19 Hereditarily countable sets

HC is the set of all hereditarily countable sets. There is a close connection between the projective hierarchy above level 2 and a natural hierarchy on the subsets of HC. A formula of set theory is Δ_0 iff it is in the smallest family of formulas containing the atomic formulas of the form "$x \in y$" or "$x = y$", and closed under conjunction, $\theta \wedge \phi$, negation, $\neg\theta$, and bounded quantification, $\forall x \in y$ or $\exists x \in y$. A formula θ of set theory is Σ_1 iff it of the form $\exists u_1, \ldots, u_n \;\; \psi$ where ψ is Δ_0.

Theorem 19.1 *A set $A \subseteq \omega^\omega$ is Σ^1_2 iff there exists a Σ_1 formula $\theta(\cdot)$ of set theory such that*

$$A = \{x \in \omega^\omega : HC \models \theta(x)\}.$$

proof:

We note that Δ_0 formulas are absolute between transitive sets, i.e., if $\psi(\cdots)$ is Δ_0 formula, M a transitive set and $\overline{y}$ a finite sequence of elements of M, then $M \models \psi(\overline{y})$ iff $V \models \psi(\overline{y})$. Suppose that $\theta(\cdot)$ is a Σ_1 formula of set theory. Then for every $x \in \omega^\omega$ we have that $HC \models \theta(x)$ iff there exists a countable transitive set $M \in HC$ with $x \in M$ such that $M \models \theta(x)$. Hence, $HC \models \theta(x)$ iff there exists $E \subseteq \omega \times \omega$ and $\overset{\circ}{x} \in \omega$ such that letting $M = (\omega, E)$ then

1. E is extensional and well-founded,
2. $M \models$ZFC*, (or just that ω exists)
3. $M \models \theta(\overset{\circ}{x})$,
4. for all $n, m \in \omega \quad x(n) = m$ iff $M \models \overset{\circ}{x}(\overset{\circ}{n}) = \overset{\circ}{m}$.

Therefore, $\{x \in \omega^\omega : HC \models \theta(x)\}$ is a Σ^1_2 set. On the other hand given a Σ^1_2 set A there exists a Π^1_1 formula $\theta(x, y)$ such that $A = \{x : \exists y \theta(x, y)\}$. But then by Mostowski absoluteness (Theorem 17.5) we have that $x \in A$ iff there exists a countable transitive set M with $x \in M$ and there exists $y \in M$ such that $M \models$ZFC* and $M \models \theta(x, y)$. But this is a Σ_1 formula for HC.
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The theorem says that $\Sigma^1_2 = \Sigma^{HC}_1$. Similarly, $\Sigma^1_{n+1} = \Sigma^{HC}_n$. Let us illustrate this with an example construction.

Theorem 19.2 *If V=L, then there exists a Δ^1_2 Luzin set $X \subseteq \omega^\omega$.*

proof:

Let $\{T_\alpha : \alpha < \omega_1\}$ (ordered by $<_c$) be all subtrees T of $\omega^{<\omega}$ whose branches $[T]$ are a closed nowhere dense subset of ω^ω. Define x_α to be the least constructed ($<_c$) element of ω^ω which is not in

$$\bigcup_{\beta<\alpha} [T_\beta] \cup \{x_\beta : \beta < \alpha\}.$$

Define $X = \{x_\alpha : \alpha < \omega_1\}$. So X is a Luzin set.

To see that X is Σ_1^{HC} note that $x \in X$ iff there exists a transitive countable M which models ZFC*+V=L such that $M \models$"$x \in X$"(i.e. M models the first paragraph of this proof).

To see that X is Π_1^{HC} note that $x \in X$ iff for all M if M is a transitive countable model of ZFC*+V=L with $x \in M$ and $M \models$"$\exists y \in X \;\; x <_c y$", then $M \models$"$x \in X$". This is true because the nature of the construction is such that if you put a real into X which is constructed after x, then x will never get put into X after this. So x will be in X iff it is already in X.

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