PREFACE TO THE SECOND EDITION

In the present second edition it has been decided not to make changes in the text; but the publishers have asked me to make here a brief statement of progress since the first edition appeared, and to comment on changes which would be desirable in the light of this progress.

When these lectures were delivered in April, 1948, there were several unanswered questions. Although work on these questions was continued while the manuscript was in press, yet an effort was made to make the text proper represent exactly the state of affairs at the time of delivery, with brief indications of later results in the footnotes. (The typing of the final copy for photography was completed in April, 1949, and minor corrections could be made until December, 1949.) The later results were then written up in [95], [96], [98], [106], and [108]; of these [95] and [108], ([98] is an abstract of these two) contain the new results in regard to negation; [96] contains a complete revision of Chapter V, including the elimination theorem for the systems LXY, as well as a reformulation and more abstract proof of the elimination theorem in general and its extension to the singular forms of LC and LK (described in V 7.3, p. 110); and [106] deals with the permutability of rules and the strengthened Gentzen Hauptsatz for the classical system. (This paper requires correction--see below.)

In addition to these papers, [103], which represents lectures delivered at Louvain in the winter of 1950-51, contains a treatment of that algebraic approach whose neglect was mentioned in the preface to the first edition, p. iv. Philosophical comment on the nature of implication, largely based on these lectures, is contained in [107] and [99]. Finally, in a proposed book [111] on which Prof. Feys and I are collaborating, applications of the Gentzen method to proving the consistency of certain systems of combinatory logic are expected to appear. The fact that such methods could be used was already mentioned in [16] and [17], and formed part of the motivation for making this study in the first place. In particular [111], in §9F4 contains the proof of a form of elimination theorem; if this method of proof were adapted to the present circumstances it would give a proof of the elimination theorem which would be valid, under certain limitations, without the rules W and K; this would be a great improvement on the proof given here or in [96]. However, this has not yet been worked out in detail.

In the meantime Gentzen's methods, and others similar to them, have interested several other writers. A French translation of Gentzen's thesis by Feys and Ladrière [112] has recently appeared; there is also an interesting study by Ladrière [123] of the Gentzen Hauptsatz and its relation to other logical theorems (Herbrand, Church, etc). The Gentzen methods themselves have been studied and improved by Quine (see [130], [131], also Hermes [114] and Stanley [143]) and Kleene [119], [120], [121]. Rosser [135], although he does not use Gentzen methods explicitly, formulates derived rules, in particular his "Rule C", which are strongly reminiscent of them in the form given by Quine. The work of Schütte [140] is a compromise between Gentzen methods and those of the ordinary predicate calculus; he shows that we can obtain the advantages of the Gentzen techniques by procedures, expressed in the ordinary notation of the predicate calculus, which is an advantage for some purposes. The work of Lorenzen [124], [126],--his book [125] is not yet available--is more distantly related; but it is similar in spirit and is suggestive. (Both Lorenzen and Schütte have other papers, not cited here, giving applications to more powerful systems.) Finally there are some works whose connection with the present program is, for one reason or another, not quite clear, viz: [116], [117], [129], [133], [134], [136], [142], [141].

The work of Kleene requires special comment. From a remark in his [121] one gathers that his attention was called to Gentzen's thesis by his observation in 1947 that the non-deducibility of certain formulas in intuitionistic predicate calculus could be established by its aid. (Cf. III §6; the proof there given dates from 1937, although my notes of that date contained an error. For other non-deducibility proofs see [119], [45], [115], [128].) In his paper [122] he gave a more thorough treatment of rule permutation than that in [106]; in fact, he showed that the theorem of the latter is not true if the second, R_2, of the rules to be permuted contains a characteristic variable which occurs in a component of the first, and this possibility has to be shown not to occur in the proof of Gentzen's strengthened Hauptsatz. In [120] and [122], Chap. XV, Kleene gave a treatment of the Gentzen technique which has many analogies with, and improvements on, that attempted here.

Some remarks are now in order concerning some items which were listed in the original bibliography. Items [21] and [23] have been published since the first edition: the former in Mind, vol. 62, pp. 172-183; the latter as a book (VIII + 75 pp) by the North Holland Publishing Co. at the end of 1951. Of the items listed in the preface as not yet examined all but [55] have since been seen. That of Ketonen [53] is especially interesting. It contains a formulation of the rule $\Lambda\ell$ as a single rule, viz.

$$\frac{\mathfrak{X},\ A,\ B \Vdash \mathfrak{Y}}{\mathfrak{X},\ A \quad B \Vdash \mathfrak{Y}}$$

The rules for the classical system are so formulated as to be reversible. There is also a proof of the completeness theorem for LK directly from the L-formulation which far surpasses anything previously given.

This concludes the general discussion of progress. We turn now to matters of detail concerning portions of the monograph.

Chapter I. This topic has been revised in several papers, in particular [103], Chap. 1. The first two chapters of [111] will be devoted to formal systems and epitheoretic methods other than those considered here. For philosophical discussion of special topics see [99], [100], [107], [109], [110].

