6 Generic G_δ

It is natural[4] to ask

"What are the possibly lengths of Borel hierarchies?"

In this section we present a way of forcing a generic G_δ.

Let X be a Hausdorff space with a countable base $\mathcal{B}$. Consider the following forcing notion.

$p \in \mathbb{P}$ iff it is a finite consistent set of sentences of the form:

1. "$B \subseteq \mathring{U}_n$" where $B \in \mathcal{B}$ and $n \in \omega$, or

2. "$x \notin \mathring{U}_n$" where $x \in X$ and $n \in \omega$, or

3. "$x \in \bigcap_{n<\omega} \mathring{U}_n$" where $x \in X$.

Consistency means that we cannot say that both "$B \subseteq \mathring{U}_n$" and "$x \notin \mathring{U}_n$" if it happens that $x \in B$ and we cannot say both "$x \notin \mathring{U}_n$" and "$x \in \bigcap_{n<\omega} \mathring{U}_n$". The ordering is reverse inclusion. A $\mathbb{P}$ filter G determines a G_δ set U as follows: Let

$$U_n = \bigcup\{B \in \mathcal{B} : \text{"}B \subseteq \mathring{U}_n\text{"} \in G\}.$$

Let $U = \bigcap_n U_n$. If G is $\mathbb{P}$-generic over V, a density argument shows that for every $x \in X$ we have that

$$x \in U \text{ iff "}x \in \bigcap_{n<\omega} \mathring{U}_n\text{"} \in G.$$

Note that U is not in V (as long as X is infinite). For suppose $p \in \mathbb{P}$ and $A \subseteq X$ is in V is such that

$$p \Vdash \mathring{U} = \check{A}.$$

Since X is infinite there exist $x \in X$ which is not mentioned in p. Note that $p_0 = p \cup \{\text{"}x \in \bigcap_{n<\omega} \mathring{U}_n\text{"}\}$ is consistent and also $p_1 = p \cup \{\text{"}x \notin \mathring{U}_n\text{"}\}$ is consistent for all sufficiently large n (i.e. certainly for U_n not mentioned in p.) But $p_0 \Vdash x \in \mathring{U}$ and $p_1 \Vdash x \notin \mathring{U}$, and since x is either in A or not in A we arrive at a contradiction.

In fact, U is not F_σ in the extension (assuming X is uncountable). To see this we will first need to prove that $\mathbb{P}$ has ccc.

Lemma 6.1 *$\mathbb{P}$ has ccc.*

proof:

Note that p and q are compatible iff $(p \cup q) \in \mathbb{P}$ iff $(p \cup q)$ is a consistent set of sentences. Recall that there are three types of sentences:

[4] 'Gentlemen, the great thing about this, like most of the demonstrations of the higher mathematics, is that it can be of no earthly use to anybody.' -Baron Kelvin

1. $B \subseteq \overset{\circ}{U}_n$

2. $x \notin \overset{\circ}{U}_n$

3. $x \in \bigcap_{n<\omega} \overset{\circ}{U}_n$

where $B \in \mathcal{B}$, $n \in \omega$, and $x \in X$. Now if for contradiction A were an uncountable antichain, then since there are only countably many sentences of type 1 above we may assume that all $p \in A$ have the same set of type 1 sentences. Consequently for each distinct pair $p, q \in A$ there must be an $x \in X$ and n such that either "$x \notin \overset{\circ}{U}_n$ " $\in p$ and "$x \in \bigcap_{n<\omega} \overset{\circ}{U}_n$ " $\in q$ or vice-versa. For each $p \in A$ let D_p be the finitely many elements of X mentioned by p and let $s_p : D_p \rightarrow \omega$ be defined by

$$s_p(x) = \begin{cases} 0 & \text{if "}x \in \bigcap_{n<\omega} \overset{\circ}{U}_n\text{ "} \in p \\ n+1 & \text{if "}x \notin \overset{\circ}{U}_n\text{ "} \in p \end{cases}$$

But now $\{s_p : p \in A\}$ is an uncountable family of pairwise incompatible finite partial functions from X into ω which is impossible. (FIN(X, ω) has the ccc, see Kunen [54].)

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If $V[G]$ is a generic extension of a model V which contains a topological space X, then we let X also refer to the space in $V[G]$ whose topology is generated by the open subsets of X which are in V.

Theorem 6.2 *(Miller [73]) Suppose X in V is an uncountable Hausdorff space with countable base $\mathcal{B}$ and G is $\mathbb{P}$-generic over V. Then in $V[G]$ the G_δ set U is not F_σ.*

proof:

We call this argument the old switcheroo. Suppose for contradiction

$$p \Vdash \bigcap_{n\in\omega} \overset{\circ}{U}_n = \bigcup_{n\in\omega} \overset{\circ}{C}_n \text{ where } \overset{\circ}{C}_n \text{ are closed in } X .$$

For $Y \subseteq X$ let $\mathbb{P}(Y)$ be the elements of $\mathbb{P}$ which only mention $y \in Y$ in type 2 or 3 statements. Let $Y \subseteq X$ be countable such that

1. $p \in \mathbb{P}(Y)$ and

2. for every n and $B \in \mathcal{B}$ there exists a maximal antichain $A \subseteq \mathbb{P}(Y)$ which decides the statement "$B \cap \overset{\circ}{C}_n = \emptyset$".

Since X is uncountable there exists $x \in X \setminus Y$. Let

$$q = p \cup \{\text{"}x \in \bigcap_{n\in\omega} \overset{\circ}{U}_n\text{ "}\}.$$

Since q extends p, clearly

$$q \Vdash x \in \bigcup_{n\in\omega} \overset{\circ}{C}_n$$

so there exists $r \leq q$ and $n \in \omega$ so that

$$r \Vdash x \in \overset{\circ}{C}_n \ .$$

Let

$$r = r_0 \cup \{``x \in \bigcap_{n\in\omega} \overset{\circ}{U}_n\ "\}$$

where r_0 does not mention x. Now we do the switch. Let

$$t = r_0 \cup \{``x \notin \overset{\circ}{U}_m\ "\}$$

where m is chosen sufficiently large so that t is a consistent condition. Since

$$t \Vdash x \notin \bigcap_{n\in\omega} \overset{\circ}{U}_n$$

we know that

$$t \Vdash x \notin \overset{\circ}{C}_n \ .$$

Consequently there exist $s \in \mathbb{P}(Y)$ and $B \in \mathcal{B}$ such that

1. s and t are compatible,
2. $s \Vdash B \cap \overset{\circ}{C}_n = \emptyset$, and
3. $x \in B$.

But s and r are compatible, because s does not mention x. This is a contradiction since $s \cup r \Vdash x \in \overset{\circ}{C}_n$ and $s \cup r \Vdash x \notin \overset{\circ}{C}_n$.

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