$x \leqq m_1$ is inductive finite. But according to a previous theorem then also the set of the $x \leqq m$ must be inductive finite. Therefore the set of all $x \leqq y$ is inductive finite for arbitrary y. Taking y then as the last element, one sees the truth of the theorem.

Using the last theorems we obtain another version of the proof of the statement that every inductive infinite set M is Dedekind infinite. However we must also use the well-ordering theorem, so that this proof depends on the axiom of choice as well. Let M be well-ordered. Then after our preceding results this well-ordering of M cannot simultaneously be an inverse well-ordering. Thus there is a subset $M_1 \supset 0$ without a last element. The set of all elements $x \leqq$ an element y of M_1 is then an initial part N of M without last element. Every element n of N has a successor $n' \epsilon N$. We may then define a mapping f of M into a proper part of M by putting $f(n) = n'$ for every $n \epsilon N$ and $f(n) = n$ for every n not ϵN.

10. The simple infinite sequence. Development of arithmetic

Let M be a Dedekind infinite set, f a one-to-one correspondence between M and a proper part M' of M. Let 0 denote an element of M not in M'. I denote generally by a' the image f(a) of a, also by P', when $P \subseteq M$, the set of all $p' = f(p)$ when p runs through P. Let N be the intersection of all subsets X of M possessing the two properties

1) $0 \epsilon X$, 2) $(x)(x \epsilon X \rightarrow x' \epsilon X)$.

Then N is called a simple infinite sequence or the f-chain from 0. We may say that it is the natural number series. It is evident that N has the properties 1) and 2). Further we have the principle of induction: A set containing 0 and for every x in it also containing x' contains N.

Theorem 46. $(y)(y \epsilon N \rightarrow (Ex)(y = x') \ \& \ (x \epsilon N) \cdot v \cdot y = 0)$.

This means that any element of N is either 0 or the f-image of another element of N. The proof is easy: Let us assume that $n \epsilon N$ and $\neq 0$ and $\neq$ every x' when $x \epsilon N$. Then $N - \{n\}$ would still possess the properties 1) and 2), which is absurd.

In order to develop arithmetic it is above all necessary to define the two fundamental operations addition and multiplication. Usually these as well as any other arithmetical functions are introduced by the so-called recursive definitions. I shall show how we are able to use here the ordinary explicit definitions which can be formulated with the aid of the predicate calculus. I shall introduce addition and multiplication by defining the sets of ordered triples (x,y,z) such that $x + y = z$ resp. $xy = z$.

We may consider the sets X of triples (a,b,c), where a,b,c are ϵN, which have the two properties:

1) All triples of the form (a,0,a) are ϵX.

2) Whenever (a,b,c) is ϵX, (a,b',c') is ϵX.

It is clear that there exist such sets X. Indeed the set X_0 of all triples (a,b,c), where a,b,c are ϵ N, is one of them.

Now let S be the intersection of all these X. I shall show that S is just the set of triples a,b,c such that a + b = c according to the usual meaning of addition. First of all it is clear that S itself is one of the sets X with the properties 1) and 2). Further, the following inversion of 2) is true:

Theorem 47. *Whenever $(a,b',c') \epsilon S$, we have $(a,b,c) \epsilon S$.*

Proof. Let us assume that we had a triple $(a,b',c') \epsilon S$ while $(a,b,c) \bar{\epsilon} S$. Then it is seen that $S - \{(a,b',c')\}$ would still have the two properties. Indeed if $(\alpha,\beta,\gamma) \epsilon S - \{a,b',c')\}$ then $(\alpha,\beta,\gamma) \epsilon S$, whence $(\alpha,\beta',\gamma') \epsilon S$, whence again $(\alpha,\beta',\gamma') \epsilon S - \{(a,b',c')\}$ unless $\alpha = a, \beta = b, \gamma = c$ which however cannot be the case, since $(a,b,c) \bar{\epsilon} S$, whereas $(\alpha,\beta,\gamma) \epsilon S$.

Using Theorem 46 we may also formulate Theorem 47 thus:

$$(x)(y)(z)[(x,y,z)\epsilon S \ \& \ (y \neq 0) \ \& \ (z \neq 0) \rightarrow (Eu)(Ev)((x,u,v)\epsilon S \ \& \ (y = u') \ \& \ (z = v'))].$$

Theorem 48. $(a,b',0) \bar{\epsilon} S$.

Proof. If, for some a,b, we had $(a,b',0) \epsilon S$, it is seen that $S - \{(a,b',0)\}$ would still satisfy the requirements 1) and 2).

