Appendix A

ANOTHER PROOF OF A LEMMA BY SCHNEIDER

The lemma by Schneider proved in §3 of Chapter 6 may also be obtained by means of a different method. This method has the advantage of leading to a slightly stronger result. It is due to my former colleague, G.E.H. Reuter, now professor of mathematics at the University of Durham.

1. A special case of Taylor's formula with Lagrange's error term states that if $f(x)$ is four times differentiable in a neighbourhood of $x=0$, then

$$f(x) = f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{IV}(\xi)\frac{x^4}{4!}$$

where ξ is a number between 0 and x. Let us apply this formula to the function $f(x)=\log\cosh x$ for non-negative values of x. Then

$$f'(x) = \tanh x, \quad f''(x) = \cosh^{-2}x, \quad f'''(x) = -2\sinh x\cosh^{-3}x$$

and

$$f^{IV}(x) = 4\cosh^{-2}x - 6\cosh^{-4}x.$$

The fourth derivative assumes its maximum when $\cosh x=\sqrt{3}$, and so

$$f^{IV}(x) \leq \frac{2}{3} \qquad \text{for all } x \geq 0.$$

It follows therefore that

$$\log\cosh x \leq \frac{1}{2}x^2 + \frac{2}{3}\cdot\frac{x^4}{24},$$

and hence that

(1): $$\cosh x \leq \exp\left(\frac{1}{2}x^2 + \frac{1}{36}x^4\right) \qquad \text{if } x \geq 0.$$

2. Let again $r_1,\ldots,r_m$ be m positive integers; let further s, $\rho_1,\ldots,\rho_m$ be $m+1$ positive numbers. We denote by N the number of sets of m integers $(i_1,\ldots,i_m)$ satisfying the inequalities

(2): $$0 \leq i_1 \leq r_1,\ldots, 0 \leq i_m \leq r_m, \quad \sum_{h=1}^{m}\frac{i_h}{\rho_h} \leq \left(\frac{1}{2}-s\right)\sum_{h=1}^{m}\frac{r_h}{\rho_h},$$

or, what is the same, the number of such sets satisfying

(3): $$0 \leq i_1 \leq r_1,\ldots, 0 \leq i_m \leq r_m, \quad \sum_{h=1}^{m}\frac{i_h}{\rho_h} \geq \left(\frac{1}{2}+s\right)\sum_{h=1}^{m}\frac{r_h}{\rho_h}.$$

That both systems (2) and (3) have the same number of integral solutions is obvious because the transformation

$$(i_1,\ldots,i_m) \rightarrow (r_1-i_1,\ldots,r_m-i_m)$$

interchanges their solutions.

3. Denote by u a positive variable, and put

$$F_h(u) = \sum_{i=0}^{r_h} \exp\left\{u\left(\frac{i}{\rho_h} - \frac{r_h}{2\rho_h}\right)\right\} \qquad (h = 1,2,\ldots, m),$$

and

$$F(u) = \sum_{i_1=0}^{r_1} \cdots \sum_{i_m=0}^{r_m} \exp\left\{u \sum_{h=1}^{m}\left(\frac{i_h}{\rho_h} - \frac{r_h}{2\rho_h}\right)\right\} .$$

Evidently,

$$F(u) = \prod_{h=1}^{m} F_h(u) .$$

In the sum for $F_h(u)$ replace i by $r_h - i$ and note that

$$\frac{r_h - i}{\rho_h} - \frac{r_h}{2\rho_h} = -\left(\frac{i}{\rho_h} - \frac{r_h}{2\rho_h}\right) .$$

It follows that

$$F_h(u) = \frac{1}{2}\sum_{i=0}^{r_h}\left(\exp\left\{u\left(\frac{i}{\rho_h} - \frac{r_h}{2\rho_h}\right)\right\} + \exp\left\{-u\left(\frac{i}{\rho_h} - \frac{r_h}{2\rho_h}\right)\right\}\right)$$

$$= \sum_{i=0}^{r_h} \cosh\left\{u\left(\frac{i}{\rho_h} - \frac{r_h}{2\rho_h}\right)\right\}$$

$$\leqslant (r_h+1) \max_{i=0,1,\ldots,r_h} \cosh\left\{u\left(\frac{i}{\rho_h} - \frac{r_h}{2\rho_h}\right)\right\} .$$

