L^2 BOUNDS FOR NORMAL DERIVATIVES OF DIRICHLET EIGENFUNCTIONS

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ABSTRACT. Suppose that M is a compact Riemannian manifold with boundary and u is an L^2-normalized Dirichlet eigenfunction with eigenvalue λ. Let ψ be its normal derivative at the boundary. Scaling considerations lead one to expect that the L^2 norm of ψ will grow as $\lambda^{1/2}$ as $\lambda \to \infty$. We sketch proofs of an upper bound of the form $\|\psi\|_2^2 \le C\lambda$ for any Riemannian manifold, and a lower bound $c\lambda \le \|\psi\|_2^2$ provided that M has no trapped geodesics (see the main Theorem for a precise statement). Here c and C are positive constants that depend on M, but not on λ. Full details will appear in [3].

1. INTRODUCTION

Let M be a smooth compact Riemannian manifold with smooth boundary $\partial M = Y$ (for example, the closure of a smooth bounded domain in Euclidean space). Let $H = -\Delta_M$ be minus the Dirichlet Laplacian on M. As is well known, H has discrete spectrum $0 < \lambda_1 < \lambda_2 \le \lambda_3 \cdots \to \infty$.

Let u_j be an L^2-normalized eigenfunction corresponding to λ_j, and let ψ_j be the normal derivative of u_j at the boundary. In this paper we consider the following question: do there exist constants c and C, depending on M but not on j, such that

$$c\lambda_j \le \|\psi_j\|^2_{L^2(Y)} \le C\lambda_j \ \ ?$$

This question was posed by Ozawa in [6]; he showed that a weaker version of this statement, obtained by summing over all eigenvalues in $[0, \lambda]$, is true.

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In this paper, we sketch proofs that the upper bound is always true on Riemannian manifolds, while the lower bound holds provided a condition of 'no trapped geodesics' holds. In addition, we give several examples to illustrate the link between failure of the geodesic condition and failure of the lower bound. Full details of proofs are given in [3].

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2. Examples

We begin by considering several examples. These examples show in particular that the lower bound does *not* always hold.

Example 1 — the disc. Let $M = \{x \in \mathbb{R}^2 \mid |x| < a\}$ for some $a > 0$. In this case we have an equality

$$\int_{S_1} \psi_j^2(\theta)\, d\theta = \frac{2\lambda_j}{a}.$$

This follows easily from an identity for eigenfunctions due to Rellich, which we discuss below.

Example 2 — the rectangle. Let $M = [0,a]\times[0,b]$, where $a \le b$. Then it is a simple matter to write down the eigenfunctions, by separating variables. A computation gives

$$\frac{4}{b}\lambda \le \|\psi\|_2^2 \le \frac{4}{a}\lambda,$$

and these bounds are the best possible.

Example 3 — the cylinder. Let $M = [0,\pi] \times S^1_{2\pi}$, the product of an interval of length π with a circle of length 2π. Then eigenfunctions take the form $\sin(mx)e^{in\theta}$. The upper bound

$$\|\psi\|_2^2 \le \frac{4}{\pi}\lambda$$

holds, but by holding m fixed and sending n to infinity, we see that the best lower bound is $O(1)$.

Example 4 — the hemisphere. Let M be the hemisphere

$$M = \{x \in \mathbb{R}^3 \mid |x| = 1,\ x\cdot(0,0,1) \ge 0\}.$$

In this case, the eigenfunctions are given by those spherical harmonics which are odd under reflection in the (x_1, x_2) plane, namely, spherical harmonics

$$u = c_{lm} Y_{lm} = c_{lm} e^{im\phi} P_{lm}(\cos\theta), \quad \lambda = l(l+1),$$

where $-l \leq m \leq l$ and $l - m$ is odd. (Here, we are using spherical polar coordinates where $(x_1, x_2, x_3) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$.) Let us consider the case when $m = l - 1$. Then the eigenfunction in this case is $u_l = ce^{i(l-1)\theta}\cos\theta(\sin\theta)^{l-1}$. A short computation gives

$$\|\psi\|_2^2 \sim l^{3/2} \sim \lambda^{3/4}, \quad l \to \infty,$$

for this class of eigenfunctions. Hence there is no nontrivial lower bound for the hemisphere.

