ON THE UNIFORM CLASSIFICATION OF $L_p(\mu)$ SPACES

by Anthony Weston *

In this paper we survey results on the uniform classification of $L_p(\mu)$ spaces, we cite several open problems and we tie some loose ends in the existing theory (i.e., Theorem 12(a), (b)).

The topological classification of Banach spaces was initiated in Mazur [17] where he proved

Theorem 1: *For $1 \leq p, q < \infty$ the real Banach spaces $L_p(0,1)$, $L_q(0,1)$ and ℓ_q are homeomorphic.*

From Mazur's work it also followed that the unit balls $B(L_p(0,1))$, $B(L_q(0,1))$ and $B(\ell_q)$ are uniformly homeomorphic. We recall that a bijection $f : X \to Y$ between metric spaces is called a uniform homeomorphism if it is uniformly continuous in both directions.

For a thorough study of the topological structure of linear metric spaces we refer the reader to Bessaga and Pelczynski [8]. We would like to mention that [8] includes a proof of the

Anderson-Kadec Theorem: *Every infinite dimensional, separable, locally convex, complete linear space is homeomorphic to the Hilbert space ℓ_2.*

The papers that led to this theorem are Kadec [13], [14], Anderson [4] and Bessaga and Pelczynski [6], [7]. Note also Torunczyk's generalization in [19]: two Banach spaces are homeomorphic if and only if they have the same density character.

In the present work our interest is in the uniform classification of $L_p(\mu)$ spaces. Combining results of Lindenstrauss [15] and Enflo [10] we get

Theorem 2: *An infinite dimensional $L_{p_1}(\mu_1)$ is not uniformly homeomorphic to $L_{p_2}(\mu_2)$ if $p_1 \neq p_2$, $1 \leq p_i < \infty$.*

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The following two theorems go somewhat in the other direction and, especially, Theorem 4 is of particular contrast to Theorem 2.

Theorem 3: (Aharoni [1]) *For* $1 \leq p \leq 2$, $L_p(0,1)$ *(respectively* ℓ_p*) is uniformly homeomorphic to a bounded subset of itself.*

Theorem 4: (Aharoni [1]) *For* $1 \leq p \leq 2$, $1 \leq q < \infty$, $L_p(0,1)$ *is uniformly homeomorphic to a bounded subset of* ℓ_q *(and therefore to a subset of* $L_q(0,1)$*).*

Surprising, then, are the next two theorems. They are due to Aharoni, Maurey and Mityagin and are to be found in [2].

Theorem 5: *For* $2 < p < \infty$, $L_p(0,1)$ *(respectively* ℓ_p*) is not uniformly homeomorphic to a bounded subset of itself.*

Theorem 6: *For* $2 < p < \infty$, $1 \leq q \leq 2$, *a* $L_p(\mu)$ *space is not uniformly homeomorphic to any subset of a* $L_q(\mu)$ *space.*

The following question is still open.

Open Problem 1: For $2 < p < \infty$, $2 < q < \infty$, is $L_p(0,1)$ uniformly homeomorphic to a bounded subset of ℓ_q?

Questions of this genre may be found in Lindenstrauss [15] and Enflo [11].

The following theorem gives examples of uniformly homeomorphic Banach spaces which are not isomorphic.

Theorem 7: *Let* $1 \leq p,\ q,\ p_n < \infty$ *be such that* $p_n \to p$ *then*

$$(\sum \oplus \ell_{p_n})_q \quad \textit{is uniformly homeomorphic to} \quad \ell_p \oplus_q (\sum \oplus \ell_{p_n})_q.$$

The case $p = 1$ is due to Ribe [18] and the generalization was obtained by Aharoni and Lindenstrauss [3]. Note that by taking $p = 1$, $q > 1$ and $p_n > 1$ (for all n) we obtain a reflexive Banach space which is uniformly homeomorphic to a non-reflexive Banach space.

The following result is due to Enflo (unpublished).

Theorem 8: $L_1(0,1)$ *and* ℓ_1 *are not uniformly homeomorphic.*

Enflo's proof used the following basic facts:

1. A uniformly continuous map T from a Banach space into a metric space satisfies a first order Lipschitz condition for large distances i.e., given $\delta > 0$, we can find a C so that
$$d(Tx, Ty) \leq C\|x - y\| \text{ whenever } \|x - y\| \geq \delta.$$
2. If $x \neq y$ in $L_1(0,1)$ we can find a sequence of metric midpoints (x_n) such that $\|x_j - x_k\| = \frac{1}{2}\|x - y\|$ whenever $j \neq k$.

