Commutator estimates in the operator L^p-spaces.

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Abstract

We consider commutator estimates in non-commutative (operator) L^p-spaces associated with general semi-finite von Neumann algebra. We discuss the difficulties which appear when one considers commutators with an unbounded operator in non-commutative L^p-spaces with $p \neq \infty$. We explain those difficulties using the example of the classical differentiation operator. *MSC (2000): 46L52, 47B47.*
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1 Introduction

Let us consider the spaces $L^p := L^p(\mathbb{R})$, $1 \leq p \leq \infty$, i.e. the spaces of all Lebesgue measurable functions with integrable p-th power, if $1 \leq p < \infty$ and which are essentially bounded, if $p = \infty$.

Let us fix a Lipschitz function $f : \mathbb{R} \mapsto \mathbb{C}$, i.e. a function for which there exists a constant $c_f > 0$, such that

$$|f(t_1) - f(t_2)| \leq c_f \, |t_1 - t_2|, \quad t_1, t_2 \in \mathbb{R}.$$

Let us take $x \in L^\infty$. We denote by $\frac{1}{i}\frac{dx}{dt}$ (or x') the derivative of x, taken in the sense of tempered distributions. Let us recall that the chain rule says that, for every Lipschitz function f,

$$\frac{1}{i}\frac{d}{dt}(f(x)) = f'(x) \cdot \frac{1}{i}\frac{dx}{dt}, \tag{1.1}$$

where f' is the derivative of the tempered distribution f. If $\frac{1}{i}\frac{dx}{dt} \in L^p$ for some $1 \leq p \leq \infty$, then the latter identity implies that $\frac{1}{i}\frac{d}{dt}(f(x)) \in L^p$ as well and

$$\left\| \frac{1}{i} \frac{d}{dt}(f(x)) \right\|_{L^p} \leq c_f \left\| \frac{1}{i} \frac{dx}{dt} \right\|_{L^p},$$

where c_f is the Lipschitz constant of the function f. The latter relation may serve as a criterion for a function f to be Lipschitz. Indeed, let us introduce the following definition.

A function $f : \mathbb{R} \mapsto \mathbb{C}$ is called p-Lipschitz, for some $1 \leq p \leq \infty$, if and only if there is a constant $c_{f,p}$ such that

$$\left\| \frac{1}{i} \frac{d}{dt}(f(x)) \right\|_{L^p} \leq c_{f,p} \left\| \frac{1}{i} \frac{dx}{dt} \right\|_{L^p} \tag{1.2}$$

for every $x \in L^\infty$ such that $\frac{1}{i}\frac{dx}{dt} \in L^p$.[1]

In the classical (function) case we have the following result.

Theorem 1.1. *Let $f : \mathbb{R} \mapsto \mathbb{C}$ be a function. The following statements are equivalent:*

a. the function f is Lipschitz;

b. the function f is p-Lipschitz, for some $1 \leq p \leq \infty$;

c. the function f is p-Lipschitz, for every $1 \leq p \leq \infty$.

Proof. The proof uses a standard argument based on integration by parts and using an approximation identity. We leave details to the reader. □

We now introduce the class of p-Lipschitz functions in the general (operator) setting.

Let $\mathcal{M}$ be a semi-finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and equipped with normal semi-finite faithful (n.s.f.) trace τ. We denote the operator norm by $\|\cdot\|$. Let $\tilde{\mathcal{M}}$ stands for the collection of all τ-measurable operators, i.e. the collection of all linear operators $x : \mathscr{D}(x) \mapsto \mathcal{H}$ affiliated

[1] The latter inequality supposed to be read as follows. If $x \in L^\infty$ and the derivative $\frac{1}{i}\frac{dx}{dt}$ is a function in L^p, then the composition $f(x)$ is a tempered distribution such that the derivative $\frac{1}{i}\frac{d}{dt}(f(x))$ is a function in L^p and the inequality (1.2) holds.

