29 Δ^1_1-codes

Using Π^1_1-reduction and universal sets it is possible to get codes for Δ^1_1 subsets of ω and ω^ω.

Here is what we mean by Δ^1_1 codes for subsets of X where $X = \omega$ or $X = \omega^\omega$.

There exists a Π^1_1 sets $C \subseteq \omega \times \omega^\omega$ and $P \subseteq \omega \times \omega^\omega \times X$ and a Σ^1_1 set $S \subseteq \omega \times \omega^\omega \times X$ such that

- for any $(e, u) \in C$

$$\{x \in X : (e, u, x) \in P\} = \{x \in X : (e, u, x) \in S\}$$

- for any $u \in \omega^\omega$ and $\Delta^1_1(u)$ set $D \subseteq X$ there exists a $(e, u) \in C$ such that

$$D = \{x \in X : (e, u, x) \in P\} = \{x \in X : (e, u, x) \in S\}.$$

From now on we will write
"e is a $\Delta^1_1(u)$-code for a subset of X"
to mean $(e, u) \in C$ and remember that it is a Π^1_1 predicate.

We also write "D is the $\Delta^1_1(u)$ set coded by e" if "e is a $\Delta^1_1(u)$-code for a subset of X" and

$$D = \{x \in X : (e, x) \in P\} = \{x \in X : (e, x) \in S\}.$$

Note that $x \in D$ can be said in either a $\Sigma^1_1(u)$ way or $\Pi^1_1(u)$ way, using either S or P.

Theorem 29.1 *(Spector-Gandy [103][31]) Π^1_1-reduction and universal sets implies Δ^1_1 codes exist.*

proof:

Let $U \subseteq \omega \times \omega^\omega \times X$ be a Π^1_1 set which is universal for all $\Pi^1_1(u)$ sets, i.e., for every $u \in \omega^\omega$ and $A \in \Pi^1_1(u)$ with $A \subseteq X$ there exists $e \in \omega$ such that $A = \{x \in X : (e, u, x) \in U\}$. For example, to get such a U proceed as follows. Let $\{e\}^u$ be the partial function you get by using the e^{th} Turing machine with oracle u. Then define $(e, u, x) \in U$ iff $\{e\}^u$ is the characteristic function of a tree $T \subseteq \bigcup_{n<\omega}(\omega^n \times \omega^n)$ and $T_x = \{s : (s, x \upharpoonright |s|) \in T\}$ is well-founded.

Now get a doubly universal pair. Let $e \mapsto (e_0, e_1)$ be the usual recursive unpairing function from ω to $\omega \times \omega$ and define

$$U^0 = \{(e, u, x) : (e_0, u, x) \in U\}$$

and

$$U^1 = \{(e, u, x) : (e_1, u, x) \in U\}.$$

The pair of sets U^0 and U^1 are Π^1_1 and doubly universal, i.e., for any $u \in \omega^\omega$ and A and B which are $\Pi^1_1(u)$ subsets of X there exists $e \in \omega$ such that

$$A = \{x : (e, u, x) \in U^0\}$$

and
$$B = \{x : (e, u, x) \in U^1\}.$$
Now apply reduction to obtain $P^0 \subseteq U^0$ and $P^1 \subseteq U^1$ which are Π^1_1 sets. Note that the by the nature of taking cross sections, $P^0_{e,u}$ and $P^1_{e,u}$ reduce $U^0_{e,u}$ and $U^1_{e,u}$. Now we define

- "e is a $\Delta^1_1(u)$ code" iff $\forall x \in X(x \in P^0_{e,u}$ or $x \in P^1_{e,u})$, and
- $P = P^0$ and $S =\sim P^1$.