Using these facts Enflo showed that a uniformly continuous bijection $T : L_1(0,1) \to \ell_1$ will map metric midpoints between (suitably chosen) x, y in $L_1(0,1)$ to "almost" metric midpoints between Tx, Ty in ℓ_1 and, further, deduced that T^{-1} cannot be uniformly continuous. Benyamini's survey [5] — on the uniform classification of Banach spaces — includes a proof of Theorem 8.

Bourgain [9] generalizes Enflo's midpoint argument to obtain

Theorem 9: *$L_p(0,1)$ and ℓ_p are not uniformly homeomorphic for $1 \leq p < 2$.*

We have the long standing

Open Problem 2: Are $L_p(0,1)$ and ℓ_p uniformly homeomorphic for $p > 2$?

So far we have been addressing the classical Banach spaces $L_p(\mu)$, $1 \leq p < \infty$. We should also like to consider the F-spaces $L_p(\mu)$, $0 < p < 1$. The usual metric on such a space is given by
$$d(f,g) := \int |f - g|^p d\mu.$$

Theorem 6.2.1 in Enflo [12] says that: if a locally bounded linear space is uniformly homeomorphic to a Banach space with roundness > 1, then it is a normable space.

An immediate corollary, for example, is

Theorem 10: *$L_p(0,1)$ is not uniformly homeomorphic to $L_q(0,1)$ if $0 < p < 1$, $1 < q < \infty$.*

For $0 < p < 1$ the analysis of uniformly continuous maps out of ℓ_p is impaired if one uses the usual metric. In Weston [20], by the introduction of a uniformly equivalent metric on $\ell_p (0 < p < 1)$, Enflo's midpoint strategy is again exploited to obtain

Theorem 11: *For $0 < p, q \leq 1$ the real F-spaces $L_p(0,1)$ and ℓ_q are not uniformly homeomorphic.*

At this point it is relevant to note that Theorem 1 and the subsequent remark about unit balls is in fact true for all $0 < p,\ q < \infty$, the understanding being that for $0 < p,\ q < 1$ we are dealing with F-spaces. In [17] Mazur introduced the bijections

$$M_{p,q} : L_p(0,1) \to L_q(0,1) : f \mapsto (\operatorname{sign} f)|f|^{p/q}$$
$$m_{p,q} : \ell_p \to \ell_q : (a_j) \mapsto ((\operatorname{sign} a_j)|a_j|^{p/q})$$

and we have, for example,

Theorem 12: *For $0 < p, q < \infty$ the unit balls $B(L_p(0,1))$, $B(L_q(0,1))$ and $B(\ell_q)$ are uniformly homeomorphic. Indeed, we have the following estimates,*

(a) *For* $0 < p < 1 \leq q < \infty$

$$\|M_{p,q}(f) - M_{p,q}(g)\| \leq 2d(f,g)^{1/q} \text{ for all } f,g \in L_p(0,1)$$

whilst

$$d(M_{q,p}(f),\ M_{q,p}(g)) \leq 2^{q-p}(\frac{q}{p})^p \|f-g\|^p \text{ for all } f,g \in B(L_q(0,1)).$$

(b) *For* $0 < p \leq q < 1$

$$d(M_{p,q}(f),\ M_{p,q}(g)) \leq 2^q d(f,g) \text{ for all } f,g \in L_p(0,1)$$

whilst

$$d(M_{q,p}(f),\ M_{q,p}(g)) \leq 2^{\frac{q-p}{p}}(\frac{q}{p})^p d(f,g)^{p/q} \text{ for all } f,g \in B(L_q(0,1)).$$

(c) *For* $1 \leq p \leq q < \infty$

$$\|(M_{p,q}(f) - M_{p,q}(g)\| \leq 2\|f-g\|^{p/q} \text{ for all } f,g \in L_p(0,1).$$

whilst

$$\|(M_{q,p}(f) - M_{q,p}(g)\| \leq (\frac{q}{p})2^{\frac{q}{p-1}}\|f-g\| \text{ for all } f,g \in B(L_q(0,1)).$$

Note: The same estimates apply for $m_{p,q}$ (and its inverse $m_{q,p}$).

We should like to give a proof of (a) (the proof of (b) is similar) but first we need to recall three inequalities. The first two are from Mazur [17] and the third is standard.

1. For real numbers a and b and for $t \geq 1$ we have

$$|(\text{sign } a)|a|^{1/t} - (\text{sign } b)|b|^{1/t}|^t \leq 2^t|a-b|.$$

2. For real numbers a and b and for $t \geq 1$ we have

$$|(\text{sign } a)|a|^t - (\text{sign } b)|b|^t| \leq t|a-b|(|a|+|b|)^{t-1}.$$

3. If $0 < p < \infty$ then, setting $\gamma_p = \max(1, 2^{p-1})$,

$$|\alpha - \beta|^p \leq \gamma_p(|\alpha|^p + |\beta|^p)$$

for arbitrary (complex) numbers α and β.