with $\mathcal{M}$ such that for every $\epsilon > 0$ there is a projection $p_\epsilon \in \mathcal{M}$ with $\tau(\mathbf{1}-p_\epsilon) < \epsilon$ and $p_\epsilon(\mathcal{H}) \subseteq \mathscr{D}(x)$. The class $\tilde{\mathcal{M}}$ is a $*$-algebra. Furthermore, there is a topology on the algebra $\tilde{\mathcal{M}}$, which is called *the measure topology.* This topology is defined by the collection of neighborhoods of the origin $\{N_{\epsilon,\delta}\}_{\epsilon,\delta>0}$, where $N_{\epsilon,\delta}$ consists of all linear operators $x : \mathscr{D}(x) \mapsto \mathcal{H}$ affiliated with $\mathcal{M}$ such that there is a projection $p_\epsilon \in \mathcal{M}$ for which $\tau(\mathbf{1}-p_\epsilon) < \epsilon$ and $\|xp\| \leq \delta$. The class $\tilde{\mathcal{M}}$ equipped with the measure topology is a complete topological algebra. We refer the reader to [19, 12, 15] for more details.

We now construct the non-commutative L^p-spaces $\mathcal{L}^p := L^p(\mathcal{M}, \tau)$, $1 \leq p \leq \infty$, see [10] and references therein. Indeed, the space $\mathcal{L}^p$, is defined by

$$\mathcal{L}^p := \{x \in \tilde{\mathcal{M}} : \quad \|x\|_{\mathcal{L}^p} < \infty\}$$

where

$$\|x\|_{\mathcal{L}^p} := \tau\left((x^*x)^{\frac{p}{2}}\right)^{\frac{1}{p}}, \text{ when } p < \infty,$$

$$\|x\|_{\mathcal{L}^\infty} := \|x\|, \ x \in \tilde{\mathcal{M}}.$$

The spaces $\mathcal{L}^p$ resemble their classical counterparts. The spaces $\mathcal{L}^\infty$ coincides with $\mathcal{M}$ and the space $\mathcal{L}^1$ is the predual of the algebra $\mathcal{M}$. Furthermore, the Hölder inequality is valid in the spaces $\mathcal{L}^p$, that is

$$\|xy\|_{\mathcal{L}^p} \leq \|x\|_{\mathcal{L}^q}\|y\|_{\mathcal{L}^s}, \quad \frac{1}{p} = \frac{1}{q} + \frac{1}{s}, \ 1 \leq p, q, s \leq \infty. \tag{1.3}$$

Remark 1.1. Let us mention two basic examples of the above construction.

a. The algebra of all complex $n \times n$-matrices acting on the sequence space ℓ_n^2 which is usually denoted by $B(\ell_n^2)$ equipped with the standard trace Tr, $n \in \mathbb{N}$. The algebra of τ-measurable operators coincides with $B(\ell_n^2)$ in this case. The space $\mathcal{L}^p$, $1 \leq p \leq \infty$ consists of all $n \times n$-matrices and the norm $\|\cdot\|_{\mathcal{L}^p}$ is given by the p-th Schatten-von Neumann norm, i.e. $\|x\|_{\mathcal{L}^p} = \|s(x)\|_{\ell^p}$, where $s(x)$ is the sequence of singular values of the operator x counted with multiplicities, see [13].

b. The algebra $\mathcal{M} = L^\infty$ acting on the space L^2, where every function $x \in L^\infty$ is considered as a multiplication operator, i.e.

$$x(\xi) := x \cdot \xi, \quad \xi \in L^2.$$

The trace τ on the algebra L^∞ is given by Lebesgue integration. The algebra $\tilde{\mathcal{M}}$ consists of all Lebesgue measurable functions which are bounded except on a set of finite measure. The spaces $\mathcal{L}^p$ turn into the classical L^p-spaces $L^p(\mathbb{R})$.