Certain changes in terminology have been made in this newer work. The word 'term' as here used conflicts with the usual uses of that word in the U-language, so beginning about 1950 the colorless word 'ob' has been substituted for it. Now I favor confining the word 'proposition' to the obs of certain formal systems whose usual interpretation makes that suitable, and the term 'statement' for what is asserted by a sentence. This usage is introduced in [99]. It is quite consistent with the notation of Chapter II ff, because on the level of the episystems what are here called propositions are actually obs.

The grammatical terminology introduced in §5 is too elaborate. Terms like 'functive' had better be dropped; and the terms in the third column of Table 3 used instead of those in the second (except 'predicate'). For ordinary logical purposes the terms 'operator', 'operation' are to be preferred to 'adjunctor', 'adjunction'. Finally, there can be no objection to using functors which are binary infixes as names for the corresponding functions, so that blanks, etc., in such cases are unnecessary.

Chapter II. The rule bl) on p. 25 seems to require more motivation. It says in effect that, in the presence of A, $A \supset B$ is at least as strong as B. It is then a constructive form of modus ponens.

On p. 26 the term 'entails' is now preferred for '$\Vdash$'. The relation $\Vdash$ is sometimes called 'entailment'.

In the formulation of the rules we can require that the A in pl) be elementary. The possibility of alternative formulations such as those in Remarks 2 and 3 needs to be considered more thoroughly. The different formulations stand more or less on a par, like Kleene's G1, G2, and G3, and are suitable for different purposes.

Under Remark 3 on page 36 the premise over Er' should be $A_1, \ldots, A_m \vdash B$; it is intended to apply to Er' only.

On page 44, Theorem 8 could be generalized to read as follows:

Theorem 8'. If all constituents of $\mathfrak{X}$ and $\mathfrak{Y}$ are elementary and (4) holds, then some constituent of $\mathfrak{Y}$ is S-deducible from $\mathfrak{X}$.

If the elimination theorem were stated in the form

$$\frac{\mathfrak{X}_1,\ A \Vdash \mathfrak{D} \qquad\qquad \mathfrak{X}_2 \Vdash A, \mathfrak{Z}}{\mathfrak{X}_1,\ \mathfrak{X}_2 \Vdash \mathfrak{D}, \mathfrak{Z}}$$

it could probably be deduced without requiring either K or W, provided that the rules were suitably reformulated. This seems reasonable in view of the proof cited from [111], but the details have not been worked out. There are interesting systems to which such a proof would apply although the present one does not. The proof should be given in a more abstract form like that of [96], so as to make the extension to further operations less hazardous.

Instead of the treatment of TC given, the singular form (LC_1 of [96]) should be introduced and proved equivalent to LC. The equivalence between LC_1 and TC would then be easy.

The case of Er was omitted from the proof of Theorem 17 inadvertently.

On p. 63, Theorem 22 applies to particular formulations of HA and HC only; the formulation for HA is that of Theorem 20; that for HC is the first of the two formulations in Theorem 21.

<u>Chapter III</u>. On p. 67 it should be explained that $q = r + s$.

The assumption A3 is related to the assumption of the predi cate calculus that there is at least one individual. Lately there has been some interest in including the empty domain (e.g. [113], [127], [132]). From that standpoint one would presumably drop this assumption.

On p. 80, Definition 6, the requirement (c) can, of course, be weakened.

On p. 82, at the end of Theorem 8 read "LA*(∘ ‿)" instead of "LA*(·𝔇)‿ ".

On p. 88, Remark 2 applies only to the stated formulations of HA* and HC*.

The improvements in technique due to Quine should certainly be mentioned in this chapter.

<u>Chapter IV</u>. For the principal corrections to this chapter see [98]. This paper is an abstract; the details are given in [95] and [108] (see also rev. [93]). The general tendency of the corrections is to do as much as possible in the L-systems.

An alternative form of the Glivenko theorem for LD, using $\neg A \supset A$ in the place of $\neg\neg A$, is given in [103] V §5.

In regard to the error of Wajsberg commented on in footnote 17, p. 107, Scholz has called my attention to the correction made by Wajsberg in [144] p. 21.

On p. 108 it should be added to footnote 21 that the completeness of the classical propositional algebra was elegantly established by Ketonen in [53]. Of the proofs existing before that time that of Kalmar [119] is well adapted to the present context.

Chapter VI. The proof of the elimination theorem in [96] requires extensive changes in this chapter. These are indicated in [96] itself.

So far as necessity is concerned, this brings the treatment for a type of necessity similar to Lewis's S4 to a degree of completeness comparable with that for the non-modal connections. One can formulate a necessity of S5 type, but I do not know of any proof of the elimination theorem for it. Whether the new approach, above indicated, to the elimination theorem will yield such a proof remains to be seen.

As to possibility, the conjectures made on pp. 119-120 have turned out to be erroneous. Some remarks as to possibility are made in [96]. These are not completely satisfactory. Indeed, a completely satisfactory theory of possibility, regarded as an independent connection, has not yet been proposed.

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