16 Covering number of an ideal

This section is a small diversion.[8] It is motivated by Theorem 11.1 of Martin and Solovay.

Define for any ideal I in Borel(2^ω)

$$\mathrm{cov}(I) = \min\{|\mathcal{I}| : \mathcal{I} \subseteq I, \bigcup \mathcal{I} = 2^\omega\}.$$

The following theorem is well-known.

Theorem 16.1 *For any cardinal κ the following are equivalent:*

1. *$MA_\kappa(ctbl)$, i.e. for any countable poset, $\mathbb{P}$, and family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \kappa$ there exists a $\mathbb{P}$-filter G with $G \cap D \neq \emptyset$ for every $D \in \mathcal{D}$, and*

2. $\mathrm{cov}(\mathrm{meager}(2^\omega)) \geq \kappa$.

proof:

$\mathrm{MA}_\kappa(\mathrm{ctbl})$ implies $\mathrm{cov}(\mathrm{meager}(2^\omega)) \geq \kappa$, is easy because if $U \subseteq 2^\omega$ is a dense open set, then

$$D = \{s \in 2^{<\omega} : [s] \subseteq U\}$$

is dense in $2^{<\omega}$.

$\mathrm{cov}(\mathrm{meager}(2^\omega)) \geq \kappa$ implies $\mathrm{MA}_\kappa(\mathrm{ctbl})$ follows from the fact that any countable poset, $\mathbb{P}$, either contains a dense copy of $2^{<\omega}$ or contains a p such that every two extensions of p are compatible.

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Theorem 16.2 *(Miller [77])* $\mathrm{cof}(\mathrm{cov}(\mathrm{meager}(2^\omega))) > \omega$*, e.g., it is impossible to have* $\mathrm{cov}(\mathrm{meager}(2^\omega)) = \aleph_\omega$.

proof:

Suppose for contradiction that $\kappa = \mathrm{cov}(\mathrm{meager}(2^\omega))$ has countable cofinality and let κ_n for $n \in \omega$ be a cofinal sequence in κ. Let $\langle C_\alpha : \alpha < \kappa\rangle$ be a family of closed nowhere dense sets which cover 2^ω. We will construct a sequence $P_n \subseteq 2^\omega$ of perfect sets with the properties that

1. $P_{n+1} \subseteq P_n$,
2. $P_n \cap \bigcup\{C_\alpha : \alpha < \kappa_n\} = \emptyset$, and
3. $\forall \alpha < \kappa \quad C_\alpha \cap P_n$ is nowhere dense in P_n.

This easily gives a contradiction, since $\bigcap_{n<\omega} P_n$ is nonempty and disjoint from all C_α, contradicting the fact that the C_α's cover 2^ω.

[8] All men's gains are the fruit of venturing. Herodotus BC 484-425.

We show how to obtain P_0, since the argument easily relativizes to show how to obtain P_{n+1} given P_n. Since cov(meager(2^ω)) $> \kappa_n$ there exists a countable sequence

$$D = \{x_n : n \in \omega\} \subseteq 2^\omega$$

such that D is dense and for every n

$$x_n \notin \bigcup_{\alpha<\kappa_n} C_\alpha.$$

Consider the following forcing notion $\mathbb{P}$.

$$\mathbb{P} = \{(H,n) : n \in \omega \text{ and } H \in [D]^{<\omega}\}$$

This is ordered by $(H,n) \leq (K,m)$ iff

1. $H \supseteq K$,
2. $n \geq m$, and
3. for every $x \in H$ there exists $y \in K$ with $x \upharpoonright m = y \upharpoonright m$.

Note that $\mathbb{P}$ is countable.

For each $n \in \omega$ define $E_n \subseteq \mathbb{P}$ by $(H,m) \in E_n$ iff

1. $m > n$ and
2. $\forall x \in H \exists y \in H\ x \upharpoonright n = y \upharpoonright n$ but $x \upharpoonright m \neq y \upharpoonright m$.

and for each $\alpha < \kappa_0$ let

$$F_\alpha = \{(H,m) \in \mathbb{P} : \forall x \in H\ \ [x \upharpoonright m] \cap C_\alpha = \emptyset\}.$$