Let us fix a linear self-adjoint operator $D : \mathscr{D}(D) \mapsto \mathcal{H}$ (not necessary affiliated with $\mathcal{M}$) such that

(D1) $e^{itD}\, x\, e^{-itD} \in \mathcal{L}^\infty$, whenever $x \in \mathcal{L}^\infty$, $t \in \mathbb{R}$;

(D2) $\tau(e^{itD}\, x\, e^{-itD}) = \tau(x)$, whenever $x \in \mathcal{L}^1 \cap \mathcal{L}^\infty$.

Let us recall that the subspace $\mathscr{D} \subseteq \mathscr{D}(D)$ is called *a core* of the operator D if and only if the closure $\overline{(D|_{\mathscr{D}})}$ coincides with D.

Definition 1.1. Let $x \in \mathcal{M}$. We say that the commutator $[D, x]$ is defined and belongs to $\mathcal{L}^p$, for some $1 \leq p \leq \infty$ if and only if there is a core $\mathscr{D} \subseteq \mathscr{D}(D)$ of the operator D such that $x(\mathscr{D}) \subseteq \mathscr{D}(D)$ and the operator $Dx - xD$, initially defined on $\mathscr{D}$, is closable, in which case the closure $\overline{Dx - xD}$ belongs to $\mathcal{L}^p$. In this case, the symbol $[D, x]$ stands for the closure $\overline{Dx - xD}$.

In the case $p = \infty$, we have the following observation.

Lemma 1.1 ([5, Proposition 3.2.55]). *Let $D : \mathscr{D}(D) \mapsto \mathcal{H}$ be a self-adjoint linear operator and $x \in \mathcal{M}$. If $[D, x]$ is bounded, then $x(\mathscr{D}(D)) \subseteq \mathscr{D}(D)$.*

The relation $x(\mathscr{D}(D)) \subseteq \mathscr{D}(D)$ in the cases $1 \leq p < \infty$ may fail as it is shown in the example with the differentiation operator below. On the other hand, the weaker relation $x(\mathscr{D}) \subseteq \mathscr{D}(D)$ for some core $\mathscr{D} \subseteq \mathscr{D}(D)$ is much easier to attack and, more importantly, is sufficient for the applications we study; see Theorems 3.2, 3.3 and 3.4.

By analogy with the beginning of the section, we introduce the following definition.

Definition 1.2. A function $f : \mathbb{R} \mapsto \mathbb{C}$ is called p-Lipschitz for some $1 \le p \le \infty$ (with respect to the couple $(\mathcal{M}, \tau)$ and the operator D) if and only if there is a constant $c_{f,p}$ such that $[D, f(x)] \in \mathcal{L}^p$ and

$$\|[D, f(x)]\|_{\mathcal{L}^p} \le c_{f,p} \, \|[D, x]\|_{\mathcal{L}^p},$$

for every $x = x^* \in \mathcal{M}$ such that $[D, x] \in \mathcal{L}^p$.

The present note is concerned with the following problem.

Problem 1.1. Which the function $f : \mathbb{R} \mapsto \mathbb{C}$ is p-Lipschitz?

Similar problems have been under considerable investigation over a long period. We refer the reader to the works [7, 14, 1, 2, 3, 4, 10, 8, 20, 17].

In this note, we shall show some sufficient criteria for a function to be p-Lipschitz stated in terms of (scalar) smoothness properties of this function. The main results, Theorems 3.2, 3.3 and 3.4, are essentially proved in [16]. The purpose of the present note is to give an additional insight in the matter and explain some interesting points about the construction of commutators in the non-commutative L^p-spaces with respect to atomless algebras using the example of the classical differentiation operator.