Theorem 49. $(x)(y)((x,0,y) \epsilon S \rightarrow (x = y))$.

Proof. Indeed, if (a,0,b) with $b \neq a$ were ϵS, then $S - \{(a,0,b)\}$ would still possess the properties 1) and 2).

Theorem 50. $(x)(y)((x,y,0) \epsilon S) \rightarrow (x = 0) \ \& \ (y = 0))$.

Proof. Let (a,b,0) be ϵS. According to theorem 48 we have b = 0 because of Theorem 46.

Then Theorem 49 yields a = 0.

Theorem 51. $(x)(y)(z)(u)(((x,y,z) \epsilon S) \ \& \ ((x,y,u) \epsilon S) \rightarrow (z = u))$.

Proof. Let P(b) be the proposition $(x)(z)(u) (((x,b,z) \epsilon S) \ \& \ ((x,b,u) \epsilon S) \rightarrow (z = u))$. Then P(0) is true. Indeed, if $(a,0,c) \epsilon S$ and $(a,0,d) \epsilon S$, it follows from Theorem 49 that c = a and d = a, whence c = d. Let us assume that P(b) is true for some b. Then, if (a,b',c) and (a,b',d) are ϵ S, we have by Theorem 47 that $c = c_1'$, $d = d_1'$ for some c_1 and d_1 while $(a,b,c_1) \epsilon S$ and $(a,b,d_1) \epsilon S$, whence because of the assumed validity of P(b) it follows that $c_1 = d_1$, whence c = d. Hence by complete induction the general validity of P(b) is proved.

Theorem 52. $(x)(y)(Ez)((z,y,z) \epsilon S)$.

Proof. Let P(b) here denote $(x)(Ez)((x,b,z) \epsilon S)$. Then P(0) is true. Let us assume that P(b) is true for some b. Then for arbitrary a there is a c such that $(a,b,c) \epsilon S$, whence $(a,b',c') \epsilon S$ so that P(b') is true. Thus the theorem is proved by complete induction.

The two last theorems show that for every x and y there is just one z such that $(x,y,z) \epsilon S$. We may therefore, instead of $(a,b,c) \epsilon S$, write c = a + b. If, further, 0' is called 1, we have a + 1 = a' and the equations

$$a' \neq 0, \ (a' = b') \leftrightarrow (a = b), \ a + 0 = a, \ a + b' = (a + b)'$$

are generally valid. As is well known we may derive the commutative and associative laws of addition by complete induction. This will be carried out later even in the more difficult case of predicative set theory based on the ramified theory of types.

Now let us consider the sets Y of triples with the two properties:

1) all triples (a,0,0) are ϵ Y

2) whenever (a,b,c) ϵ Y and (c,a,d) ϵ S, we have (a,b',d) ϵ Y.

It is evident that such sets of triples exist. Indeed the set of all triples is such a Y. Now let P be the intersection of all these Y. Then it is clear that P is again such a Y, but we can also prove the following inversions of the properties 1) and 2):

Theorem 53. *If (a,O,b)* ϵ P, *then* $b = 0$.

Proof. Indeed, if (a,0,b) were ϵ P, $b \neq 0$, then P - {(a,0,b)} would not only have the property 1), which is immediately seen, but also 2). Let (α,β,γ) be ϵ P - {(a,0,b)} and (γ,α,δ) ϵ S. Then (α,β,γ) ϵ P together with (γ,α,δ) ϵ S yields (α,β',δ) ϵ P, whence (α,β',δ) ϵ P - {(a,0,b)} because (α,β',δ) cannot coincide with (a,0,b).

Theorem 54. *If (a,b',c)* ϵ *P, then (Ez)((a,b,z)* ϵ *P & (z,a,c)* ϵ *S).*

Proof. Let us assume that we had (a,b',c) ϵ P, while for all z either (a,b,z) $\bar{\epsilon}$ P or (z,a,c) $\bar{\epsilon}$ S. Let us consider the set P' = P - {(a,b',c)}. This set has obviously the property 1). Now let (α,β,γ) be ϵ P' and therefore ϵ P. As proved above, there exists a unique δ such that (γ,α,δ) ϵ S. Then (α,β',δ) ϵ P and therefore also (α,β',δ) ϵ P' unless $\alpha = a, \beta = b, \delta = c$. This is impossible, however, because in such a case we should have (a,b,γ) ϵ P and (γ,a,c) ϵ S. Thus P' would also possess the property 2), and that is absurd.

Theorem 55. (x)(y)(z)(u) ((x,y,z) ϵ P. & (x,y,u) ϵ P $\rightarrow$ (z = u)).

