A remark on the H^∞-calculus

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Abstract

If A, B are sectorial operators on a Hilbert space with the same domain and range, and if $\|Ax\| \approx \|Bx\|$ and $\|A^{-1}x\| \approx \|B^{-1}x\|$, then it is a result of Auscher, McIntosh and Nahmod that if A has an $H^\infty-$calculus then so does B. On an arbitrary Banach space this is true with the additional hypothesis on B that it is almost R-sectorial as was shown by the author, Kunstmann and Weis in a recent preprint. We give an alternative approach to this result.
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1 Introduction

In [1] the authors showed that if X is a Hilbert space and A, B are sectorial operators with the same domain and range and satisfying estimates

$$\|Ax\| \approx \|Bx\| \qquad x \in \text{Dom } (A) \tag{1.1}$$

and

$$\|A^{-1}x\| \approx \|B^{-1}x\| \qquad x \in \text{Ran } (A) \tag{1.2}$$

then if one of (A, B) admits an $H^\infty-$calculus then so does the other. Results of this type are useful in applications and were studied in [7] for arbitrary Banach spaces. In that paper, a similar result (Theorem 5.1) is proved under the additional hypothesis that A is almost R-sectorial.

In this note we give a rather different approach to this result. We replace the almost R-sectoriality assumption by the technically weaker assumption of almost U-sectoriality, although this is probably not of great significance. However, our approach here is perhaps a little simpler. We also point out

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that some additional assumption is necessary in arbitrary Banach spaces; there are examples of sectorial operators A, B satisfying (1.1) and (1.2) but such that only one has an $H^\infty-$calculus.

It is possible to consider estimates on fractional powers and our results can be extended in this direction (as in [7]); however to keep the exposition simple we will not discuss this point. We also point out that our approach is really based on an interpolation method, known as the Gustavsson-Peetre method [5] (see also [4]); but to avoid certain technicalities we have not made this explicit.

2 U-bounded collections of operators

Let X be a complex Banach space. A family $\mathcal{T}$ of operators $T : X \to X$ is called *U-bounded* if there is a constant C such that if $(x_j)_{j=1}^n \subset X,\ (x_j^*)_{j=1}^n \subset X^*,\ (T_j)_{j=1}^n \subset \mathcal{T}$,

$$\sum_{j=1}^n |\langle T_j x_j, x_j^* \rangle| \le C \sup_{|a_j|=1} \| \sum_{j=1}^n a_j x_j \| \sup_{|a_j|=1} \| \sum_{j=1}^n a_j x_j^* \|.$$

The best such constant C is called the U-bound for $\mathcal{T}$ and is denoted $U(\mathcal{T})$. This concept was introduced in [8].

We recall that $\mathcal{T}$ is called *R-bounded* if there is a constant C such that if $(x_j)_{j=1}^n \subset X,\ (T_j)_{j=1}^n \subset \mathcal{T}$,

$$(\mathbb{E}\| \sum_{j=1}^n \epsilon_j T x_j \|^2)^{1/2} \le C(\mathbb{E}\| \sum_{j=1}^n \epsilon_j x_j \|^2)^{1/2}.$$

Here $(\epsilon_j)_{j=1}^n$ is a sequence of independent Rademachers. The best such constant C is called the R-bound for $\mathcal{T}$ and is denoted $R(\mathcal{T})$. An R-bounded family is automatically U-bounded [8].

We will need the following elementary property:

Proposition 2.1. *Suppose $F : (0,\infty) \to \mathcal{L}(X)$ is a continuous function and that $\mathcal{T} = \{F(t) :\ 0 < t < \infty\}$ is U-bounded with U-bound $U(F)$. Suppose $g \in L_1(\mathbb{R}, dt/t)$. Then the family of operators*

$$G(s) = \int_0^\infty g(st)F(t)\frac{dt}{t} \qquad 0 < s < \infty$$

is U-bounded with constant at most $U(F)\int_0^\infty |g(t)|dt/t$.