3. Main theorem

The upper bound holds in all the examples in the previous section, but the lower bound fails for the last two. To see more heuristically why the lower bound fails, it helps to consider a more dynamic picture, by considering the wave equation

$$\frac{\partial^2 v(x,t)}{\partial t^2} = -Hv$$

on the cylinder (Example 3). If u is an eigenfunction with eigenvalue λ, then $v = e^{i\sqrt{\lambda}t}u$ is a solution to the wave equation. Thus in the case of the cylinder, a particular solution to the wave equation is

$$2i\sin(mx)e^{in\theta}e^{i\sqrt{m^2+n^2}t} = e^{imx}e^{in\theta}e^{i\sqrt{m^2+n^2}t} - e^{-imx}e^{in\theta}e^{i\sqrt{m^2+n^2}t}.$$

The wavefronts are at $\pm mx + n\theta =$ constant, and energy moves 'normal' to the wavefronts. When we hold m fixed and send n to infinity, the energy is moving more and more along lines (that is, geodesics) where x is constant, and so does not 'reach' the boundary. Thus, the failure seems to be related to the existence of a family of 'trapped' geodesics in the interior of the manifold, which never reach the boundary.

In the case of the hemisphere, there is no geodesic trapped in the interior, but any Riemannian extension N of M would have a trapped geodesic, namely the boundary of M. Note that the lower bound is not violated as severely for the hemisphere, which is consistent with it being a borderline case.

Our main result is that the heuristic above is correct:

Theorem 3.1. *Let M be a smooth compact Riemannian manifold with boundary. Then the upper bound holds for some C independent of j.*

The lower bound holds provided that M can be embedded in the interior of a compact manifold with boundary, N, of the same dimension, such that every geodesic in M eventually meets the boundary of N. In particular, the lower bound holds if M is a subdomain of Euclidean space.

In Section 7, we give some further examples which show that the degree of failure of the bounds is related to the extent to which geodesics are trapped.

4. Upper bound

Our proof is based on the following Lemma which we call a Rellich-type estimate.

Lemma **4.1.** *Let u be a Dirichlet eigenfunction of H. Then for any differential operator A,*

$$\int_M \langle u, [H, A]u \rangle \, dg = \int_Y \frac{\partial u}{\partial \nu} Au \, d\sigma. \tag{4.1}$$

Proof. Let λ be the eigenvalue corresponding to u. We write $[H, A] = [H - \lambda, A]$ and use the fact that $(H - \lambda)u = 0$ to write the integral over M as

$$\int_M \langle (H - \lambda)u, Au \rangle - \langle u, (H - \lambda)Au \rangle \, dg.$$

Then we use Green's formula, and the fact that u vanishes at the boundary, to deduce (4.1). □

The upper bound is now easily deduced. Let us choose coordinates (r, y) near the boundary of M, where r is distance to the boundary (which is a smooth function for $r < \delta$, for some sufficiently small $\delta > 0$) and y are local coordinates on $Y = \partial M$, extended to be constant along geodesics perpendicular to the boundary. Let $\chi(r)$ be a smooth function which is supported in $[0, \delta/2]$ and with $\chi(0) = 1$, and let

$$A = \chi(r)\partial_r.$$

Then the right hand side of (4.1) becomes $\|\psi\|_2^2$, while the left hand side is bounded by

$$C \int_M \big(|\nabla u|^2 + 1 \big) = C(\lambda + 1),$$

proving the upper bound.

Remark. Notice that this argument actually gives a bound of

$$C_\epsilon \|Hu\|_{L^2(\{r \le \epsilon\})} = C_\epsilon \lambda \|u\|_{L^2(\{r \le \epsilon\})}$$

for any $\epsilon > 0$.

5. Lower bound for Euclidean domains

In the case of subdomains of Euclidean space, there is a very simple proof of the lower bound based on Lemma 4.1. We set

$$A = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}.$$

Then $[H, A] = 2H$, so now the left hand side is 2λ, and we obtain the identity

$$2\lambda = \int\limits_Y \frac{\partial u}{\partial \nu} Au \, d\sigma = \int\limits_Y \nu \cdot x \, \big(\frac{\partial u}{\partial \nu}\big)^2 \, d\sigma$$

which immediately implies the lower bound. (Rellich published this identity in 1940 [7].)

6. The lower bound in general

It is considerably harder to prove the lower bound for general manifolds satisfying the ‘no trapped geodesic’ condition of Theorem 3.1. We use the method for Euclidean domains as a guide, and seek a first order operator A having a positive commutator with H. It is impossible to find a vector field A with this property, in general (we will show this later), so we look for a first order pseudodifferential operator. A is essentially determined by its principal symbol, a, a function on S^*M, the cosphere bundle of M.