2 Commutators with the differentiation operator $\frac{1}{i}\frac{d}{dt}$

In the present section, we fix $\mathcal{M} = L^\infty$ (see Remark 1.1) and $\tau(\cdot) = \int (\cdot)\, dt$. Let us consider the operator $D := \frac{1}{i}\frac{d}{dt} : \mathscr{D}(D) \mapsto L^2$ with the domain given by

$$\mathscr{D}(D) := \left\{ \xi \in L^2 : \ \ \frac{1}{i}\frac{d\xi}{dt} \in L^2 \right\}.$$

The operator D is self-adjoint and the unitary group $\{e^{itD}\}_{t\in\mathbb{R}}$ is given by the translations, i.e.

$$e^{itD}(\xi)(s) = \xi(s+t), \quad s \in \mathbb{R}. \tag{2.1}$$

Consequently,

$$\begin{aligned}(e^{itD}xe^{-itD}\xi)(s) &= (xe^{-itD}\xi)(s+t) = x(s+t)(e^{-itD}\xi)(s+t)\\ &= x(s+t)\xi(s), \quad \xi \in L^2,\ t,s \in \mathbb{R}.\end{aligned}$$

Therefore, for every $x \in L^\infty$, the operator $e^{itD}xe^{-itD}$ is a multiplication operator on L^2 induced by the translated function $x(\cdot + t) \in L^\infty$. The latter readily yields the fact that the operator D satisfies (D1)–(D2).

Let $x \in L^\infty$ be such that $[D, x] \in L^p$, $1 \le p \le \infty$. By Definition 1.1, there is a core $\mathscr{D} \subseteq \mathscr{D}(D)$ such that $x(\mathscr{D}) \subseteq \mathscr{D}(D)$ and

$$(Dx - xD)(\xi) = \frac{1}{i}\frac{d}{dt}(x \cdot \xi) - x \cdot \frac{1}{i}\frac{d\xi}{dt} = \frac{1}{i}\frac{dx}{dt} \cdot \xi, \quad \xi \in \mathscr{D}. \tag{2.2}$$

Thus, if the derivative $\frac{1}{i}\frac{dx}{dt}$ is a function, then the operator $Dx - xD$ acts as a multiplication operator on $\mathscr{D}$. Clearly, $Dx - xD$ is closable and the closure $\overline{Dx - xD} \in L^p$ if and only if $\frac{1}{i}\frac{dx}{dt} \in L^p$.

In other words, by Definition 1.1, the operator $[D, x]$ belongs to L^p, $1 \le p \le \infty$, for a given $x \in L^\infty$ if and only if there is a core $\mathscr{D} \subseteq \mathscr{D}(D)$ such that

$$x(\mathscr{D}) \subseteq \mathscr{D}(D) \quad \text{and} \quad \frac{1}{i}\frac{dx}{dt} \in L^p. \tag{2.3}$$

Furthermore, let us note that the inclusion $x(\mathscr{D}) \subseteq \mathscr{D}(D)$ means that for every function $\xi \in \mathscr{D}$, the function $x \cdot \xi$ is differentiable and

$$\frac{1}{i}\frac{d}{dt}(x \cdot \xi) \in L^2. \tag{2.4}$$

Since $x \cdot \frac{1}{i}\frac{d\xi}{dt} \in L^2$, for every $\xi \in \mathscr{D}(D)$, $x \in L^\infty$, it follows from the last identity in (2.2) that (2.4) is equivalent to $\frac{1}{i}\frac{dx}{dt} \cdot \xi \in L^2$. The latter means that, if $\mathscr{D} \subseteq \mathscr{D}(D)$ is a core, then

$$x(\mathscr{D}) \subseteq \mathscr{D}(D) \iff \frac{1}{i}\frac{dx}{dt}(\mathscr{D}) \subseteq L^2. \tag{2.5}$$

Thus, we can restate (2.3) as $[D, x] \in L^p$, $1 \le p \le \infty$ for a given $x \in L^\infty$ if and only if there exists a core $\mathscr{D} \subseteq \mathscr{D}(D)$ such that

$$\frac{1}{i}\frac{dx}{dt}(\mathscr{D}) \subseteq L^2 \quad \text{and} \quad \frac{1}{i}\frac{dx}{dt} \in L^p. \tag{2.6}$$