For G a $\mathbb{P}$-filter, define $X \subseteq D$ by

$$X = \bigcup\{H : \exists n\ (H,n) \in G\}$$

and let $P = cl(X)$. It easy to check that the E_n's are dense and if G meets each one of them, then P is perfect (i.e. has no isolated points). The F_α for $\alpha < \kappa_0$ are dense in $\mathbb{P}$. This is because $D \cap C_\alpha = \emptyset$ so given $(H,n) \in \mathbb{P}$ there exists $m \geq n$ such that for every $x \in H$ we have $[x \upharpoonright m] \cap C_\alpha = \emptyset$ and thus $(H,m) \in F_\alpha$. Note that if $G \cap F_\alpha \neq \emptyset$, then $P \cap C_\alpha = \emptyset$. Consequently, by Theorem 16.1, there exists a $\mathbb{P}$-filter G such that G meets each E_n and all F_α for $\alpha < \kappa_0$. Hence $P = cl(X)$ is a perfect set which is disjoint from each C_α for $\alpha < \kappa_0$. Note also that for every $\alpha < \kappa$ we have that $C_\alpha \cap D$ is finite and hence $C_\alpha \cap X$ is finite and therefore $C_\alpha \cap P$ is nowhere dense in P. This ends the construction of $P = P_0$ and since the P_n can be obtained with a similar argument, this proves the Theorem.

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Question 16.3 *(Fremlin) Is the same true for the measure zero ideal in place of the ideal of meager sets?*

Some partial results are known (see Bartoszynski, Judah, Shelah [7][8][9]).

Theorem 16.4 *(Miller [77]) It is consistent that* $\text{cov}(\text{meager}(2^{\omega_1})) = \aleph_\omega$.

proof:

In fact, this holds in the model obtained by forcing with $\text{FIN}(\aleph_\omega, 2)$ over a model of GCH.

$\text{cov}(\text{meager}(2^{\omega_1})) \geq \aleph_\omega$: Suppose for contradiction that

$$\{C_\alpha : \alpha < \omega_n\} \in V[G]$$

is a family of closed nowhere dense sets covering 2^{ω_1}. Define

$$E_\alpha = \{s \in \text{FIN}(\omega_1, 2) : [s] \cap C_\alpha = \emptyset\}.$$

Using ccc, there exists $\Sigma \in [\aleph_\omega]^{\omega_n}$ in V with

$$\{E_\alpha : \alpha < \omega_n\} \in V[G \restriction \Sigma].$$

Let $X \subseteq \aleph_\omega$ be a set in V of cardinality ω_1 which is disjoint from Σ. By the product lemma $G \restriction X$ is $\text{FIN}(X, 2)$-generic over $V[G \restriction \Sigma]$. Consequently, if $H : \omega_1 \to 2$ corresponds to G via an isomorphism of X and ω_1, then $H \notin C_\alpha$ for every $\alpha < \omega_n$.

$\text{cov}(\text{meager}(2^{\omega_1})) \leq \aleph_\omega$: Note that for every uncountable $X \subseteq \omega_1$ with $X \in V[G]$ there exists $n \in \omega$ a $Z \in [\omega_1]^{\omega_1} \cap V[G \restriction \omega_n]$ with $Z \subseteq X$. To see this note that for every $\alpha \in X$ there exists $p \in G$ such that $p \Vdash \alpha \in X$ and $p \in \text{FIN}(\omega_n, 2)$ for some $n \in \omega$. Consequently, by ccc, some n works for uncountably many α.

Consider the family of all closed nowhere dense sets $C \subseteq 2^{\omega_1}$ which are coded in some $V[G \restriction \omega_n]$ for some n. We claim that these cover 2^{ω_1}. This follows from above, because for any $Z \subseteq \omega_1$ which is infinite the set

$$C = \{x \in 2^{\omega_1} : \forall \alpha \in Z \;\; x(\alpha) = 1\}$$

is nowhere dense.

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Theorem 16.5 *(Miller [77]) It is consistent that there exists a ccc σ-ideal I in* $\text{Borel}(2^\omega)$ *such that* $\text{cov}(I) = \aleph_\omega$.

proof:

Let $\mathbb{P} = \text{FIN}(\omega_1, 2) * \overset{\circ}{\mathbb{Q}}$ where $\overset{\circ}{\mathbb{Q}}$ is a name for the Silver forcing which codes up generic filter for $\text{FIN}(\omega_1, 2)$ just like in the proof of Theorem 11.1.

