SOME NEW APPLICATIONS OF RIESZ PRODUCTS

Gavin Brown

1. INTRODUCTION.

The results which I will describe arise from joint work with Bill Moran, and with Dani Berend, Charles Pearce and Andy Pollington. The common theme is the formalism of Riesz product measures, for these have proved to be exceptionally useful in the study of normal numbers. Elsewhere in these proceedings, Tony Dooley describes some of our joint work on the related class of G-measures (G for Gibbs) which we introduced, following work of Keane, [6], to generalize certain important features of Riesz products. I am confident that G-measures will also have significant applications – but, curiously, it is those properties of Riesz products which are sacrificed in that generalization which are the ones which matter here.

In fact the simplest case is when m denotes Haar measure on the circle $\mathbb{T}$ and we consider

$$\mu = \lim_{n\to\infty} \prod_{n=1}^{N} (1 + \cos 2\pi t^n x).m(dx),$$

for some integer t greater than 3. The limit is taken in the weak * topology (evaluation on continuous functions) and the resulting measure μ is a probability distribution i.e. is positive and has mass one. The important fact is that the Fourier transform $\mu\hat{}$ vanishes off words of the form

$$\sum_{i=1}^{M} \epsilon_i t^i, \quad \epsilon_i \in \{0, \pm 1\}$$

1991 *Mathematics Subject Classification.* 42A55.

and, on such a word, takes the value

$$\prod_{i=1}^{M}\left(\frac{1}{2}\right)^{|\epsilon_i|}.$$

2. NORMAL NUMBERS IN ONE DIMENSION.

The first application to be discussed was treated in the sequence of papers [2],[3],[4]. It took us a long time to see how simple the arguments can be made! The original motivating question was raised by Steinhaus and solved by Cassels in [5] – are there numbers normal to base 2 which are not normal to base 3? The answer is indeed yes and the result was extended to all rationally independent basis pairs by Schmidt (see [9], [10]).

Let me recall that x **is normal to base** t if, for every f in $C(\mathbb{T})$,

$$\lim_{N\to\infty}\sum_{n=1}^{N}\frac{f(t^n x)}{N}=\int f\, dm.$$

It is a consequence, therefore, of the ergodic theorem, that almost all x are t-normal. It is also straightforward to see that x is t-normal if and only if every digit (in fact every finite block of digits) occurs in the base t expansion with the correct frequency.

It is very easy to see how to adapt base t expansions to produce examples of non-normal numbers. In particular, when $t = 3$, we could be drastic and eliminate the digit 1 from all ternary expansions. The numbers produced belong to Cantor's middle third set, which, of course, has a naturally associated probability distribution – Lebesgue's singular measure on the Cantor set. Intuitively, it is reasonable to suppose that conditioning the base 3 expansion – even as violently as here – has no influence on the base 2 expansion in some generic sense. In other words, we might well conjecture that almost all numbers with respect to Lebesgue's singular measure on the triadic Cantor set are, in fact, normal in base 2. We can even go further and suggest that this will be confirmed by an estimation of Fourier transforms. That is pretty much what Cassels did (an explicit description of the work in those terms appears in [7]) and is the base upon which Schmidt also built. Our contribution was the simple technical device of replacing infinite convolutions (such as the singular Cantor measure) by Riesz products. The perturbation to produce non-normality is marginally harder to check, but the estimation of the Fourier transform is very significantly easier.

Let us suppose then that we are given positive integers s, t such that no power of s is a power of t. We define the Riesz product μ (depending on t) as in the last section. A version of the Strong Law of Large Numbers (exploiting weak correlation) shows us that

$$\frac{1}{N}\sum_{n=1}^{N}\exp(2\pi i t^n x) \to \frac{1}{2} \quad (\mu\ a.e.)$$

Using the exponentials as test-functions in the definition of normality, makes it clear to us that t- normality is violated μ almost everywhere.

According to the stratagem which I sketched, it remains to verify that s-normality occurs μ almost everywhere. The convenient tool is another variant of the Strong Law of Large Numbers which goes back to Davenport, Erdös and Le Veque but which I learned from Russ Lyons, who was probably influenced by the account in Rauzy, [8].

