Introduction

Secondary characteristic classes are one of the most invariant tools in studying foliations. The Godbillon–Vey class is the most significant, and is extensively studied. It is well-known that it admits continuous deformations [65], [41]. On the other hand, when foliations are assumed to admit certain transversal structures, it often become rigid or trivial [12], [63], [20], [14], [44], [57]. In this monograph, secondary characteristic classes (secondary classes for short) of transversely holomorphic foliations are studied. It is known that the Godbillon–Vey class is decomposed into a product of another secondary class and a power of the Chern class ([64], [3]). We will call this secondary class the imaginary part of the Bott class. The Bott class is a complex secondary class in the sense that it is an invariant of transversely holomorphic foliations, and that it is a cohomology class with coefficients in $\mathbb{C}$ or $\mathbb{C}/\mathbb{Z}$. The definition of the Bott class is quite similar to that of the Godbillon–Vey class, however, it can be found already in [17] (see also [18, p. 49]). A lot of examples of non-trivial Godbillon–Vey classes are known for real foliations [65], [41]. On the other hand, such examples are barely known for transversely holomorphic foliations. There is a paper of Rasmussen [64], where some examples are given by using actions of complex Lie groups such as $\mathrm{SL}(n;\mathbb{C})$. The construction makes use of the existence of certain lattices of which the existence seems unknown [51]. In this monograph, we will first show that this difficulty can be avoided in a natural way, and construct some examples of the same kind by using other semisimple Lie groups. More precisely, we will show the following in Chapter 3.

THEOREM A.

1) *For each q, there are transversely holomorphic foliations of complex codimension q of which the Godbillon–Vey classes are non-trivial.*
2) *If q is odd and $q \geq 3$, then there are at least two transversely holomorphic foliations of complex codimension q which are non-cobordant as real foliations of codimension $2q$. If $q = 5$, then there are at least three transversely holomorphic foliations such that none of them are cobordant as real foliations of real codimension 10.*

Moreover, these foliations can be realized as locally homogeneous foliations.

Two foliations or Γ-structures $\mathcal{F}_1$ of M_1 and $\mathcal{F}_2$ of M_2 are said to be *cobordant* if there is a manifold M equipped with a (transversely holomorphic) foliation $\mathcal{F}$ such that $\partial M = M_1 \amalg M_2$ and that $\mathcal{F}|_{M_1} = \mathcal{F}_1$, $\mathcal{F}|_{M_2} = \mathcal{F}_2$.

We will compute the Godbillon–Vey class by using the theory due to Kamber–Tondeur [49], which says that the characteristic mapping from $H^*(\mathrm{WO}_{2q})$ or $H^*(\mathrm{WU}_q)$ is factored through the Lie algebra cohomology (Theorems 3.1.8 and 3.1.14). For completeness of the exposition, we will give a proof by following Baker [12].

Once the non-triviality is shown, the rigidity makes sense. Indeed, it is known that the Bott class admits continuous deformations [17], [19]. Therefore, one can expect that the Godbillon–Vey class also admits continuous deformations in the category of transversely holomorphic foliations. It is however not the case. Indeed, the above-mentioned formula for the Godbillon–Vey class implies that it is in the image of secondary classes of higher codimensional foliations. For real foliations, it is well-known that there is a natural map from $H^*(\mathrm{WO}_{q+1})$ to $H^*(\mathrm{WO}_q)$ such that the image consists of rigid classes, namely, classes rigid under deformations. A counterpart for transversely holomorphic foliations is the mapping $H^*(\mathrm{WU}_{q+1})$ to $H^*(\mathrm{WU}_q)$, and it is also well-known that the image consists of rigid classes (see Chapter 4 for the definition of deformations of transversely holomorphic foliations). The Godbillon–Vey class is an element of $H^*(\mathrm{WO}_q)$, and not in the image of $H^*(\mathrm{WO}_{q+1})$. On the other hand, the Godbillon–Vey class for transversely holomorphic foliations is an element of $H^*(\mathrm{WU}_q)$, and in the image of $H^*(\mathrm{WU}_{q+1})$. We are interested not only in actual deformations but infinitesimal deformations. Derivatives of secondary classes can be defined with respect to actual deformations as well as infinitesimal deformations in a unified manner. They are studied by Heitsch [39], [40], [42]. In the last paper, derivatives of real secondary classes with respect to infinitesimal deformations are obtained. Also, derivatives of certain type of *cocycles* are obtained for complex secondary classes. We will complete his definitions, and introduce derivatives of *classes* in $H^*(\mathrm{WU}_q)$. In Chapter 4, the following is shown.

THEOREM B. *The Godbillon–Vey class is rigid under both actual and infinitesimal deformations in the category of transversely holomorphic foliations.*

Indeed, Theorem B is valid for classes which belong to the image of the natural map $H^*(\mathrm{WU}_{q+1}) \to H^*(\mathrm{WU}_q)$ (Theorems B1 and B2).

This monograph is organized as follows. First of all, basic notions and general constructions of secondary classes are recalled. In Chapter 2, we review relation between real and complex secondary classes. In Chapter 3, Theorem A is shown in steps. Firstly, the theory of Kamber–Tondeur is recalled in Section 3.1. As a result, it will be shown that secondary classes of locally homogeneous foliations are realized in the Lie algebra cohomology. Some related known results in the real category are also recalled. Calculations of Lie algebra cohomology using the unitary trick will be explained in Section 3.2. The construction of examples is carried out in Section 3.3. They are constructed on the complex simple groups of type A_n, B_n, C_n and G_2. These examples will have some common properties and it will be shown that the groups of type D_n, E_n and F_4 cannot have foliations having these properties.

In Chapter 4, Theorem B is shown. The proof is separately given for smooth deformations and for infinitesimal deformations.

In Chapter 5, relations with the residue of Heitsch [41], [43] are discussed. A relation of infinitesimal derivatives of the Bott class and a certain cohomological invariant introduced by Fuks [28], Lodder [56] and Kotschick [55] is also shown (Theorem 5.14). The rigidity of the Godbillon–Vey class also follows from the theorem. It is not difficult but we think that it is new.

In the last chapter, an attempt to formulate a version of Duminy's theorem for transversely holomorphic foliations is briefly given. Duminy's theorem for real codimension-one foliations is significant because it deeply relates the Godbillon–Vey class with dynamical properties of foliations. We will explain that there is an analogue for transversely holomorphic foliations of complex codimension one, though it is quite weaker than the original one.

The most part of this monograph is based on a preprint [4]. Basic materials and definition of infinitesimal derivatives are added. Alternative proofs of the rigidity of the Godbillon–Vey classes are also added. One is Corollary 4.3.30, which can be found in [10], and the other one is Theorem 5.14.