Thus, in general, a verification of the statement $[D, x] \in L^p$, $1 \leq p < \infty$ consists of two steps whose nature is quite different. A verification of the condition $\frac{1}{i}\frac{dx}{dt} \in L^p$ is carried out in the literature almost exclusively via methods related to Banach space geometry (Schur multipliers, double operator integrals, vector-valued Fourier multipliers [9, 6, 11, 10]). However, the first condition in (2.6) has an operator-theoretical nature and does not correspond to the methods listed above. We outline an approach to this problem when $D = \frac{1}{i}\frac{d}{dt}$.

Let us first consider $[D, x] \in L^p$ when $2 \leq p < \infty$. We shall show that in the present setting, the required core $\mathscr{D}$ appears very naturally due to the fact that the underlying Hilbert space L^2 possesses the additional Banach structure induced by the L^p-scale. Indeed, let us set

$$\mathscr{D} := \mathscr{D}(D) \cap L^q, \quad \text{where} \quad \frac{1}{2} = \frac{1}{p} + \frac{1}{q}. \tag{2.7}$$

Clearly, the Hölder inequality implies that (2.6) holds for the subset $\mathscr{D}$ and any $x \in L^\infty$ such that $\frac{1}{i}\frac{dx}{dt} \in L^p$. We shall verify that $\mathscr{D}$ is a core of D in Theorem 3.3 below. What we would like to emphasize is that the core $\mathscr{D}$ is found purely by a Banach space construction. Thus, we see that in the case $2 \leq p < \infty$, we have

$$[D, x] \in L^p \iff \frac{1}{i}\frac{dx}{dt} \in L^p.$$

Finally, we comment on the case $1 \leq p < 2$. Here, the problem of finding the core $\mathscr{D}$ satisfying the first condition in (2.6) cannot be resolved by a purely Banach space approach as in (2.7) above. Indeed, let $C(\mathbb{R})$ be the class of all continuous functions on $\mathbb{R}$. We note that $\mathscr{D}(D) \subseteq C(\mathbb{R})$, [18, Theorem 2, p. 124]. If we now consider the function $x \in L^\infty$ such that

$$\frac{1}{i}\frac{dx}{dt} \in L^p, \quad \text{but} \quad \frac{1}{i}\frac{dx}{dt} \notin L^2_{loc},$$

then

$$\frac{1}{i}\frac{dx}{dt} \cdot \xi \notin L^2, \quad \text{for every} \quad \xi \in \mathscr{D}(D), \ \xi \not\equiv 0.$$

That means that despite the fact that the derivative $\frac{1}{i}\frac{dx}{dt}$ exists in the sense of tempered distributions and belongs to L^p, *there is no core* such that the commutator $[D, x]$ may be defined according to Definition 1.1.

3 Main result

As we have seen in the example with the operator $D = \frac{1}{i}\frac{d}{dt}$, a meaningful resolution of Problem 1.1 requires locating a core $\mathscr{D}$ of the operator D satisfying the first condition in (2.5). As we indicated in that example, a possible candidate on the role of such $\mathscr{D}$ is the space

$$\mathscr{D}(D) \cap \mathcal{L}^1 \cap \mathcal{L}^\infty.$$

Unfortunately, in general, the domain $\mathscr{D}(D) \subseteq \mathcal{H}$ may have an empty intersection with the space $\mathcal{L}^1 \cap \mathcal{L}^\infty$. We shall show below that this is not the case when $\mathcal{M}$ is taken in the left regular representation (see Theorem 3.3).

3.1 The left regular representation

Let $\mathcal{M}$ be a semi-finite von Neumann algebra equipped with n.s.f. trace τ and let $\mathcal{L}^p := L^p(\mathcal{M}, \tau)$, $1 \leq p \leq \infty$ be the corresponding non-commutative L^p-spaces.