Let $\prod_{\alpha<\aleph_\omega} \mathbb{P}$ be the direct sum (i.e. finite support product) of $\aleph_\omega$ copies of $\mathbb{P}$. Forcing with the direct sum adds a filter $G = \langle G_\alpha : \alpha < \aleph_\omega\rangle$ where each G_α is $\mathbb{P}$-generic. In general, a direct sum is ccc iff every finite subproduct is ccc. This follows by a delta-system argument. Every finite product of $\mathbb{P}$ has ccc, because $\mathbb{P}$ is σ-centered, i.e., it is the countable union of centered sets.

Let V be a model of GCH and $G = \langle G_\alpha : \alpha < \aleph_\omega\rangle$ be $\prod_{\alpha<\aleph_\omega} \mathbb{P}$ generic over V. We claim that in $V[G]$ if I is the σ-ideal given by Sikorski's Theorem 9.1 such that $\prod_{\alpha<\aleph_\omega} \mathbb{P}$ is densely embedded into $\text{Borel}(2^\omega)/I$ then $\text{cov}(I) = \aleph_\omega$.

First define, $m_{\mathbb{P}}$, to be the cardinality of the minimal failure of MA for $\mathbb{P}$, i.e., the least κ such that there exists a family $|\mathcal{D}| = \kappa$ of dense subsets of $\mathbb{P}$ such that there is no $\mathbb{P}$-filter meeting all the $D \in \mathcal{D}$.

Lemma 16.6 *In $V[\langle G_\alpha : \alpha < \aleph_\omega\rangle]$ we have that $m_{\mathbb{P}} = \aleph_\omega$.*

proof:

Note that for any set $D \subset \mathbb{P}$ there exists a set $\Sigma \in [\aleph_\omega]^{\omega_1}$ in V with $D \in V[\langle G_\alpha : \alpha \in \Sigma\rangle]$. So if $|\mathcal{D}| = \omega_n$ then there exists $\Sigma \in [\aleph_\omega]^{\omega_n}$ in V with $\mathcal{D} \in V[\langle G_\alpha : \alpha \in \Sigma\rangle]$. Letting $\alpha \in \aleph_\omega \setminus \Sigma$ we get G_α a $\mathbb{P}$-filter meeting every $D \in \mathcal{D}$. Hence $m_{\mathbb{P}} \geq \aleph_\omega$.

On the other hand:

Claim: For every $X \in [\omega_1]^{\omega_1} \cap V[\langle G_\alpha : \alpha < \aleph_\omega\rangle]$ there exists $n \in \omega$ and $Y \in [\omega_1]^{\omega_1} \cap V[\langle G_\alpha : \alpha < \aleph_n\rangle]$ with $Y \subseteq X$.

proof:

For every $\alpha \in X$ there exist $p \in G$ and $n < \omega$ such that $p \Vdash \check{\alpha} \in \overset{\circ}{X}$ and domain$(p) \subseteq \aleph_n$. Since X is uncountable there is one n which works for uncountably many $\alpha \in X$.

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It follows from the Claim that there is no H which is $\text{FIN}(\omega_1, 2)$ generic over all the models $V[\langle G_\alpha : \alpha < \aleph_n\rangle]$, but forcing with $\mathbb{P}$ would add such an H and so $m_{\mathbb{P}} \leq \aleph_\omega$ and the Lemma is proved.

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Lemma 16.7 *If $\mathbb{P}$ is ccc and dense in the cBa* $\text{Borel}(2^\omega)/I$, *then* $m_{\mathbb{P}} = \text{cov}(I)$.

proof:

This is the same as Lemma 11.2 equivalence of (1) and (3), except you have to check that m is the same for both $\mathbb{P}$ and $\text{Borel}(2^\omega)/I$.

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Kunen [56] showed that least cardinal for which MA fails can be a singular cardinal of cofinality ω_1, although it is impossible for it to have cofinality ω (see Fremlin [27]). It is still open whether it can be a singular cardinal of cofinality greater than ω_1 (see Landver [59]). Landver [60] generalizes Theorem 16.2 to the space 2^κ with basic clopen sets of the form $[s]$ for $s \in 2^{<\kappa}$. He uses a generalization of a characterization of cov(meager(2^ω)) due to Bartoszynski [6] and Miller [78].