PROPOSITION 1. *Suppose that (X_n) is a sequence of random variables of bounded modulus and that*

$$\sum_{N=1}^{\infty} N^{-1} E\left(|Y_N|^2\right) < \infty,$$

where

$$Y_N = N^{-1}\sum_{n=1}^{N} X_n.$$

Then almost surely $Y_N \to 0$.

By choosing $X_n(x) = \exp(2\pi i r s^n x)$ and the underlying probability distribution as the Riesz product μ based on the powers of t, as in the introduction, we translate Proposition 1 into the following practical criterion:

PROPOSITION 2. *Suppose that, for all integers r,*

$$\sum_{N=1}^{\infty} N^{-3} \sum_{k=1}^{N}\sum_{j=1}^{k-1} |\hat{\mu}(rs^k - rs^j)| < \infty.$$

Then μ almost all numbers are s-normal.

Now we have successfully converted a number theory problem to Fourier analysis. Let us now convert it back to number theory by simply counting the number of times that $\hat{\mu}$ is non-zero. In fact let A_N denote the number of $k \leq N$ such that for some j with $0 \leq j \leq k - \log N$ and some $\epsilon_i \in \{0, \pm 1\}$, the following equation holds:

$$r(s^k - s^j) = \sum_{i=1}^{a} \epsilon_i t^i. \tag{1}$$

In view of Proposition 2, it will follow that μ almost all numbers are s-normal provided that

$$\sum N^{-2} A_N < \infty. \tag{2}$$

At this point it is appropriate to recall that s-normality and s^K-normality coincide, where K is a positive integer. Moreover we intend to exploit Baker's estimates for linear forms in logarithms by using the existence of a constant B such that

$$|n \log s - m \log t| > B^{-\log N}, \tag{3}$$

whenever $0 < \max(|n|, |m|) \leq N$. Note that we can (and do) replace s, t by suitable powers s^K, t^K to ensure

$$\min(s, t) \geq B^2, \tag{4}$$

without alteration to (3).

Observe that, from (1), we have

$$s^k/t^a = r^{-1}(1 - s^{j-k})^{-1} \sum_{i=1}^{a} \epsilon_i t^{i-a}. \tag{5}$$

T he first term on the right side of (5) is constant and the second factor has order of magnitude $1 + O(B^{-2\log N})$. Thus, for any **fixed** set of ϵ_i, two solutions s^k/t^a and $s^{k'}/t^{a'}$ would yield the estimate

$$|(k \log s - a \log t) - (k' \log s - a' \log t)| = O(B^{-2\log N}), \tag{6}$$

and (6) contradicts (3).

It follows that we need concentrate only on the variation of ϵ_i in the expression

$$\sum_{i=1}^{a} \epsilon_i t^{i-a}.$$

For definiteness let us suppose $\epsilon_a = 1$. Let us also rewrite the expression as

$$1 + \sum_{i=1}^{\infty} \eta_i t^{-1}, \quad \eta_i \in \{0, \pm 1\}. \tag{7}$$

Let us now fix the first [log N] choices of η_i in (7). Thus, after making at most $3^{\log N}$ choices the expression labelled (7) becomes

$$(1+C)\left(1+(1+C)^{-1}\sum_{i=[\log N]+1}^{\infty}\epsilon_i t^{-i}\right), \tag{8}$$

where

$$C=\sum_{i=1}^{[\log N]}\epsilon_i t^{-i}.$$

It is now obvious that the quantity labelled (8) is a constant multiple of a number whose magnitude is $1+O(B^{-2\log N})$. Again we threaten to violate (3) and so we deduce that

$$A_N \leq 3^{\log N} = N^{\log 3}.$$

Thus (2) is verified, and we have a complete proof that μ almost all numbers are indeed s- normal.

In combination with some well-known and elementary facts about normal numbers these comments establish the basic theorem.