Let us consider the mapping $L : \mathcal{M} \mapsto B(\mathcal{L}^2)$, given by $L(x) := L_x$, $x \in \mathcal{M}$, where the operator $L_x \in B(\mathcal{L}^2)$ is given by

$$L_x(\xi) := x \cdot \xi, \quad \xi \in \mathcal{L}^2.$$

The image $\mathcal{M}_L := L(\mathcal{M})$ is a von Neumann algebra acting on $\mathcal{L}^2$. The mapping L is a $*$-isomorphism between the algebras $\mathcal{M}$ and $\mathcal{M}_L$. The algebra $\mathcal{M}_L$ is equipped with n.s.f trace $\tau_L := \tau \circ L^{-1}$. With this definition of τ_L, the mapping L becomes a trace preserving $*$-isomorphism. Consequently, it extends to a $*$-homeomorphism between topological $*$-algebras $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}_L := (\mathcal{M}_L)^\sim$. We shall denote the latter extension by L also. Alternatively, the mapping $L : \tilde{\mathcal{M}} \mapsto \tilde{\mathcal{M}}_L$ is given by $L(x) = L_x$, where $L_x : \mathscr{D}(L_x) \mapsto \mathcal{L}^2$ is an operator given by

$$\mathscr{D}(L_x) = \{\xi \in \mathcal{L}^2 : \quad x \cdot \xi \in \mathcal{L}^2\} \text{ and } L_x(\xi) = x \cdot \xi, \quad \xi \in \mathscr{D}(L_x).$$

Since the mapping $L : \tilde{\mathcal{M}} \mapsto \tilde{\mathcal{M}}_L$ is trace preserving, its restriction to the space $\mathcal{L}^p$ becomes an isometry between the spaces $\mathcal{L}^p$ and $\mathcal{L}^p_L := L^p(\mathcal{M}_L, \tau_L)$, for every $1 \leq p \leq \infty$.

3.1.1 Approximation of the commutator $[D, x]$

In the present section we shall consider the construction of an approximation of the commutator $[D, x]$ by means of the corresponding unitary group $\{e^{itD}\}_{t\in\mathbb{R}}$.

For illustration, let us again consider the example of the differentiation operator. If $x \in L^\infty(\mathbb{R})$ and $D = \frac{1}{i}\frac{d}{dt}$, then we have the well known relations

$$x(t+s) - x(s) = i\int_0^t \frac{1}{i}\frac{dx}{dt}(s+\tau)\,d\tau, \quad t, s \in \mathbb{R}, \tag{3.1}$$

$$\frac{1}{i}\frac{dx}{dt}(s) = \lim_{t\to 0}\frac{x(s+t) - x(s)}{it}. \tag{3.2}$$

An operator version of (3.1) and (3.2), in the case $p = \infty$ may be found in [5, Section 3.2.5]

Theorem 3.1. *Let $D : \mathscr{D}(D) \mapsto \mathcal{H}$ be a self-adjoint linear operator, satisfying (D1)–(D2) and let $x \in \mathcal{M}$. If $[D, x] \in \mathcal{L}^\infty$, then*

a. $e^{itD}xe^{-itD} - x = i\int_0^t e^{isD}[D, x]e^{-isD}\,ds$, $t \in \mathbb{R}$;

b. $\left\|\dfrac{e^{itD}xe^{-itD} - x}{t}\right\|_{\mathcal{L}^\infty} \leq \|[D, x]\|_{\mathcal{L}^\infty}$;

c. $\lim_{t\to 0}\dfrac{e^{itD}xe^{-itD} - x}{t} = i[D, x]$;

where the integral and the limit converge with respect to the weak operator topology.

The natural framework to deal with the commutator $[D,x] \in \mathcal{L}^p$ when $p < \infty$ is the setting of the left regular representation. Thus, from now on, we consider the algebra $\mathcal{M}_L$ with the n.s.f. trace τ_L. We denote by $\mathcal{L}^p_L := L^p(\mathcal{M}_L, \tau_L)$, $1 \leq p \leq \infty$ the corresponding non-commutative L^p-space.