Proof. Suppose $(x_j)_{j=1}^n \subset X$, $(x_j^*)_{j=1}^n \subset X^*$ with

$$\sup_{|a_j|=1} \| \sum_{j=1}^n a_j x_j \|, \sup_{|a_j|=1} \| \sum_{j=1}^n a_j x_j^* \| \leq 1.$$

Then for $s_1, \ldots, s_n \in \mathbb{R}$ we have

$$\sum_{j=1}^n |\langle G(s_j)x_j, x_j^* \rangle| \leq \sum_{j=1}^n \int_0^\infty |g(t)| |\langle F(s_j^{-1}t)x_j, x_j^* \rangle| \frac{dt}{t}$$
$$\leq U(F) \int_0^\infty |g(t)| \frac{dt}{t}.$$

□

3 Sectorial operators

Let X be a complex Banach space and let A be a closed operator on X. A is called *sectorial* if A has dense domain Dom (A) and dense range Ran $(A) =$ Dom (A^{-1}) and for some $0 < \varphi < \pi$ the resolvent $(\lambda - A)^{-1}$ is bounded for $|\arg \lambda| \geq \varphi$ and satisfies the estimate

$$\sup_{|\arg \lambda| \geq \varphi} \|\lambda(\lambda - A)^{-1}\| < \infty.$$

The infimum of such angles φ is denoted $\omega(A)$.

Let Σ_φ be the open sector $\{z \neq 0 : |\arg z| < \varphi\}$. If $f \in H^\infty(\Sigma_\varphi)$ we say that $f \in H_0^\infty(\Sigma_\varphi)$ if there exists $\delta > 0$ such that $|f(z)| \leq C \max(|z|^\delta, |z|^{-\delta})$. For $f \in H_0^\infty(\Sigma_\varphi)$ where $\varphi > \omega(A)$ we can define $f(A)$ by a contour integral, which converges as a Bochner integral in $\mathcal{L}(X)$.

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma_\nu} f(\lambda)(\lambda - A)^{-1} d\lambda$$

where Γ_ν is the contour $\{|t|e^{-i\nu \mathrm{sgn}\, t} : -\infty < 0 < \infty\}$ and $\omega(A) < \nu < \varphi$. We can then estimate $\|f(A)\|$ by

$$\|f(A)\| \leq C_\varphi \int_{\Gamma_\nu} |f(\lambda)| \frac{|d\lambda|}{|\lambda|}.$$

If we have a stronger estimate

$$\|f(A)\| \leq C\|f\|_{H^\infty(\Sigma_\varphi)} \qquad f \in H_0^\infty(\Sigma_\varphi)$$

then we say that A *has an* $H^\infty(\Sigma_\varphi)-$*calculus*; in this case we may extend the functional calculus to define $f(A)$ for every $f \in H^\infty(\Sigma_\varphi)$. The infimum of all such angles φ is denoted by $\omega_H(A)$.

We will need a criterion for the existence of an H^∞-calculus. It will be convenient to use the notation $f_\lambda(z) = f(\lambda z)$ and to let $u(z) = z(1+z)^{-2}$ so that $u \in H_0^\infty(\Sigma_\varphi)$ for all $\varphi < \pi$. The following criterion goes back to [2] and [3]. A simple proof is given in [10].

Proposition 3.1. *Let A be a sectorial operator and suppose $0 < \varphi < \pi$. Then the following are equivalent:*
(i) There is a constant C so that

$$\int_0^\infty |\langle u_\mu(tA)x, x^*\rangle| \frac{dt}{t} \leq C\|x\|\|x^*\| \qquad |\arg\mu| = \varphi,\ x \in X, x^* \in X^*.$$

(ii) A has an $H^\infty-$calculus with $\omega_H(A) \leq \pi - \varphi$.

Remark. (i) is equivalent by the Maximum Modulus Principle to

$$\int_0^\infty |\langle u_\mu(tA)x, x^*\rangle| \frac{dt}{t} \leq C\|x\|\|x^*\| \qquad |\arg\mu| \leq \varphi,\ x \in X, x^* \in X^*.$$

If A is sectorial we can define a closed operator A^* on X^* by $A^*x^* = x^* \circ A$ with domain Dom (A^*) consisting of all x^* such that $x \to x^*(Ax)$ extends to a bounded linear functional on X. Then A^* need not be sectorial since it need not have dense domain or range. Note that

$$\|A^*x^*\| = \sup_{\substack{\|A^{-1}x\| \leq 1 \\ x \in \text{Ran }(A)}} |\langle x, x^*\rangle| \qquad x^* \in \text{Dom }(A^*)$$

and

$$\|(A^*)^{-1}x\| = \sup_{\substack{\|Ax\| \leq 1 \\ x \in \text{Dom }(A)}} |\langle x, x^*\rangle| \qquad x^* \in \text{Ran }(A^*).$$

Thus if A and B are sectorial operators satisfying (1.1) and (1.2) they will also satisfy Dom $(A^*) =$ Dom (B^*), Ran $(A^*) =$ Ran (B^*) and

$$\|A^*x^*\| \approx \|B^*x^*\| \qquad x^* \in \text{Dom }(A^*) \tag{3.1}$$

and

$$\|(A^*)^{-1}x^*\| \approx \|(B^*)^{-1}x^*\| \qquad x^* \in \text{Ran }(A^*) \tag{3.2}$$

If A is a sectorial operator and $\varphi > \omega(A)$ we shall that $f \in H_0^\infty(\Sigma_\varphi)$ is U-bounded (respectively R-bounded) for A if the family of operators $\{f(tA) : 0 < t < \infty\}$ is a U-bounded (respectively R-bounded) collection.