Now $\cosh x$ is decreasing for $x \leqslant 0$ and increasing for $x \geqslant 0$. The maximum is thus attained both when $i=0$ and when $i=r_h$, and hence

$$F_h(u) \leqslant (r_h+1) \cosh \frac{r_h u}{2\rho_h} .$$

Therefore, by (1),

$$F(u) \leqslant (r_1+1)\ldots(r_m+1) \prod_{h=1}^{m} \cosh \frac{r_h u}{2\rho_h}$$

$$\leqslant (r_1+1)\ldots(r_m+1) \exp\left\{\frac{1}{2}\sum_{h=1}^{m}\left(\frac{r_h u}{2\rho_h}\right)^2 + \frac{1}{36}\sum_{h=1}^{m}\left(\frac{r_h u}{2\rho_h}\right)^4\right\},$$

that is,

$$(4): \qquad F(u) \leqslant (r_1+1)\ldots(r_m+1) \exp\left\{\frac{u^2}{8}\sum_{h=1}^{m}\left(\frac{r_h}{\rho_h}\right)^2 + \frac{u^4}{576}\sum_{h=1}^{m}\left(\frac{r_h}{\rho_h}\right)^4\right\} .$$

4. By definition, the inequalities (3) have N integral solutions $(i_1, \ldots, i_m)$. These inequalities may also be written as

$$0 \leq i_1 \leq r_1, \ldots, 0 \leq i_m \leq r_m, \quad \sum_{h=1}^{m} \left(\frac{i_h}{\rho_h} - \frac{r_h}{2\rho_h} \right) \geq s \sum_{h=1}^{m} \frac{r_h}{\rho_h},$$

and they therefore imply that

$$(5): \qquad \exp\left\{ u \sum_{h=1}^{m} \left(\frac{i_h}{\rho_h} - \frac{r_h}{2\rho_h} \right) \right\} \geq \exp\left(su \sum_{h=1}^{m} \frac{r_h}{\rho_h} \right).$$

On the other hand, all terms in the multiple sum for $F(u)$ are positive. It follows then from (5) that

$$F(u) \geq N \exp\left(su \sum_{h=1}^{m} \frac{r_h}{\rho_h} \right).$$

On combining this inequality with (4), we find that

$$(6): \quad N \leq (r_1+1)\ldots(r_m+1) \exp\left\{ -su \sum_{h=1}^{m} \frac{r_h}{\rho_h} + \frac{u^2}{8} \sum_{h=1}^{m} \left(\frac{r_h}{\rho_h} \right)^2 + \frac{u^4}{576} \sum_{h=1}^{m} \left(\frac{r_h}{\rho_h} \right)^4 \right\}.$$

To simplify this estimate, put

$$c_1 = \frac{1}{m} \sum_{h=1}^{m} \frac{r_h}{\rho_h}, \qquad c_2 = \frac{1}{m} \sum_{h=1}^{m} \left(\frac{r_h}{\rho_h} \right)^2, \qquad c_4 = \frac{1}{m} \sum_{h=1}^{m} \left(\frac{r_h}{\rho_h} \right)^4,$$

and fix u in terms of s by

$$u = \frac{4c_1 s}{c_2}.$$

The inequality (6) then takes the form

$$(7): \qquad N \leq (r_1+1)\ldots(r_m+1) \exp\left\{ -m \left(\frac{2c_1^2}{c_2} s^2 - \frac{4}{9} \frac{c_1^4 c_4}{c_2^4} s^4 \right) \right\}.$$

For the applications it suffices to consider values of s with

$$0 \leq s \leq \frac{1}{2} \qquad \text{and hence} \qquad s^4 \leq \frac{1}{4} s^2.$$

