\ p' + (\frac{p}{T_o} - \frac{\rho_o D}{T_o}) &= -\omega E , && (3.2) \\ G_K(D'' - \alpha^2 D) + (1 - \frac{\gamma}{\Gamma})E &= \omega(\gamma\rho_o D + (1-\gamma)p) , && (3.3) \end{aligned}$$

subject to the boundary conditions $E = D = 0$ *at* $z = 0, z_T$.

Again the eigenvalue, ω, appears nonlinearly and this system was studied in detail by Antia,Chitre and Kale [2]. These authors cast Equations (17) - (19) into a system of first order differential equations and used finite differencing with a truncation error of O(h) to solve a system of algebraic equations of the form $C\mathbf{X} = \mathbf{0}$. The complex eigenvalues were found using Muller's method on *det* $C = 0$.

To turn the system into companion matrix form we introduce two new variables $F = \omega p$ and $G = \omega D$. Hence we obtain

$$\begin{aligned} L_3\mathbf{Y} &= \omega \mathrm{M}_3 \mathbf{Y} && (3.4) \\ \text{where} & \\ \mathbf{Y} &= (\mathrm{p}, \mathrm{E}, \mathrm{D}, \mathrm{F}, \mathrm{G})^{\mathrm{T}}. && (3.5) \end{aligned}$$

Obviously extensive results may be generated by varying the parameters which appear in the system of equations. For the sake of comparison with some of the results reported by Antia, Chitre and Kale we restrict attention to one particular set of values and choose; $G_K = 5.28 \times 10^{-2}$, $\frac{T_T}{T_B} = 0.1$, $\gamma = 1.01$, $\alpha = 1$ *and* $\Gamma = 1.66$. For these parameter values $\frac{z_T}{H} = 2.26$.

Table 1 displays the lowest complex eigenvalues for two different grid spacings (we also considered two other spacings between those shown in the table). The growth or decay of

TABLE 1. Oscillatory acoustic - gravity modes

h=0.023		h=0.009	
ω_R	ω_I	ω_R	ω_I
-0.107×10^{-1}	0.853	-0.107×10^{-1}	0.853
0.220×10^{-1}	1.251*	0.220×10^{-1}	1.251
0.939×10^{-4}	1.580	0.142×10^{-4}	1.577
0.525×10^{-1}	2.066*	0.525×10^{-1}	2.067
0.235×10^{-3}	2.297	0.264×10^{-4}	2.288
0.275×10^{-1}	2.935*	0.278×10^{-1}	2.939

$*$ Muller's method - Antia, Chitre and Kale [2]

the mode depends on the sign of $Re\ \omega (= \omega_R)$ while $Im\ \omega\ (= \omega_I)$ gives the frequency of the wave. The results calculated by Antia, Chitre and Kale are indicated in the table. They also found a decaying mode (in Table 1 this is the mode $[-0.107 \times 10^{-1}, 0.853]$) but did not report a value since they were explicitly concerned with overstable modes ($\omega_R > 0$). As Table 1 indicates the matrix method gives eigenvalues not reported previously. For these modes the oscillatory part, ω_I , is stable with respect to h. The growth term, ω_R , is less than the truncation error, $O(h^2)$ and so the values must be viewed with caution. To determine whether these eigenvalues are purely imaginary requires smaller values of h. The results do suggest however that any search should include the imaginary axis.

Table 2 shows the three largest purely growing modes (convective modes), which are stable for all the values of h considered. These particular modes have not been reported before.

TABLE 2. Convective Modes

$0.023 \le h \le 0.009$
ω_R
0.249
0.508×10^{-1}
0.129×10^{-1}

4. Atmospheres with a Vertical Magnetic Field

For an atmosphere containing a vertical magnetic field it is possible, within the context of the system given by Equations (11) - (14) to consider two types of oscillatory modes. One type (adiabatic oscillations) is obtained by putting $G_K = 0$ and the equations reduce to a fourth order system which may be written formally as,

$$L_4 \begin{pmatrix} A \\ E \end{pmatrix} = -\omega^2 \begin{pmatrix} A \\ E \end{pmatrix},$$

where now E is the z - component of the velocity. This system is solved for the boundary conditions $A' = E = 0$ *at* $z = 0$ *and* z_T. Adiabatic oscillations for polytropic atmospheres have been studied by Gore [7] for various sets of boundary conditions. For the boundary conditions adopted here the adiabatic modes are purely oscillatory.

