The Arithmetical Hierarchy Theorem shows that there are no inclusions among the classes Π_n^0 and Σ_n^0 other than those given by 13.2.

The Arithmetical Enumeration Theorem is false for Δ_n^0 relations; for if it were true, we could use the proof of the Arithmetical Hierarchy Theorem to show that there is a Δ_n^0 relation which is not Δ_n^0.

Let Φ be a set of total functions. If Q is any concept defined in terms of recursive functions, we can obtain a definition of Q in Φ or relative to Φ by replacing recursive everywhere in the definition of Q by recursive in Φ. For example, R is arithmetical in Φ if it has a definition (1) where P is recursive in Φ; and R is Π_n^0 in Φ if it has such a definition in which the prefix is Π_n^0. We shall assume that this is done for all past and future definitions.

Now let us consider how the results of this section extend to the relativized case. Up to the Enumeration Theorem, everything extends without problems. The rest extends to finite Φ but not to arbitrary Φ. For example, if Φ is the set of all reals, then every unary relation is recursive in Φ and hence Π_n^0 and Σ_n^0 in Φ for all n. Thus the Hierarchy Theorem fails. Since the Hierarchy Theorem is a consequence of the Enumeration Theorem, the Enumeration Theorem also fails.

14. Recursively Enumerable Relations

A relation R is semicomputable if there is an algorithm which, when applied to the inputs $\vec{x}$, gives an output iff $R(\vec{x})$. If F is the function computed by the algorithm, then the algorithm applied to $\vec{x}$ gives an output iff $\vec{x}$ is in the domain of F. Hence R is semicomputable iff it is the domain of a computable function.

As an example, let A be the set of n such that $x^n + y^n = z^n$ holds for some positive integers x, y, and z. Then A is semicomputable; the algorithm with input n tests each triple (x,y,z) in turn to see if $x^n + y^n = z^n$. On the other hand, it is not known if A is computable.

A relation is recursively enumerable (abbreviated RE) if it is the domain of a recursive function. By the above and Church's Thesis, a relation is RE iff it is semicomputable.

We let W_e be the domain of the function $\{e\}$. We say that e is an index of the relation R if R is W_e. (Note that this is not the same as being an index of the function χ_R.) Clearly a relation has an index iff it is RE. By the Normal Form Theorem, we have

$$(1) \qquad W_e(\vec{x}) \leftrightarrow \exists y T_k(e,\vec{x},y).$$

14.1. PROPOSITION. A relation is RE iff it is Σ_1^0.

Proof. If R is RE, it is W_e for some e and hence Σ_1^0 by (1). Suppose that R is Σ_1^0; say $R(\vec{x}) \leftrightarrow \exists y P(\vec{x},y)$ with P recursive. Then R is the domain of the recursive function F defined by $F(\vec{x}) \simeq \mu y P(\vec{x},y)$ and hence is RE. □

We often use 14.1 tacitly. In particular, we use it to apply the results of the last section to RE relations. By the Enumeration Theorem, there is a $(k+1)$–ary RE relation R which enumerates the class of k–ary RE relations. In fact, we can define such an R by $R(\vec{x},e) \leftrightarrow W_e(\vec{x})$; this is RE by (1).

By the Arithmetical Hierarchy Theorem, there is an RE set which is not recursive. In fact, the proof of that theorem shows that such a set D is defined by $D(e) \leftrightarrow W_e(e)$.

We let $W_{e,s}$ be the domain of $\{e\}_s$. Then $W_e(\vec{x})$ iff $W_{e,s}(\vec{x})$ for some s; in this case, $W_{e,s}(\vec{x})$ for all $s \geq y$, where y is the computation number of $\{e\}(\vec{x})$. By 8.4, $W_{e,s}(\vec{x})$ is a recursive relation of e,s, and $\vec{x}$. Note also that if $W_{e,s}(\vec{x})$, then each x_i is less than the computation number of $\{e\}(\vec{x})$ and hence less than s. Thus $W_{e,s}$ is finite.

14.2. RE PARAMETER THEOREM. If R is a $(k+m)$–ary RE relation, there is a recursive total function S such that

$$W_{S(y_1,\ldots,y_m)}(\vec{x}) \leftrightarrow R(\vec{x},y_1,\ldots,y_m)$$

for all $\vec{x},y_1,\ldots,y_m$.