It follows in this case from (7) that

$$(8): \qquad N \leq (r_1+1)\ldots(r_m+1) \exp(-Cms^2)$$

where C denotes the expression

$$C = \frac{2c_1^2}{c_2} - \frac{c_1^4 c_4}{9c_2^4}.$$

5. We finally impose on r_h and ρ_h the additional conditions

$$\left| \frac{r_h}{\rho_h} - 1 \right| \leq \frac{1}{10} \qquad (h = 1, 2, \ldots, m)$$

These inequalities evidently imply that

$$\frac{9}{10} \leqslant c_1 \leqslant \frac{11}{10}, \qquad \left(\frac{9}{10}\right)^2 \leqslant c_2 \leqslant \left(\frac{11}{10}\right)^2, \qquad \left(\frac{9}{10}\right)^4 \leqslant c_4 \leqslant \left(\frac{11}{10}\right)^4,$$

and hence that

$$\frac{2}{3} < \frac{81}{121} = \frac{\left(\frac{9}{10}\right)^2}{\left(\frac{11}{10}\right)^2} \leqslant \frac{c_1^2}{c_2} \leqslant \frac{\left(\frac{11}{10}\right)^2}{\left(\frac{9}{10}\right)^2} = \frac{121}{81} < \frac{3}{2},$$

$$\frac{c_4}{c_2^2} \leqslant \frac{\left(\frac{11}{10}\right)^4}{\left(\frac{9}{10}\right)^4} = \left(\frac{121}{81}\right)^2 < \frac{9}{4} < 3.$$

It follows therefore that

$$C = \frac{2c_1^2}{c_2}\left\{1 - \frac{1}{18}\cdot\frac{c_1^2}{c_2}\cdot\frac{c_4}{c_2^2}\right\} \geqslant 2\cdot\frac{2}{3}\cdot\left(1 - \frac{1}{18}\cdot\frac{3}{2}\cdot 3\right) = 1.$$

Thus the following result has been proved.

Theorem 1: Let $r_1,\ldots,r_m$ *be* m *positive integers, and let* s, $\rho_1,\ldots,\rho_m$ *be* $m+1$ *positive numbers such that*

$$0 \leqslant s \leqslant \frac{1}{2}; \qquad \left|\frac{r_h}{\rho_h} - 1\right| \leqslant \frac{1}{10} \qquad (h = 1,2,\ldots,m).$$

There are at most

$$(r_1+1)\ldots(r_m+1)e^{-ms^2}$$

integral solutions $(i_1,\ldots,i_m)$ *of the inequalities*

$$0 \leqslant i_1 \leqslant r_1,\ldots, 0 \leqslant i_m \leqslant r_m, \quad \sum_{h=1}^{m} \frac{i_h}{\rho_h} \leqslant \left(\frac{1}{2} - s\right) \sum_{h=1}^{m} \frac{r_h}{\rho_h}$$

or, what is the same, of the inequalities

$$0 \leqslant i_1 \leqslant r_1,\ldots, 0 \leqslant i_m \leqslant r_m, \quad \sum_{h=1}^{m} \frac{i_h}{\rho_h} \geqslant \left(\frac{1}{2} + s\right) \sum_{h=1}^{m} \frac{r_h}{\rho_h}.$$

Let us compare this estimate with that given by the Lemma 2 of Chapter 6 in the special case when $\rho_1 = r_1,\ldots,\rho_m = r_m$! The notation is slightly distinct at the two places. If we return to that of Lemma 2, then, by this lemma, the inequalities

$$0 \leqslant i_1 \leqslant r_1,\ldots, 0 \leqslant i_m \leqslant r_m, \quad \sum_{h=1}^{m} \frac{i_h}{r_h} \leqslant \frac{1}{2}(m-s) \quad \left(\text{or} \geqslant \frac{1}{2}(m+s)\right)$$

have not more than

$$(r_1+1)\ldots(r_m+1)\,\frac{\sqrt{2m}}{s}$$

integral solutions, and by Theorem 1 not more than

$$(r_1+1)\dots(r_m+1)e^{-\left(\frac{s}{\sqrt{m}}\right)^2} .$$

It is easily verified that always

$$e^{-\left(\frac{s}{\sqrt{m}}\right)^2} < \frac{\sqrt{2m}}{s} .$$

Hence Theorem 1 is not only more general, but also a little stronger than Lemma 2. Unfortunately, this improvement does not seem to be of great use in Roth's theory.