Proof. Let S (b) denote the statement (x)(z)(u) ((x,b,z) ϵ P & (x,b,u) ϵ P $\rightarrow$ (z = u)). Then S(0) is true because (x,0,z) ϵ P $\rightarrow$ (z = 0) and (x,0,u) ϵ P $\rightarrow$ (u = 0) (see Theorem 53). Let us assume that S(b) is true, and let us look at the conjunction (a,b',c_1) ϵ P & (a,b',c_2) ϵ P. If this condition is fulfilled, we have according to Theorem 54, that x and y exist such that (a,b,x) ϵ P & (a,b,y) ϵ P together with (x,a,c_1) ϵ S & (y,a,c_2) ϵ S. Because of the validity of S(b) this yields first x = y, whence $c_1 = c_2$ by Theorem 51.

Theorem 56. (x)(y)(Ez) ((x,y,z) ϵ P).

Proof. Let S(b) here be the statement (x)(Ez) ((x,b,z) ϵ P). Then S(0) is obviously true. Let S(b) be true and let us assume (a,b,c) ϵ P. Then by Theorem 52 there exists a d such that (c,a,d) ϵ S, whence (a,b',d) ϵ P.

The two last theorems show that to every a,b there exists a unique c such that (a,b,c) ϵ P. Therefore we may instead of (a,b,c) ϵ P write c = ab, c being a function of a and b. Further, we have besides the earlier formulas $a' \neq 0$, $(a' = b') \leftrightarrow (a = b)$, $a + 0 = a$, $a + b' = (a + b)'$ also

$$a \cdot 0 = 0, \quad ab' = ab + a.$$

These, together with $(a = b) \rightarrow (a = c \rightarrow b = c)$, beside the principle of induction and the predicate calculus, constitute, however, the axiom system for formal number theory, see, for example, R.L. Goodstein, Mathematical Logic, p. 44. Thus we see that the development of ordinary arithmetic is possible in the Zermelo-Fraenkel set theory.

The method I used here to replace the recursive definition of addition and multiplication by explicit definitions can be used quite generally for other recursive definitions. The primitive recursive schema, for example, is:

$$f(0, a_2, \ldots, a_n) = g(a_2, \ldots, a_n)$$

$$f(a_1 + 1, a_2, \ldots, a_n) = h(f(a_1, \ldots, a_n), a_1, \ldots, a_n)$$

Here g and h are previously defined functions with $n-1$ respectively $n+1$ arguments, while f is the function to be defined. From the set-theoretic standpoint we may replace this recursive definition by the following explicit one. That g and h are already known may be expressed by saying that we have a set G of n-tuples and a set H of $(n+2)$-tuples of elements of N such that for arbitrary $a_1, \ldots, a_{n-1}$ there is just one b such that $(a_1, \ldots, a_{n-1}, b) \epsilon G$ and for arbitrary $a_1, \ldots, a_{n+1}$ there is just one b such that $(a_1, \ldots, a_{n+1}, b) \epsilon H$. Then we consider all sets of $n+1$-tuples of elements of N which possess the two properties:

1) Whenever $(a_2, \ldots, a_n, b) \epsilon G$, we have $(0, a_2, \ldots, a_n, b) \epsilon X$.

2) Whenever $(a_1, a_2, \ldots, a_n, b) \epsilon X$ and $(b, a_1, \ldots, a_n, c) \epsilon H$, we have $(a_1 + 1, a_2, \ldots, a_n, c) \epsilon X$. Then the intersection F of all sets X of this kind yields the function f, namely, as often as $(a_1, \ldots, a_n, b)$ is ϵF, we have $b = f(a_1, \ldots, a_n)$, and inversely.

But also other kinds of recursions may be treated in the same way. As a further example we may take the definition of the Ackermann-Péter function, namely:

$$\phi(0,n) = n + 1$$

$$\phi(m+1, 0) = \phi(m, 1)$$

$$\phi(m+1, n+1) = \phi(m, \phi(m+1, n)).$$

We consider here the sets Z of triples with the three properties:

1) All triples $(0,n,n+1)$ are ϵZ

2) Whenever $(m,1,n)$ is ϵZ, so is $(m+1, 0, n)$

3) For arbitrary m, n, h, k we have

$$(m+1, n, h) \epsilon Z . \& . (m, h, k) \epsilon Z \rightarrow (m+1, n+1, k) \epsilon Z.$$

If ϕ is the intersection of all these sets Z, one proves easily that to every pair a, b there is just one c such that $(a,b,c) \epsilon \phi$. Thus c is a function ϕ of a,b, and this ϕ is just the function defined by the recursive schema.