The nonadiabatic modes are the result of evaluating the eigenvalues for the full sixth order system given by Equations (11) - (14). The parameters used to obtain the results were; $G_B = 0.0239$ (corresponding to a magnetic field of 1000G), $G_K = 0.0393$, $\frac{T_T}{T_B} = 0.33, \gamma = 1.1, \alpha = 1, \Gamma = 1.56$ *and* $\lambda = 3$. For these paramters $\frac{z_T}{H} = 1.869$.

The results for both adiabatic and nonadiabatic modes are shown in Table 3. The modes are numbered as shown to facilitate a later comparison with a root finding technique. The eigenvalues for both spectra were evaluated for various values of h to confirm the numerical stability of the results. As Table 3 shows the nonadiabatic spectrum contains modes which are not present in the spectrum for the adiabatic oscillations (modes 5a and 9a).For these two anomalous modes we found that ω_R varied with changing h and at all times was less than the truncation error. The imaginary parts(ω_I) for 5a and 9a were numerically stable for changing h. It is worth pointing out that mode 9 also has ω_R less than the truncation error but this mode not only appears as a purely oscillatory adiabatic mode but also ω_R is numerically stable with respect to changes in h.

TABLE 3. Adiabatic and Nonadiabatic Modes

	Adiabatic (h=0.0047)	Nonadibatic (h=0.0074)	
mode	ω_I	ω_R	ω_I
1	0.294	0.484×10^{-1}	0.367
2	0.753	0.785×10^{-2}	0.762
3	0.940	-0.143×10^{-1}	0.936
4	1.135	0.147×10^{-2}	1.135
5	1.510	0.243×10^{-3}	1.508
5a		0.179×10^{-5}	1.586
6	1.701	0.312×10^{-2}	1.719
7	1.886	0.444×10^{-3}	1.884
8	2.261	0.134×10^{-3}	2.258
9	2.636	0.248×10^{-4}	2.633
9a		-0.379×10^{-4}	2.800
9b	2.944	0.722×10^{-2}	2.934
10	3.011	0.326×10^{-3}	3.009

Overstable magnetoacoustic modes, corresponding to the nonadiabatic modes shown in Table 3, have been reported by Antia and Chitre [1] and Chitre [4]. Unfortunately, in their calculations and incorrect value for the gravitational acceleration was used and consequently their results are for unphysical parameters. Nevertheless we have used their parameters in the matrix method to compare the output from this method with their eigenvalues which were computed using a root finding technique due to Delves and Lyness [5]. We have not listed the results, though for the sake of discussion we have numbered the modes in the same way as shown in Table 3. We find that within the first 13 eigenvalues there is close agreement between the two methods for all but the modes labelled 5a, 9a and 9b. These modes were not found by the root finding technique. The real parts for the eigenvalues labelled 5a and 9a are numerically unstable with respect to changes in h and have values less than the truncation errors. Note that for a meaningful comparison it was necessary to calculate the adiabatic spectrum using an incorrect value of g.

5. Conclusion

In this paper we have used a generalized eigenvalue method to find approximate solutions to coupled differential equations in which the eigenvalue appears nonlinearly. The advantage of the matrix method is that it is easily implemented and requires no initial approximation to compute all the eigenvalues of a particular discretization. The fact that there is not complete agreement between the results presented here and the root finding techniques used in previous works requires further investigation. The appearance of new modes with real parts decreasing with finer grid sizes suggests such modes are purely oscillatory and are due to the presence of thermal diffusion. Hopefully forthcoming investigations will clarify the nature of these modes.

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