6. In Chapter 6 the Lemma 2 enabled us to prove the existence of the approximation polynomial $A(x_1,\dots,x_m)$ which played such an important role in the proof of Roth's Theorem and the more general Approximation Theorems. Theorem 1 allows to construct a more general approximation polynomial. There is no need for giving the details of the proof which is just like that in Chapter 6. The final result is as follows.

Theorem 2: *Let*

$$F(x) = F_0x^f + F_1x^{f-1} + \dots + F_f , \quad \textit{where} \quad f \geq 1,\ F_0 \neq 0,\ F_f \neq 0,$$

be a polynomial with integral coefficients which has no multiple factors and does not vanish at $x=0$. *There exists a positive constant* c *depending only on* $F(x)$ *as follows. Let* m *be a positive integer,* s *a real number such that*

$$0 \leq s \leq \frac{1}{2}, \quad ms^2 \geq \log(4f);$$

let $r_1,\dots,r_m$ *be* m *positive integers; and let* $\rho_1,\dots,\rho_m,\sigma_1,\dots,\sigma_m,\tau_1,\dots,\tau_m$ *be* $3m$ *positive numbers satisfying*

$$\left|\frac{r_h}{\rho_h} - 1\right| \leq \frac{1}{10}, \left|\frac{r_h}{\sigma_h} - 1\right| \leq \frac{1}{10}, \left|\frac{r_h}{\tau_h} - 1\right| \leq \frac{1}{10} \qquad (h = 1,2,\dots,m).$$

Then there exists a polynomial

$$A(x_1,\dots,x_m) = \sum_{i_1=0}^{r_1} \dots \sum_{i_m=0}^{r_m} a_{i_1\dots i_m} x_1^{i_1}\dots x_m^{i_m} \neq 0$$

with the following properties.

(i): *The coefficients are integers satisfying*

$$|a_{i_1\dots i_m}| \leq c^{r_1+\dots+r_m},$$

and each coefficient $a_{i_1\dots i_m}$ *vanishes unless both*

$$\sum_{h=1}^{m} \frac{i_h}{\rho_h} > \left(\frac{1}{2} - s\right) \sum_{h=1}^{m} \frac{r_h}{\rho_h} \quad \text{and} \quad \sum_{h=1}^{m} \frac{i_h}{\sigma_h} < \left(\frac{1}{2} + s\right) \sum_{h=1}^{m} \frac{r_h}{\sigma_h} .$$

(ii): $A_{j_1 \ldots j_m}(x,\ldots,x)$ *is divisible by* $F(x)$ *for all suffixes* $j_1,\ldots,j_m$ *such that*

$$0 \leq j_1 \leq r_1,\ldots,\ 0 \leq j_m \leq r_m,\ \sum_{h=1}^{m} \frac{j_h}{\tau_h} \leq \left(\frac{1}{2} - s\right) \sum_{h=1}^{m} \frac{r_h}{\tau_h} .$$

(iii): *The following majorants hold,*

$$A_{j_1 \ldots j_m}(x_1,\ldots,x_m) \ll c^{r_1+\ldots+r_m}(1+x_1)^{r_1}\ldots(1+x_m)^{r_m},$$

$$A_{j_1 \ldots j_m}(x,\ldots,x) \ll c^{r_1+\ldots+r_m}(1+x)^{r_1+\ldots+r_m} .$$

Should it be possible to replace Roth's Lemma in Chapter 5 by a stronger result, then Theorem 2 might become of importance.