Proposition 3.2. *Suppose A has an H^∞-calculus and that $\varphi > \omega_H(A)$. Then for any $f \in H_0^\infty(\Sigma_\varphi)$ we have that f is R-bounded (and thus U-bounded) for A.*

Proof. Suppose $\omega(A) < \psi < \varphi$. Then the map $\lambda \to f(\lambda A)$ is analytic on $\Sigma_{\varphi-\psi}$ and extends continuously to the boundary. The operators $\{f(2^k te^{\pm i(\varphi-\psi)}A)\}_{k\in\mathbb{Z}}$ are R-bounded (uniformly in $0 < t < \infty$) by Theorem 3.3 of [8] and the result follows by Lemma 3.4 of the same paper. □

Suppose A is a sectorial operator on X and $\varphi > \omega(A)$. We will say that A is *almost U-sectorial* (respectively almost R-sectorial) if there is an angle φ such that the set of operators $\{\lambda AR(\lambda, A)^2 : \ |\arg\lambda| \geq \varphi\}$ is U-bounded (respectively R-bounded). If we define $u(z) = z(1+z)^{-2}$ this implies that the functions $u_\lambda(z) = u(\lambda z)$ are uniformly U-bounded (respectively uniformly R-bounded) for $|\arg\lambda| \leq \pi - \varphi$. The infimum of such angles is denoted $\tilde{\omega}_U(A)$. By Lemma 3.4 of [8] this definition is equivalent to

$$\tilde{\omega}_U(A) = \pi - \sup\{\theta : \ u_{e^{\pm i\theta}} \text{ is U-bounded}\}$$

or, respectively

$$\tilde{\omega}_R(A) = \pi - \sup\{\theta : \ u_{e^{\pm i\theta}} \text{ is R-bounded}\}.$$

Proposition 3.3. *Suppose A admits an H^∞-calculus. Then A is almost R-sectorial (and hence almost U-sectorial) and $\tilde{\omega}_U(A) \leq \tilde{\omega}_R(A) \leq \omega_H(A)$.*

Proof. This follows from Proposition 3.2. □

Lemma 3.1. *Suppose A is almost U-sectorial and $\varphi > \nu > \tilde{\omega}_U(A)$. Then there is a constant $C = C(\varphi)$ so that if $f \in H_0^\infty(\Sigma_\varphi)$ then f is U-bounded for A with U-bound*

$$U(f) \leq C \int_{\Gamma_\nu} |f(\lambda)| \frac{|d\lambda|}{|\lambda|}.$$

Proof. Fix $\varphi > \psi > \nu > \omega_U(A)$. We may write $f(tA)$ in the form

$$f(tA) = \frac{1}{2\pi i}\int_{\Gamma_\psi} f(t\lambda)\lambda^{-1/2}A^{1/2}(\lambda - A)^{-1}d\lambda.$$

Therefore the result follows from Lemma 2.1 once we show that the two families of operators $\{h(e^{\pm i\theta}tA) : \ 0 < t < \infty\}$ are U-bounded where $\theta = \pi - \psi$ and $h(z) = z^{1/2}(1+z)^{-1}$.

Consider

$$g(z) = -i\log\frac{1+iz^{1/2}}{1-iz^{1/2}} - \pi\frac{z}{1+z} \qquad |\arg z| < \pi.$$

Then $g \in H_0^\infty(\Sigma_\pi)$. Furthermore

$$g'(z) = z^{-1/2}(1+z)^{-1} - \pi(1+z)^{-2}.$$

Hence $g_{e^{\pm i\theta}} \in H_0^\infty(\Sigma_\psi)$. For convenience we consider the case of $+\theta$. Thus if

$$T_t = -\frac{1}{2\pi i}\int_{\Gamma_\nu} g(te^{i\theta}\lambda)A(\lambda - A)^{-2}d\lambda$$

the family of operators $\{T_t : \ 0 < t < \infty\}$ is U-bounded, again by Lemma 2.1. Now integration by parts shows that

$$\begin{aligned}
T_t &= \frac{te^{i\theta}}{2\pi i}\int_{\Gamma_\nu} ((te^{i\theta}\lambda)^{-1/2}(1+te^{i\theta}\lambda)^{-1} - \pi(1+te^{i\theta}\lambda)^{-2})\lambda(\lambda - A)^{-1}d\lambda \\
&= \frac{1}{2\pi i}\int_{\Gamma_\nu} (h(te^{i\theta}\lambda) - \pi u(te^{i\theta}\lambda))(\lambda - A)^{-1}d\lambda \\
&= h(te^{i\theta}A) - \pi u(te^{i\theta}A).
\end{aligned}$$