THEOREM 1. *Let s,t be integers strictly greater than one. Either some integral power of s is an integral power of t, in which case s-normality and t-normality coincide; or else s,t are multiplicatively independent, in which case there are infinitely many s-normal numbers which are not t-normal and vice-versa.*

The basic theorem can be elaborated in various ways. It can be shown, for example, that if S,T are (possibly infinite) collections of integer bases such that every base in S is rationally independent of all bases in T; then every real number is the sum of two numbers both of which are s-normal for all s in S and which fail to be t-normal for every t in T.

3. NON-INTEGER BASES.

With some simple adaptation, Riesz products can be applied to handle also non-integer bases. A cavalier remark in [4] adds to this the observation that normality to base θ coincides with normality to base θ^K, in general. In fact, that was not known, although to the best of my knowledge no counter-examples have been produced until now. (Fortunately all the main assertions of [4] remain true and can be proved by alternative means). Berend, Moran, Pollington and I now have a body of results which follow the principle that any assertion about powers of bases which is not obviously true is, in fact, false! Let me sketch one of these.

As usual, in this area it is very difficult to make statements about individual numbers e.g. is e normal in base 10, is $\sqrt{10}$ normal in base 10, does the decimal expansion of $\sqrt{10}$ have infinitely many ones? We may hope to do a little better by specifying the base but allowing generic examples of normal numbers. Even that seems difficult, so allow me to finesse part of the difficulty by inserting a hypothesis ({ } denotes fractional part):
(H) *for every positive integer k there exists some positive integer n such that* $\{k10^n\sqrt{10}\} \notin [-1/9,\ 1/9]$.

It is evident that (H) is a much weaker statement than the assertion that $\sqrt{10}$ is normal to base 10. It would be possible to work with an arbitrary small number in place of $\frac{1}{9}$, but that number is convenient for the exposition. In fact we require that the decimal expansion of $k\sqrt{10}$ cannot be achieved using only the three digits $0, 1, -1$. A simple size estimate for the tail shows that we may deduce this from (H).

We shall consider normality to base $\sqrt{10}$. There is a simple result (strangely it is one which does not appear to have been discovered before) to the effect that normality to some root of an integer implies normality to that integer base. Let me take that for granted and show you the interesting fact that, under (H), infinitely many numbers normal to base 10 fail to be normal to base $\sqrt{10}$.

The Riesz product to be used is given by

$$d\mu = \lim_{N\to\infty} \prod_{n=1}^{N} \left(1 + \cos 2\pi(\sqrt{10})^{2n+1}x\right) \left(\frac{1-\cos x}{x^2}\right) \frac{dx}{\pi}. \tag{9}$$

The kernel $(1 - \cos x)/x^2$ has the effect of producing straight-line interpolation of the Fourier transform between words of the form

$$\sum_{i=0}^{a} \epsilon_i \left(\sqrt{10}\right)^{2i+1} \qquad \epsilon_i \in \{0, \pm 1\}.$$

In order to imitate the argument of section 2, we consider relations of the type

$$r(10^n - 10^m) \sim \sum_{i=0}^{a} \epsilon_i \left(\sqrt{10}\right)^{2i+1}, \tag{10}$$

where $\sim$ denotes "is within distance one of," and r is an integer. Simple rearrangement of (10) shows that we require $r\sqrt{10}$ to be very close to $\sum_{i=0}^{a} \epsilon_i 10^{i-n+1}$. Here r is fixed, and n, a, ϵ_i are allowed to vary. It follows that the decimal expansion of $r\sqrt{10}$ uses only the digits 0,1,-1 and this contradicts (H).

We have now sketched the harder part of the proof of the main theorem of this section.

THEOREM 2. *Every number normal to base $\sqrt{10}$ is normal to base 10 but, under (H), there are uncountably many numbers normal to base 10 which are not normal to base $\sqrt{10}$.*

We have several results related to the theorem. In particular we can demonstrate the existence of bases θ for which θ-normality does **not** imply θ^2- normality. Indeed, we show that generically (in the sense of category or of measure), for almost all bases, there is no implication in either direction between θ-normality and θ^2-normality.