Note that e is a $\Delta^1_1(u)$ code is a Π^1_1 statement in (e, u). Also if e is a $\Delta^1_1(u)$ code, then $P_{(e,u)} = S_{e,u}$ and so its a $\Delta^1_1(u)$ set. Furthermore if $D \subseteq X$ is a $\Delta^1_1(u)$ set, then since U^0 and U^1 were a doubly universal pair, there exists e such that $U^0_{e,u} = D$ and $U^1_{e,u} =\sim D$. For this e it must be that $U^0_{e,u} = P^0_{e,u}$ and $U^1_{e,u} = P^1_{e,u}$ since the P's reduce the U's. So this e is a $\Delta^1_1(u)$ code which codes the set D.
■

Corollary 29.2 $\{(x, u) \in P(\omega) \times \omega^\omega : x \in \Delta^1_1(u)\}$ *is* Π^1_1.

proof:
$x \in \Delta^1_1(u)$ iff $\exists e \in \omega$ such that

1. e is a $\Delta^1_1(u)$ code,
2. $\forall n$ if $n \in x$, then n is in the $\Delta^1_1(u)$-set coded by e, and
3. $\forall n$ if n is the $\Delta^1_1(u)$-set coded by e, then $n \in x$.

Note that clause (1) is Π^1_1. Clause (2) is Π^1_1 if we use that $(e, u, n) \in P$ is equivalent to "n is in the $\Delta^1_1(u)$-set coded by e". While clause (3) is Π^1_1 if we use that $(e, u, n) \in S$ is equivalent to "n is in the $\Delta^1_1(u)$-set coded by e".
■

We say that $y \in \omega^\omega$ is $\Delta^1_1(u)$ iff its graph $\{(n, m) : y(n) = m\}$ is $\Delta^1_1(u)$. Since being the graph a function is a Π^0_2 property it is easy to see how to obtain $\Delta^1_1(u)$ codes for functions $y \in \omega^\omega$.

Corollary 29.3 *Suppose* $\theta(x, y, z)$ *is a* Π^1_1 *formula, then*
$$\psi(y, z) = \exists x \in \Delta^1_1(y)\ \theta(x, y, z)$$
is a Π^1_1 *formula.*

proof:
$\psi(y, z)$ iff
$\exists e \in \omega$ such that

1. e is a $\Delta^1_1(y)$ code, and
2. $\forall x$ if x is the set coded by (e, y), then $\theta(x, y, z)$.

This will be Π^1_1 just in case the clause "x is the set coded by (e, y)" is Σ^1_1. But this is Δ^1_1 provided that e is a $\Delta^1_1(y)$ code, e.g., for $x \subseteq \omega$ we just say: $\forall n \in \omega$

1. if $n \in x$ then $(e, y, n) \in S$ and

2. if $(e, y, n) \in P$, then $n \in x$.

Both of these clauses are Σ^1_1 since S is Σ^1_1 and P is Π^1_1. A similar argument works for $x \in \omega^\omega$.

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The method of this corollary also works for the quantifier

$$\exists D \subseteq \omega^\omega \text{ such that } D \in \Delta(y) \ \ \theta(D, y, z).$$

It is equivalent to say $\exists e \in \omega$ such that e is a $\Delta^1_1(y)$ code for a subset of ω^ω and $\theta(\ldots, y, z)$ where occurrences of the "$q \in D$" in the formula θ have been replaced by either $(e, y, q) \in P$ or $(e, y, q) \in S$, whichever is necessary to makes θ come out Π^1_1.

Corollary 29.4 *Suppose $f : \omega^\omega \to \omega^\omega$ is Borel, $B \subseteq \omega^\omega$ is Borel, and f is one-to-one on B. Then the image of B under f, $f(B)$, is Borel.*

proof:

By relativizing the following argument to an arbitrary parameter we may assume that the graph of f and the set B are Δ^1_1. Define

$$R = \{(x, y) : f(x) = y \text{ and } x \in B\}.$$

Then for any y the set

$$\{x : R(x, y)\}$$

is a $\Delta^1_1(y)$ singleton (or empty). Consequently, its unique element is Δ^1_1 in y. It follows that

$$y \in f(B) \text{ iff } \exists x\ R(x, y) \text{ iff } \exists x \in \Delta^1_1(y)\ R(x, y)$$

and so $f(B)$ is both Σ^1_1 and Π^1_1.

■

Many applications of the Gandy-Spector Theorem exist. For example, it is shown (assuming V=L in all three cases) that

1. there exists an uncountable Π^1_1 set which is concentrated on the rationals (Erdos, Kunen, and Mauldin [21]),

2. there exists a Π^1_1 Hamel basis (Miller [83]), and

3. there exists a topologically rigid Π^1_1 set (Van Engelen, Miller, and Steel [18]).