Thus it follows that the family $\{h(te^{i\theta}A) : \ 0 < t < \infty\}$ is U-bounded. □

4 The main results

If A is sectorial then the space Dom $(A)\cap$Ran (A) is a Banach space (densely) embedded into X under the norm $\|Ax\|+\|A^{-1}x\|+\|x\|$; similarly Dom $(A^*)\cap$ Ran (A^*) is a Banach space embedded into X^* under the norm $\|A^*x^*\| + \|(A^*)^{-1}x^*\| + \|x^*\|$.

Theorem 4.1. *Suppose A is a sectorial operator. In order that A have an H^∞-calculus with $\omega_H(A) = \varphi$ it is necessary and sufficient that:*
(i) A is almost U-sectorial with $\tilde{\omega}_U(A) = \varphi$.
(ii) There exists a constant C_1 so that for each $x \in X$ there is a continuous function $\xi : (0, \infty) \to Dom\ (A) \cap Ran\ (A)$ such that

$$\| \sum_{k=-N}^{N} a_k 2^{jk} t^j A^j \xi(2^k t) \| \le C_1 \|x\|,\ j = -1, 0, 1,\ |a_k| \le 1,\ N = 1, 2, \ldots,\ 0 < t < \infty$$

and

$$\langle x, x^* \rangle = \int_0^\infty \langle \xi(t), x^* \rangle \frac{dt}{t} \qquad x^* \in X^*.$$

(iii) There exists a constant C_2 so that for each $x^* \in X^*$ there is a continuous function $\xi^* : (0, \infty) \to Dom\ (A^*) \cap Ran\ (A^*)$ such that

$$\| \sum_{k=-N}^{N} a_k 2^{jk} t^j (A^j)^* \xi^*(2^k t) \| \le C_2 \|x^*\|,\ j = -1, 0, 1,\ |a_k| \le 1,\ N = 1, 2, \ldots,\ 0 < t < \infty$$

and

$$\langle x, x^* \rangle = \int_0^\infty \langle x, \xi^*(t) \rangle \frac{dt}{t} \qquad x \in X.$$

Proof. Let us assume (i), (ii) and (iii). Suppose $|\theta| < \pi - \varphi$ and $\|x\| \le 1,\ \|x^*\| \le 1$. Let $\xi(t), \xi^*(t)$ be chosen according to (ii) and (iii). We define

$$\tilde{\xi}(t) = tA\xi(t) + t^{-1}A^{-1}\xi(t) + 2\xi(t), \quad \tilde{\xi}^*(t) = tA^*\xi^*(t) + t^{-1}A^*\xi^*(t) + 2\xi^*(t).$$

Thus we have

$$\| \sum_{k=-N}^{N} a_k 2^{jk} \tilde{\xi}(2^k t) \| \le 3C_1,\ j = -1, 0, 1,\ |a_k| \le 1,\ N = 1, 2, \ldots,\ 0 < t < \infty$$

and

$$\| \sum_{k=-N}^{N} a_k 2^{jk} \tilde{\xi}^*(2^k t) \| \le 3C_2,\ j = -1, 0, 1,\ |a_k| \le 1,\ N = 1, 2, \ldots,\ 0 < t < \infty.$$

Note that $\tilde{\xi} : (0, \infty) \to X$ and $\tilde{\xi}^* : (0, \infty) \to X^*$ are both continuous and

$$\xi(t) = u(tA)\tilde{\xi}(t) \qquad 0 < t < \infty$$
$$\xi^*(t) = (u(tA))^*\tilde{\xi}^*(t) \qquad 0 < t < \infty.$$

If $\pi - |\arg\mu| > \nu > \varphi$ we have

$$\int_0^\infty |< u_\mu(rA)x, x^* >|\frac{dr}{r} \le \int_0^\infty\int_0^\infty\int_0^\infty |\langle u_\mu(rA)\xi(s), \xi^*(t)\rangle|\frac{dt}{t}\frac{ds}{s}\frac{dr}{r}$$
$$= \int_0^\infty\int_0^\infty\int_0^\infty |\langle u_\mu(rtA)\xi(st), \xi^*(t)\rangle|\frac{dt}{t}\frac{ds}{s}\frac{dr}{r}$$