Proof. This is easily proved from the Parameter Theorem. □

We can use implicit definitions to define RE relations. Thus suppose that we want to find an RE relation R with an index e such that $R(\vec{x}) \leftrightarrow P(\vec{x},e)$, where P is RE. Let P be the domain of the recursive function G. By the Recursion Theorem, we can find a recursive F with an index e such that $F(\vec{x}) \simeq G(\vec{x},e)$. We then take R to be the domain of F.

A <u>selector</u> for a $(k+1)$–ary relation R is a k–ary function F such that for each $\vec{x}$, $F(\vec{x})$ is defined iff $\exists y R(\vec{x},y)$; and, in this case, $F(\vec{x})$ is a y such that $R(\vec{x},y)$.

14.3. SELECTOR THEOREM. Every $(k+1)$–ary RE relation has a recursive selector.

Proof. Let $R(\vec{x},y) \leftrightarrow \exists z P(\vec{x},y,z)$ with P recursive. Then a recursive selector F for R is defined by

$$F(\vec{x}) \simeq (\mu w P(\vec{x},(w)_0,(w)_1))_0. \ \square$$

If F is k–ary, the <u>graph</u> of F, designated by $\mathcal{G}_F$, is the $(k+1)$–ary relation defined by

$$\mathcal{G}_F(\vec{x},y) \leftrightarrow F(\vec{x}) \simeq y.$$

The next theorem shows how to characterize recursive functions in terms of recursive relations.

14.4. GRAPH THEOREM. A function F is recursive iff its graph is RE. A total function F is recursive iff its graph is recursive.

Proof. Let F be recursive and let e be an index of F. Then

$$\mathcal{G}_F(\vec{x},y) \leftrightarrow \{e\}(\vec{x}) \simeq y$$
$$\leftrightarrow \exists s(\{e\}_s(\vec{x}) \simeq y).$$

Thus $\mathcal{G}_F$ is RE by 8.4. If F is total, then the definition

$$\mathcal{G}_F(\vec{x},y) \leftrightarrow F(\vec{x}) = y$$

shows that $\mathcal{G}_F$ is recursive. If $\mathcal{G}_F$ is RE, then it has a recursive selector. But the only selector for $\mathcal{G}_F$ is F. □

As an application, we prove a more general result on definition by cases of recursive functions.

14.5. PROPOSITION. Let $R_1, \ldots, R_n$ be RE relations such that for every $\vec{x}$, at most one of $R_1(\vec{x}), \ldots, R_m(\vec{x})$ is true. Let $F_1, \ldots, F_n$ be recursive functions, and define F by

$$\begin{aligned} F(\vec{x}) &\simeq F_1(\vec{x}) \quad \text{if } R_1(\vec{x}), \\ &\simeq \ldots \\ &\simeq F_n(\vec{x}) \quad \text{if } R_n(\vec{x}), \end{aligned}$$

where it is understood that $F(\vec{x})$ is undefined if none of $R_1(\vec{x}), \ldots, R_n(\vec{x})$ is true. Then F is recursive.

Proof. We have

$$\mathcal{G}_F(\vec{x},y) \leftrightarrow (\mathcal{G}_{F_1}(\vec{x},y) \mathbin{\&} R_1(\vec{x})) \vee \ldots \vee (\mathcal{G}_{F_n}(\vec{x},y) \mathbin{\&} R_n(\vec{x})).$$

By the Graph Theorem and the table, the right side is RE; so F is recursive by the Graph Theorem. □

The next result characterizes recursive relations in terms of RE relations.

14.6. PROPOSITION. A relation R is recursive iff both R and $\neg R$ are RE.

Proof. If R is recursive, then $\neg R$ is recursive; so R and $\neg R$ are RE by 13.2. Now

$$\mathcal{G}_{\chi_R}(\vec{x},y) \leftrightarrow (R(\vec{x}) \mathbin{\&} y = 0) \vee (\neg R(\vec{x}) \mathbin{\&} y = 1).$$

If R and $\neg R$ are RE, this equivalence and the table show that $\mathcal{G}_{\chi_R}$ is RE; so R is recursive by the Graph Theorem. □

14.7. PROPOSITION. A non–empty set A is RE iff it is the range of a recursive real. An infinite set A is RE iff it is the range of a one–one recursive real.