We shall discuss the extension of Theorem 3.1 to the spaces $\mathcal{L}^p_L$, $1 \leq p < \infty$.

To explain the next step, let us note that the proof of Theorem 3.1 crucially depends on the fact that the domain $\mathscr{D}(D)$ where the commutator $[D,x]$, initially defined, according to Definition 1.1 and Lemma 1.1, is invariant with respect to the group $\{e^{itD}\}_{t\in\mathbb{R}}$. On the other hand, the core $\mathscr{D}$ in Definition 1.1 lacks this invariance when $p < \infty$. We now extend Definition 1.1.

Definition 3.1. Let $x \in \mathcal{M}_L$ and let $D : \mathscr{D}(D) \mapsto \mathcal{L}^2$ be a linear self-adjoint operator. We shall say that the commutator $[D,x]$ is defined and belongs to $\mathcal{L}^p_L$, for some $1 \leq p \leq \infty$ if and only if

a. there is a core $\mathscr{D} \subseteq \mathcal{L}^1 \cap \mathcal{L}^\infty$ of the operator D such that $e^{itD}(\mathscr{D}) \subseteq \mathscr{D}$, for every $t \in \mathbb{R}$, and $x(\mathscr{D}) \subseteq \mathscr{D}(D)$;

b. the operator $Dx - xD$, initially defined on $\mathscr{D}$, is closable;

c. the closure $\overline{Dx - xD}$ belongs to $\mathcal{L}^p$. In this case, the symbol $[D,x]$ stands for the closure $\overline{Dx - xD}$.

The next result provides an extension of Theorem 3.1 over the spaces $\mathcal{L}^p_L$, $1 \leq p < \infty$.

Theorem 3.2. *Let $D : \mathscr{D}(D) \mapsto \mathcal{L}^2$ be a self-adjoint linear operator, satisfying (D1)–(D2) and let $x \in \mathcal{M}_L$. If $[D,x] \in \mathcal{L}^p_L$, for some $1 \leq p < \infty$, then*

a. $e^{itD}xe^{-itD} - x = i\int_0^t e^{isD}[D,x]e^{-isD}\,ds$, $t \in \mathbb{R}$;

b. $\left\|\dfrac{e^{itD}xe^{-itD} - x}{t}\right\|_{\mathcal{L}^p_L} \leq \|[D,x]\|_{\mathcal{L}^p_L}$;

c. $\lim_{t\to 0} \dfrac{e^{itD}xe^{-itD}-x}{t} = i[D,x]$;

where the integral and the limit converge with respect to the norm topology in $\mathcal{L}^p_L$.

3.1.2 Commutator estimates

Let us recall that we have fixed the pair $(\mathcal{M},\tau)$ and we consider the left regular representation $(\mathcal{M}_L,\tau_L)$. Let $D:\mathscr{D}(D)\mapsto\mathcal{L}^2$ be a linear self-adjoint operator satisfying (D1)–(D2).

Let us again consider the subspace

$$\mathscr{D}_0(D) := \mathscr{D}(D)\cap\mathcal{L}^1\cap\mathcal{L}^\infty \subseteq \mathcal{L}^2. \tag{3.3}$$

Unfortunately, in general case when the operator D is not affiliated with the algebra $\mathcal{M}_L$, there is no hope to expect that the latter subspace will be a core of the operator D. To single out the class of operators D for which the subspace $\mathscr{D}_0(D)$ is a core let us introduce the assumption

(D3) the unitary group $\{e^{itD}\}_{t\in\mathbb{R}}$ is a $\sigma(\mathcal{L}^1\cap\mathcal{L}^\infty,\mathcal{L}^1+\mathcal{L}^\infty)$-continuous group of contractions in the space $\mathcal{L}^1\cap\mathcal{L}^\infty$.