The principal symbol of $i[H, A]$ is given by $V_h(a)$, where V_h is the Hamilton vector field of the symbol of H. Recall that H is minus the Laplacian. It is well known that V_h is the generator of geodesic flow on S^*M. We want $i[H, A]$ positive, which amounts to finding a smooth function a on S^*M which is *increasing along all geodesics.* Note that this is clearly impossible if trapped/periodic geodesics are present.

Given the ‘no trapped geodesics’ condition on M in the theorem, such an A can easily be constructed. To show this, we first observe that, given a geodesic γ on M, one can construct an A on the larger manifold $N \supset M$, properly supported in the interior of N, such that the principal symbol of $i[H, A]$ is everywhere nonnegative, and strictly positive in a neighbourhood of γ (at least that part of it lying over M). To do *this* we simply define the symbol a of A to be linearly increasing along γ, extend it in a natural way, and then cut off. Finally, a compactness argument shows that one can add together finitely many such operators to produce one where the principal symbol of $i[H, A]$ is strictly positive on S^*M. We note that it is easy to arrange that, in addition, A satisfies the *transmission condition* (see [5], section 18.2)); for the principal symbol a of A, this condition is simply that a is odd: $a(x, -\xi) = -a(x, \xi)$.

We then follow the strategy of the previous proof. First, we need a version of Lemma 4.1 which is valid for pseudodifferential operators. For first order pseudodifferential operators A satisfying the transmission condition, with symbol a, we have the following

Lemma **6.1.** *Let u be a Dirichlet eigenfunction for H. Then*

$$\int\limits_M \langle u, [H, A]u \rangle \, dg = 2 \ \mathrm{Im} \int\limits_Y \frac{\partial u}{\partial \nu} A u \, d\sigma - \int\limits_Y \Big(\frac{\partial u}{\partial \nu}\Big)^2 c \, d\sigma, \tag{6.1}$$

where $c(y) = \lim_{\rho \to \infty} \rho^{-1} a(0, y, \rho, 0)$.

This is just as good as (4.1) for our purposes. We then follow the strategy of the previous proof. The left hand side causes few problems; it is rather easy to show that the left hand side is at least as big as a constant times λ. The right hand side, though, is more difficult. It is sufficient to show that

$$\|Au\|_{L^2(Y)} \le C\sqrt{\lambda}, \tag{6.2}$$

since then we can estimate the first term on the right hand side of (6.1) by

$$\int_Y \langle \psi, Au \rangle d\sigma \le C(\epsilon) \|\psi\|_2^2 + \epsilon\lambda, \tag{6.3}$$

and the $\epsilon\lambda$ term may then be taken to the left hand side. The estimate (6.2) is nontrivial, since A is a nonlocal operator; the restriction of Au to the boundary depends on values of u in the interior of M.

Our proof of (6.2) uses ideas from both harmonic analysis and microlocal analysis. There are two main ingredients. One is a uniform estimate for u near the boundary:

$$\int_{Y_r} u_j^2 \, d\sigma(y) \le C\lambda_j r^2 \text{ for all } r \in [0, \delta] \tag{6.4}$$

where Y_r is the set of points at distance r from the boundary, and both C and δ are *independent of j*. We prove this by looking at the quantity

$$L(r) = \int_{Y_r} u^2 d\sigma_r.$$

Here $d\sigma_r$ is the measure on Y_r induced by the Riemannian measure on M. The eigenfunction equation for u leads to the differential inequality

$$L'' \geq \frac{(L')^2}{L} - C\lambda, \tag{6.5}$$

for $L(r)$, for some constant C depending only on the manifold M. This differential inequality implies exponential increase of $L(r)$ if it ever happens that $L'(r)^2 \geq 2C\lambda L(r)$. However, this would contradict the bound

$$\int_0^\delta L(r)dr \leq 1$$

which follows from the L^2 normalization of u. Hence $L'(r)^2 \leq 2C\lambda L(r)$, and from this we deduce (6.4).

The second ingredient is expressing $Au_{|Y}$ as an integral of kernels A_r acting on the functions $u_{|Y_r}$. Here, A_s has kernel $A_s(y, y')$ which is the restriction of the kernel of A to $(r = 0, y, r' = s, y')$. Then results of Boutet de Monvel [1] and Višik and Eškin [8] give bounds on the L^2 operator norm of A_r, acting from $L^2(Y_r)$ to $L^2(Y)$. Their result is if B_r is an operator of order $-1 + k$, satisfying the transmission condition, where $k = 0, 1, 2, \dots$, then there is a bound on the operator norm of B_r of the form Cr^{-k}, where C is independent of r. In particular, for $k = 0$, the B_r are uniformly bounded as $r \to 0$.