4. MULTIDIMENSIONAL RESULTS.

In 1964, Schmidt extended the basic normality results to integer matrices. Let me follow his definition of the somewhat wider class of **almost integer matrices.** These are $n \times n$ invertible matrices with rational entries, all of whose eigenvalues are algebraic integers. We shall consider with the subclass of so-called **almost ergodic** matrices which have no roots of unity as eigenvectors. Such a matrix S has a **denominator** d, which is an integer such that dS^n has integer entries for every $n = 1, 2, \ldots$. Moreover, the dynamical system (S, m) is ergodic in the sense that, for m almost all vectors x in $\mathbf{T}^n$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(S^k x) = \int f \, dm, \tag{10}$$

for $f \in C(\mathbf{T}^n)$ and m Haar measure on $\mathbf{T}^n$. Vectors x for which (10) holds are called S-normal. It is relatively easy to see that if S, T are almost ergodic matrices and $S^p = T^q$, for integers p, q, then S-normality and T-normality coincide. Very much more interesting is the converse question. Schmidt showed in [11] that if $ST = TS$ and every eigenvalue of S has modulus greater than one, then some integer power of S is an integer power of T. Moran and I have now removed the hypothesis on the eigenvalues. It follows that there is a complete extension of the one-dimensional result to commuting (almost) ergodic integer matrices.

To give some idea of this work let me sketch the argument for the (very) special case in which the algebra $\mathcal{A}(S, T)$ generated by S, T is irreducible.

The first step is to modify the basic Riesz product and that is unrelated to irreducibility. In fact we take

$$\mu = \lim_{K\to\infty} \prod_{k=1}^{K} (1 + \cos 2\pi\alpha T^k x)m, \tag{11}$$

where α is an integer vector multiplied by the denominator of T and m is Haar measure on $\mathbb{T}^n$. In order that these products be well-defined we require dissociateness in the sense that the equations

$$\sum_{i=1}^{K} \epsilon_i \alpha T^i = 0 \quad (\epsilon_i \in \{0, \pm 1, \pm 2\}) \tag{12}$$

force $\epsilon_i = 0$ for all i.

That can be achieved for two reasons. In the first place we may replace T by a suitable power because for almost integer matrices T-normality and T^p-normality coincide. Secondly **some** eigenvalue of T must have modulus greater than one. (If all eigenvalues of T had modulus one then, by Dirichlet's theorem, some root of unity would be an eigenvalue, in contradiction of ergodicity. Then this existence of a denominator for T demonstrates that not all eigenvalues can have modulus not greater than one.) It is entirely believable that we may project on an appropriate eigenspace to verify the triviality of (12).

Now that the countable family of μ has been constructed according to (11), we note that μ almost all numbers fail to be T-normal. It remains to check that μ almost all numbers are in fact S-normal. To this end, let us suppose that $\mathbb{Q}^n$ has no invariant subspace for the algebra $\mathcal{A}(S, T)$ and that for each α in $\mathbb{Q}^n$ there exists β in $\mathbb{Q}^n$ with $\sum N^{-2} A_N(\beta) = \infty$, where $A_N(\beta)$ is the number of $k \leq N$ such that for some j with

$0 \le j \le k - \log N$ and some $\epsilon_1, \epsilon_2, \cdots \epsilon_m$ ($\epsilon_m \in \{0, \pm 1\}$),

$$\beta(S^k - S^j) = \sum_{i=1}^{m} \epsilon_i \alpha T^i. \tag{13}$$

We suppose further that for any eigenvalues σ of S and τ of T with $\sigma^p \ne \tau^q$ and $\max\,(|p|, |q|) \le N$, the inequality

$$|p \log \sigma - q \log \tau| > B^{-\log N}$$

holds and, moreover, that every eigenvalue of S or T whose modulus exceeds one, has in fact modulus greater than B^2.

LEMMA 1. *There exists an $n \times n$ matrix U over $\mathbb{Q}$ such that $U \in \mathcal{A}(S, T)$ and such that the number A_N of k for which*

$$U(S^k - S^j) = \sum_{i=1}^{m} \epsilon_i T^i \tag{14}$$

for some $0 \le j \le k - \log N$ and $\epsilon_i \in \{0, \pm 1\}$ satisfies $\sum N^{-2} A_N = \infty$.