If $D=\frac{1}{i}\frac{d}{dt}$, then the assumption (D3) is clearly satisfied, since $\{e^{itD}\}_{t\in\mathbb{R}}$ is a group of translations, see (2.1). Also, if D is affiliated with $\mathcal{M}_L$, then (D3) holds, due to the fact that $e^{itD}=L(u_t)$, for every $t\in\mathbb{R}$, where $\{u_t\}_{t\in\mathbb{R}}\subseteq\mathcal{M}$ is a group of unitaries.

Theorem 3.3. *If $D:\mathscr{D}(D)\mapsto\mathcal{L}^2$ is a linear self-adjoint operator satisfying (D1)–(D3), then the subspace $\mathscr{D}_0(D)$ is a core of the operator D.*

To state the main result, let us first recall that a Borel function $f:\mathbb{R}\mapsto\mathbb{C}$ is called of *bounded β-variation*, $1\le\beta<\infty$ if and only if

$$\|f\|_{V_\beta} := \sup\left[\sum_{j=-\infty}^{+\infty}|f(t_j)-f(t_{j+1})|^\beta\right]^{\frac{1}{\beta}} < \infty, \tag{3.4}$$

where the supremum is taken over all possible increasing two-sided sequences $\{t_j\}_{j=-\infty}^{+\infty} \subseteq \mathbb{R}$. V_β will stand for the class of all functions of bounded β-variation, $1 \leq \beta < \infty$. The class V_β is equipped with the norm $\|\cdot\|_{V_\beta}$ defined in (3.4). We also define V_∞ to be the collection of all bounded Borel functions equipped with the uniform norm.

Let us next state the main result of the text. Its proof consists of a combination of the technique developed in [8] with the approach explained above. In the special case $\mathcal{M} = B(\mathcal{H})$, the result which follows gives an alternative (and simpler) proof of [4, Example III]. Let us note that the result distinguishes two different cases $p < 2$ and $p \geq 2$ as discussed in the example of Section 2.

Theorem 3.4. *Let $D : \mathscr{D}(D) \mapsto \mathcal{L}^2$ be a linear self-adjoint operator satisfying (D1)–(D3) and let $x = x^* \in \mathcal{M}_L$. Let a function $f : \mathbb{R} \mapsto \mathbb{C}$ be such that $f' \in V_\beta$ for some $1 \leq \beta \leq \infty$.*

a. For every $2 \leq p < \frac{2\beta}{\beta-1}$ there is a constant c_p' such that if $[D, x] \in \mathcal{L}_L^p$, then $[D, f(x)] \in \mathcal{L}_L^p$ and

$$\|[D, f(x)]\|_{\mathcal{L}_L^p} \leq c_p' \, \|f'\|_{V_\beta} \, \|[D, x]\|_{\mathcal{L}_L^p}.$$

b. For every $\frac{2\beta}{\beta+1} < p < 2$ there is a constant c_p'' such that if $[D, x] \in \mathcal{L}_L^p \cap \mathcal{L}_L^2$, then $[D, f(x)] \in \mathcal{L}_L^p \cap \mathcal{L}_L^2$ and

$$\|[D, f(x)]\|_{\mathcal{L}_L^p} \leq c_p'' \, \|f'\|_{V_\beta} \, \|[D, x]\|_{\mathcal{L}_L^p}.$$

Now we state the answer to Problem 1.1 in the setting of the left regular representation.

Theorem 3.5. *Any function $f : \mathbb{R} \mapsto \mathbb{C}$ such that $f' \in V_\beta$, for some $1 \leq \beta \leq \infty$ is p-Lipschitz for every $2 \leq p < \frac{2\beta}{\beta-1}$, with respect to any operator $D : \mathscr{D}(D) \mapsto \mathcal{L}^2$ and every semi-finite von Neumann algebra $(\mathcal{M}_L, \tau_L)$.*

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