Unfortunately, this result does not quite give the result directly, since if we combine the operator bound Cr^{-2} for A_r and (6.4) for u and integrate in r, we encounter a logarithmic divergence. However, a small modification of this approach does the trick. If we let H^{-1} denote the inverse of the operator H on N, with Dirichlet boundary conditions, then H^{-1} is a pseudodifferential operator of order -2 when localized away from the boundary of N. Moreover, it satisfies the transmission condition. Since $Hu = \psi\delta_Y + \lambda u$, we may write

$$Au_{|Y} = AH^{-1}\big(\psi\delta_Y + \lambda u\big).$$

The first term is given by $(AH^{-1})_0\psi$ which is a L^2-bounded operator applied to ψ. By the upper bound for ψ, we see that this term satisfies (6.2). For the second term, we write u as the sum of a 'close' part and a 'far' part with respect to the boundary, relative to the length scale $\lambda^{-1/2}$. We can integrate in r as described above to get the bound for the close part, while a similar argument works for the far part. This completes the proof of the lower bound.

7. Periodic geodesics

In this section, we explore the relationship between trapped geodesics and the failure of the lower bound by considering two examples. The first is a hyperbolic cylinder, that is, the manifold $M = [-a, a] \times S^1_\theta$, with metric

$$g = dx^2 + (\cosh x)^2 d\theta^2.$$

This has a single periodic geodesic, at $x = 0$, which is unstable (geodesics are always unstable in manfolds with negative curvature). Let $\epsilon > 0$ be given, and let F be the set $\{|x| \geq \epsilon\}$; thus F excludes a neighbourhood of the trapped geodesic. Then Colin de Verdière and Parisse showed that there is a sequence of normalized Dirichlet eigenfunctions u_{k_j}, $k_j \to \infty$ as $j \to \infty$, such that

$$\int_F |u_{k_j}|^2 dg \sim \frac{1}{\log \lambda_{k_j}}, j \to \infty.$$

Then, applying our *upper* bound argument, which only uses the norm of eigenfunctions in a neighbourhood of the boundary (see the remark at the end of Section 4), we see that for some C

$$\|\psi_{k_j}\|_2^2 \leq C \frac{\lambda_{k_j}}{\log \lambda_{k_j}}.$$

This shows that the lower bound is violated. However, we can actually show that this is the true order of growth of $\|\psi_{k_j}\|_2^2$. Indeed, we shall show that for every normalized eigenfunction u,

$$\|\psi\|_2^2 \geq c \frac{\lambda}{\log \lambda}. \tag{7.1}$$

To show (7.1), we first observe that for every eigenfunction u, we have

$$\int_F |u|^2 dg \geq \frac{C}{\log \lambda}. \tag{7.2}$$

This follows by looking at a basis of eigenfunctions $u_{l,\lambda}$ of the form

$$u_{l,\lambda} = e^{il\theta} (\cosh x)^{-1/2} v_{l,\lambda}.$$

Then $v_{l,\lambda}$ satisfies

$$\Big(D_x^2 + \frac{1}{2} - \frac{1}{4} (\tanh x)^2 + l^2 (\operatorname{sech} x)^2 \Big) v_{l,\lambda} = \lambda v_{l,\lambda},$$

which we rewrite in the form

$$\big(h^2 D_x^2 + V \big) v_{l,\lambda} = E v_{l,\lambda}, \tag{7.3}$$

with

$$h^{-2} = l^2 + \frac{1}{4}, \quad V(x) = (\operatorname{sech} x)^2 - 1, \quad E = \frac{\lambda - l^2 - 1/2}{l^2 + 1/4}.$$

Then Theorem 20 of [2] shows that for $|E| < C$, (7.2) holds. For $E > C$, direct analysis of equation (7.3) shows that $v_{l,\lambda}$ has a uniform L^∞ bound, independent of h and E. Thus, in this case (7.2) holds *a fortiori*. In the remaining case, $E < -C < 0$, the origin is in the classically forbidden region and the result follows immediately from Agmon-type exponential decay estimates.

To complete the proof of (7.1), we construct an operator A which has a *nonnegative* commutator with H, and which is strictly positive in a neighbourhood of F. We are able to do this because of the special properties of geodesic flow on the hyperbolic cylinder. Letting G be the complement of F, and writing $F = F_+ \cup F_-$ for the two components of F, labelled according to the sign of x, a geodesic that passes from F_+, say, to G either stays over G for all subsequent time, or emerges into the region F_- and eventually reaches the boundary of M; it cannot happen that a geodesic starts in G, then moves into the set F, and returns to G. Thus, given a geodesic γ we can define an operator A with a symbol which is linearly increasing when γ is above the set F, and vanishing in some neighbourhood $U \subset\subset G$ of the periodic geodesic. As before, we can (by compactness) find a finite number of such operators whose sum has the desired property.