Proof of Theorem 12(a): Suppose $0 < p < 1 \leq q < \infty$. Set $t := \frac{q}{p} > 1$. Given $f, g \in L_p(0,1)$ we see that

$$\begin{aligned}
&\|M_{p,q}(f) - M_{p,q}(g)\| \\
&:= (\textstyle\int_0^1 |(\text{sign } f)|f|^{p/q} - (\text{sign } g)|g|^{p/q}|^q dx)^{1/q} \\
&\leq (\textstyle\int_0^1 2^q|f-g|^p dx)^{1/q} \quad \text{by 3.} \\
&= 2d(f,g)^{1/q}.
\end{aligned}$$

Given $f, g \in B(L_q(0,1))$ we see that

$$\begin{aligned}
&d(M_{q,p}(f),\, M_{q,p}(g)) \\
&:= \textstyle\int_0^1 |(\text{sign } f)|f|^{q/p} - (\text{sign } g)|g|^{q/p}|^p dx \\
&\leq (\tfrac{q}{p})^p \textstyle\int_0^1 |f-g|^p(|f|+|g|)^{q-p} dx \quad \text{by 4.} \\
&\leq (\tfrac{q}{p})^p (\textstyle\int_0^1 |f-g|^q dx)^{p/q} (\int_0^1 (|f|+|g|)^q dx)^{\frac{q-p}{q}}
\end{aligned}$$

by applying Hölder's inequality with exponent q/p. Hence

$$\begin{aligned}
d(M_{q,p}(f),\, M_{q,p}(g)) &\leq (\tfrac{q}{p})^p \|f-g\|^p (\textstyle\int_0^1 2^{q-1}(|f|^q + |g|^q) dx)^{\frac{q-p}{q}} \text{ by 5.} \\
&\leq 2^{q-p}(\tfrac{q}{p})^p \|f-g\|^p \text{ as } f, g \in B(L_q(0,1)). \quad \square
\end{aligned}$$

In [16] Lövblom studies uniform homeomorphisms between $B(L_p(0,1))$ and $B(\ell_q)$ for $1 \leq p, q < \infty$ and the next two theorems are from this paper. But first recall that if X and Y are metric linear spaces and $T : B(X) \to B(Y)$ is a uniform homeomorphism then the modulus of continuity δ_T is defined by

$$\delta_T(\epsilon) := \sup\{d(Tx, Ty) | d(x,y) \leq \epsilon\}.$$

Theorem 13: *Let $1 \leq p < q \leq 2$ and let T be a uniform homeomorphism $B(L_p(0,1)) \to B(L_q(0,1))$, $B(L_p(0,1)) \to B(\ell_q)$ or $B(\ell_p) \to B(\ell_q)$. Then there is a constant $K > 0$ such that*

$$\delta_{T^{-1}}(\delta_T(\epsilon)) \geq K\epsilon^{p/q} \text{ for all } \epsilon \leq 1.$$

Lövblom's proof of Theorem 10 uses the fact that $L_p(\mu)$ has roundness p for $1 \leq p \leq 2$ and hence does not generalise to $p > 2$ or $0 < p < 1$.

Open Problem 3: Can Theorem 13 be established for all $0 < p, q < \infty$.

Theorem 14: *Let $1 \leq p < q < \infty$ and let T be $M_{p,q}$ or $m_{p,q}$ restricted to the appropriate unit ball. Then there exists a constant $K > 0$ such that*

$$\delta_{T^{-1}}(\delta_T(\epsilon)) \leq K\epsilon^{p/q} \text{ for all } \epsilon \leq 1.$$

Note that from the estimates in Theorem 12 it is clear that Theorem 14 holds for all $0 < p < q < \infty$. Note also that Theorem 13 is sharp in the case of uniform homeomorphisms $B(L_p(0,1)) \to B(L_q(0,1))$ or $B(\ell_p) \to B(\ell_q)$, $1 \leq p < q \leq 2$, as a result of the Theorem 12(c) estimates on the Mazur maps.

We conclude this paper with two more open problems.

Open Problem 4: Are ℓ_p and ℓ_q uniformly homeomorphic for $0 < p < q < 1$?

Open Problem 5: Are $L_p(0,1)$ and $L_q(0,1)$ uniformly homeomorphic for $0 < p < q < 1$?

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