Proof. If F is a recursive real, its range A is defined by $y \in A \leftrightarrow$

$\exists x(F(x) = y)$; so A is RE. Now let A be an RE set. Let e be an index of A and let $a \in A$. Define a recursive real F by

$$F(x) \simeq (x)_0 \quad \text{if } T_1(e,(x)_0,(x)_1),$$
$$\simeq a \qquad \text{otherwise.}$$

Clearly the range of F is A. Now suppose that A is also infinite. Define $G(n) = F(H(n))$ where $H(n)$ is the least x such that $F(x) \neq F(H(m))$ for all $m < n$. Then H is defined by course-of-values recursion using only recursive symbols and hence is recursive; so G is recursive. Clearly G is one–one and has range A. □

All of the results of this section relativize to any finite Φ. For Φ a finite sequence of reals, we let W_e^{Φ} be the domain of $\{e\}^{\Phi}$, and say that e is a <u>Φ–index</u> of W_e^{Φ}. Then (1) becomes

$$(2) \qquad W_e^{\Phi}(\vec{x}) \leftrightarrow \exists y T_k^{\Phi}(e,\vec{x},y).$$

We can use (1) of §12 to rewrite this as

$$(3) \qquad W_e^{\Phi}(\vec{x}) \leftrightarrow \exists y T_{k,m}(e,\vec{x},y,\bar{\Phi}(y)).$$

Using 12.2, we see that a relation is RE in Φ iff it is RE in a finite subset of Φ; and similarly for Σ_1^0. It follows that 14.1 relativizes to arbitrary Φ. Similarly, 14.3 through 14.6 relativize to arbitrary Φ.

14.8. PROPOSITION (POST). A relation is Σ_{n+1}^0 iff it is RE in Π_n^0.

Proof. If R is Σ_{n+1}^0, then $R(\vec{x}) \leftrightarrow \exists y P(\vec{x},y)$ where P is Π_n^0. Then R is RE in P and hence in Π_n^0.

Now suppose that R is RE in Π_n^0. By 12.2, R is RE in a finite subset of Π_n^0. By the remark after 13.3 and 12.6, we may suppose the relations in Φ are unary. To simplify the notation, suppose that Φ consists of one relation P. By (3), we have for some recursive Q

$$R(\vec{x}) \leftrightarrow \exists y Q(\overline{\chi_P}(y),\vec{x},y)$$
$$\leftrightarrow \exists y \exists z(z = \overline{\chi_P}(y) \ \& \ Q(z,\vec{x},y)).$$

If we can show that $z = \overline{\chi_P}(y)$ is Σ_{n+1}^0, it will follow by the table that R is Σ_{n+1}^0. Now

$$z = \overline{\chi_P}(y) \leftrightarrow Seq(z) \mathbin{\&} lh(z) = y \mathbin{\&} (\forall i < y)((z)_i = \chi_P(i)).$$

Hence by the table, it will suffice to show that $w = \chi_P(i)$ is Σ^0_{n+1}. Since P is Π^0_n, this follows from

$$w = \chi_P(i) \leftrightarrow (w = 1 \mathbin{\&} P(i)) \vee (w = 0 \mathbin{\&} \neg P(i))$$

and the table. □

14.9. COROLLARY. A relation is Δ^0_{n+1} iff it is recursive in Π^0_n.

Proof. A relation R is Δ^0_{n+1} iff both R and $\neg R$ are Σ^0_{n+1}; hence, by Post's Theorem, iff both R and $\neg R$ are RE in Π^0_n. By the relativized version of 14.6, this holds iff R is recursive in Π^0_n. □

Since $\neg R$ is recursive in R and $R = \neg\neg R$ is recursive in $\neg R$, 12.4 and the table show that we can replace Π^0_n by Σ^0_n in both Post's Theorem and its corollary.

15. Degrees

If F and G are total functions, we let $F \leq_R G$ mean that F is recursive in G. By 12.5,

$$(1) \qquad F \leq_R F;$$

and by the Transitivity Theorem

$$(2) \qquad F \leq_R G \mathbin{\&} G \leq_R H \rightarrow F \leq_R H.$$

Let $F \equiv_R G$ mean $F \leq_R G \mathbin{\&} G \leq_R F$. It follows from (1) and (2) that $\equiv_R$ is an equivalence relation. The equivalence class of F is called the degree of F and is designated by dg F. By a degree, we mean the degree of some total function. We use small boldface letters, usually **a**, **b**, **c**, and **d**, for degrees.

We let dg(F) $\leq$ dg(G) if $F \leq_R G$. By (2), this depends only on dg(F) and dg(G), not on the choice of F and G in these equivalence classes. It follows from (1) and (2) that $\leq$ is a partial ordering of the degrees, i.e., that

$$\mathbf{a} \leq \mathbf{a},$$

$$\mathbf{a} \leq \mathbf{b} \mathbin{\&} \mathbf{b} \leq \mathbf{a} \rightarrow \mathbf{a} = \mathbf{b},$$