Thus, for a given eigenfunction u, let $M(u)$ denote the quantity on the left hand side of (7.2). If we go back to (6.1), then we find that the left hand side is at least as big as

$$c\lambda M(u) - C\lambda^{1/2} \geq c'\lambda M(u).$$

On the other hand, the argument above applied to Au shows that the right hand side is no bigger than

$$C\|\psi\|_2 \lambda^{1/2} M(u)^{1/2}.$$

We get the normalization factor $M(u)^{1/2}$ (instead of 1) since all arguments are localized near the boundary of M. The combination of these two estimates yields (7.1).

The second example we analyze is the spherical cylinder, that is, $M = (-a, a) \times S^1$, for $0 < a < 1$, with metric

$$g = dx^2 + (\cos x)^2 d\theta^2.$$

This has a periodic geodesic at $x = 0$. However, in this case, the geodesic is stable, and indeed every nearby geodesic is periodic. Thus, this case is the opposite extreme where there is an open set of periodic geodesics. We shall see that, correspondingly, the lower bound is violated in an extreme fashion.

This example may be analyzed in a similar way to the hyperbolic cylinder. We may separate variables, so there is a basis of eigenfunctions $u_{l,\lambda}$ of the form

$$e^{il\theta}(\cos x)^{-1/2} v_{l,\lambda}.$$

Then $v_{l,\lambda}$ satisfies the equation

$$\big(h^2 D_x^2 + V\big) v_{l,\lambda} = E v_{l,\lambda}, \tag{7.4}$$

where now

$$h^{-2} = l^2 - \frac{1}{4}, \quad V(x) = (\sec x)^2 - 1, \quad E = \frac{\lambda - l^2 + 1/2}{l^2 - 1/4}.$$

Here, V has a nondegenerate global mininum at $x = 0$. Let us consider a sequence of eigenfunctions u_{k_j} with $l_{k_j} \to \infty$ and $E(\lambda_{k_j}, l_{k_j}) \le E_1$, where $0 < E_1 < V(a)$, so that the boundary, $|x| = a$, is in the classically forbidden region. Then Agmon-type exponential decay estimates (see [4], chapter 3) hold, giving for some $\epsilon > 0$

$$|u_{k_j}(x,\theta)| \le C e^{-\epsilon \lambda_{k_j}^{1/2}}, \quad |x| \ge \delta > 0, \quad \text{for some } \epsilon > 0.$$

The upper bound argument then gives

$$\|\psi_{k_j}\|_2 \le C e^{-\epsilon' \lambda_{k_j}^{1/2}}, \quad \text{for some } \epsilon' > 0,$$

so we actually have exponential *decrease*, rather than $O(\lambda^{1/2})$ increase, in the L^2 norm of the normal derivative for any such sequence of eigenfunctions.

8. Vector fields are not enough

Finally, we remark that the following example shows that one cannot expect to find a first order *differential* operator A having positive commutator with H.

First we analyze what it means for a vector field to have a positive commutator with H. Let the symbol of A be $a_i(x)\xi_i$. The Hamilton vector field of H is $\xi_i \partial_{x_i}$, so having a positive commutator requires that

$$\xi_i \xi_j \frac{\partial a_j}{\partial x_i} > 0 \text{ for } |\xi| \ne 0. \tag{8.1}$$

Thus, the matrix $\partial_{x_i} a_j$ must be positive definite, or in other words a_1 is increasing in direction e_1, etc.

Now consider the manifold with boundary shown in the figure (the corners should be assumed to have been smoothed out so that it has smooth boundary). In the figure, the topmost and bottom horizontal dotted lines, and the leftmost and rightmost dotted lines, are identified.

This manifold has no trapped geodesics.

Suppose that there is a vector field A whose commutator with H has a positive symbol. Notice that the two points p and q are such that there are three geodesics from p to q: one in the direction e_1+e_2, one in the direction $-e_1$ and one in the direction $-e_2$. Write $A = a_1e_1 + a_2e_2$. Then a_1 is increasing in direction e_1, and a_2 is increasing in direction e_2. Hence $a_1(p) > a_1(q)$, and $a_2(p) > a_2(q)$. On the other hand, $a_1 + a_2$ is increasing in direction e_1+e_2, so this yields $a_1(p)+a_2(p) < a_1(q)+a_2(q)$, which is a contradiction.

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