PROOF. Choose some cyclic vector α_0 (in $\mathbb{Q}^n$) for $\mathcal{A}(S, T)$ and note that there exists β_0 such that

$$\beta_0(S^k - S^j) = \sum_{i=1}^{m} \epsilon_i \alpha_0 T^i \tag{15}$$

has $G_N(\beta_0)$ solutions k with corresponding j, m, ϵ_i. Multiplying both sides of (15) by arbitrary A in $\mathcal{A}(S, T)$ and using cyclicity and commutativity we obtain

$$\beta(S^k - S^j) = \sum_{i=1}^{m} \epsilon_i \alpha T^i \tag{16}$$

where α runs through all $\mathbb{Q}^n$. For any particular choice of k, j, ϵ_i we may define

$$U = (S^k - S^j)^{-1} \sum_{i=1}^{m} \epsilon_i T^i, \tag{17}$$

using ergodicity of S, and then we find

$$\beta = \alpha U. \tag{18}$$

The combination of (16) and (18) now yields (14) for all compatible choices of k, j, m, ϵ_i.

LEMMA 2. *There is a field automorphism $\phi : \mathcal{A}(S,T) \to F$ where F is a finite extension of $\mathbb{Q}$ and such that $\phi(A)$ is an eigenvalue of A for every A in $\mathcal{A}(S,T)$. Moreover given any fixed eigenvalue σ of S, we may choose $\phi(S) = \sigma$.*

PROOF. Every non-zero A in $\mathcal{A}(S,T)$ is invertible or else its kernel would be a proper invariant subspace and this gives the existence of the automorphism ϕ. The Cayley-Hamilton theorem shows that $\phi(A)$ is an eigenvalue of A. Because the minimal polynomial m_S of S is irreducible, the Galois group acts transitively on the roots – hence we may fix $\phi(S) = \sigma$.

THEOREM 3. *Under the hypotheses of this section, there are integers p, q such that $S^p = T^q$.*

PROOF. Choose the map ϕ of the last lemma so that $|\phi(S)| > B^2$ and evaluate both sides of (14) under ϕ. This produces an analogue of (1) with $\phi(U) = r,\ \phi(S) = s,\ \phi(T) = t$. We can apply the one-dimensional argument to get $s^p = t^p$, and the result follows because ϕ is an automorphism.

The result just sketched forms the basis of an inductive proof of the full commutative theorem which will appear in [1].

REFERENCES.

1. G. Brown and W. Moran, *Schmidt's conjecture on normality for commuting matrices,* to appear.

2. G. Brown, W. Moran, and C.E.M. Pearce, *Riesz products and normal numbers*, J. London Math. Soc. 32(1985), 12-18.

3. G. Brown, W. Moran and C.E.M. Pearce, *A decomposition theorem in which the summands have prescribed normality properties,* J. of Number Theory 24(1986), 259-271.

4. G. Brown, W. Moran and C.E.M. Pearce, *Riesz products, Hausdorff dimension and normal numbers*, Math. Proc. Camb. Phil. Soc. 101(1987), 529-540.

5. J.W.S. Cassels, *On a problem of Steinhaus about normal numbers,* Colloq. Math. 7(1059), 95-101.

6. M.S. Keane, *Strongly mixing g-measures,* Invent. Math. 16(1971), 309-324.

7. C.E.M. Pearce and M.S. Keane, *On normal numbers* J. Aust. Math. Soc. (A) 32(1982), 79-87.

8. G. Rauzy, "Propriétés statistiques de suites arithmétiques," PUF 1976.

9. W.M. Schmidt, *On normal numbers,* Pac. J. Math. 10(1960), 661-672.

10. W.M. Schmidt, *Über die Normalität von Zahlen zu verschiedenen Basen,* Acta Arith. 7(1962), 299- 301.

11. W.M. Schmidt, *Normalität bezuglich Matrizen,* J. für die reine und angewandte Math. 214/5(1964), 227-260.

School of Mathematics,
University of New South Wales,
P.O. Box 1, Kensington
Australia 2033.