For fixed r, s

$$\int_0^\infty |\langle u_\mu(rtA)\xi(st), \xi^*(t)\rangle|\frac{dt}{t} = \int_0^\infty |\langle u_\mu(rtA)u(stA)\tilde{\xi}(st), (u(tA))^*\tilde{\xi}^*(t)\rangle|\frac{dt}{t}$$
$$= \int_1^2 \sum_{j\in\mathbb{Z}} |\langle u_{r\mu}(2^j tA)u_s(2^j tA)u(2^j tA)\tilde{\xi}(s2^j t), \tilde{\xi}^*(2^j t)\rangle|\frac{dt}{t}$$
$$\le 9C_1C_2U(u_{r\mu}u_s u)$$
$$\le C\int_{\Gamma_\nu} |u(r\mu\lambda)u(s\lambda)u(\lambda)|\frac{|d\lambda|}{|\lambda|},$$

where C is constant independent of x, x^*. Integrating over r, s gives:

$$\int_0^\infty |< u_\mu(rA)x, x^* >|\frac{dr}{r} \le C\left(\int_{\Gamma_\nu} |u_\mu(\lambda)|\frac{|d\lambda|}{|\lambda|}\right)\left(\int_{\Gamma_\nu} |u(\lambda)|\frac{|d\lambda|}{|\lambda|}\right)^2.$$

This estimate shows, by Proposition 3.1, that A has an $H^\infty-$calculus with $\omega_H(A) \le \varphi$. Since $\tilde{\omega}_U(A) \le \omega_H(A)$ by Proposition 3.3 we have equality.

To complete the proof we show that if A has an $H^\infty-$calculus then (i), (ii) and (iii) hold and that $\tilde{\omega}_U(A) \le \omega_H(A)$.

To show (ii) and (iii) we observe that

$$12\int_0^\infty (u(tz))^2\frac{dt}{t} = 1.$$

Note that $z^j u(z)^2 \in H_0^\infty(\Sigma_\varphi)$ for $j = -1, 0, 1$. It follows easily that if $x \in X$ and $x^* \in X^*$ then

$$\xi(t) = 12u(tA)^2 x, \qquad \xi^*(t) = 12(u(tA)^2)^* x^*$$

give the required functions.

For (i) observe that $\tilde{\omega}_U(A) \le \omega_H(A)$ but the first part of the proof shows equality.

□

Theorem 4.2. *Suppose A and B are sectorial operators such that Dom $(A) =$ Dom (B), Ran $(A) =$ Ran (B) and for a suitable constant C we have*

$$C^{-1}\|Ax\| \le \|Bx\| \le C\|Ax\| \qquad x \in Dom\ (A)$$

and

$$C^{-1}\|A^{-1}x\| \le \|B^{-1}x\| \le C\|A^{-1}x\| \qquad x \in Ran\ (A).$$

Suppose A has an $H^\infty-$calculus. Then the following are equivalent:
(i) B has an $H^\infty-$calculus with $\omega_H(B) = \varphi$.
(ii) B is almost U-sectorial and $\tilde{\omega}_U(B) = \varphi$.

Proof. This is now immediate from Theorem 4.1 using (3.1) and (3.2). □

If X is a Hilbert space then the assumption that B is almost U-sectorial is redundant and this reduces to the result of Auscher, McIntosh and Nahmod [1]. However, in general this assumption cannot be eliminated. It suffices to take a sectorial operator A with an $H^\infty-$calculus with $\omega_H(A) > \omega(A)$. Such examples exist [6]; in fact examples are known on subspaces of L_p when $1 < p < 2$ [9]. Now fix θ with $\pi - \omega_H(A) < \theta < \pi - \omega(A)$. Thus $e^{\pm i\theta}A$ are sectorial with $\omega(e^{\pm i\theta}A) \le \omega(A)+\pi-\theta$. However if both have an $H^\infty-$calculus we would deduce that for a suitable constant C

$$\int_0^\infty |\langle u(te^{\pm i\theta}A)x, x^*\rangle|\frac{dt}{t} \le C\|x\|\|x^*\| \qquad x \in X,\ x^* \in X^*$$

which would imply that $\omega_H(A) \le \pi - \theta$. This contradiction implies that at least one of $e^{\pm i\theta}A$ fails to have an $H^\infty-$calculus. However if $B = e^{\pm i\theta}A$ then (1.1) and (1.2) are trivially satisfied.

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