5 Martin's Axiom

The following result is due to Rothberger [92] and Solovay [44][72]. The forcing we use is due to Silver. However, it is probably just another view of Solovay's 'almost disjoint sets forcing'.

Theorem 5.1 *Assuming Martin's Axiom if X is any second countable Hausdorff space of cardinality less than the continuum, then* $\mathrm{ord}(X) \leq 2$ *and, in fact, every subset of X is G_δ.*

proof:

Let $A \subseteq X$ be arbitrary and let $\mathcal{B}$ be a countable base for the topology on X. The partial order $\mathbb{P}$ is defined as follows. $p \in \mathbb{P}$ iff p is a finite consistent set of sentences of the form

1. "$x \notin \overset{\circ}{U}_n$" where $x \in X \setminus A$ or
2. "$B \subseteq \overset{\circ}{U}_n$" where $B \in \mathcal{B}$ and $n \in \omega$.

Consistent means that there is not a pair of sentences "$x \notin \overset{\circ}{U}_n$", "$B \subseteq \overset{\circ}{U}_n$" in p where $x \in B$. The ordering on $\mathbb{P}$ is reverse containment, i.e. p is stronger than q, $p \leq q$ iff $p \supseteq q$. The circle in the notation $\overset{\circ}{U}_n$'s means that it is the name for the set U_n which will be determined by the generic filter. For an element x of the ground model we should use $\check{x}$ to denote the canonical name of x, however to make it more readible we often just write x. For standard references on forcing see Kunen [54] or Jech [43].

We call this forcing *Silver forcing.*

Claim: $\mathbb{P}$ satisfies the ccc.

proof:

Note that since $\mathcal{B}$ is countable there are only countably many sentences of the type "$B \subseteq \overset{\circ}{U}_n$". Also if p and q have exactly the same sentences of this type then $p \cup q \in \mathbb{P}$ and hence p and q are compatible. It follows that $\mathbb{P}$ is the countable union of filters and hence we cannot find an uncountable set of pairwise incompatible conditions.

■

For $x \in X \setminus A$ define

$$D_x = \{p \in \mathbb{P} : \exists n \text{ "}x \notin \overset{\circ}{U}_n\text{"} \in p\}.$$

For $x \in A$ and $n \in \omega$ define

$$E_x^n = \{p \in \mathbb{P} : \exists B \in \mathcal{B}\ x \in B \text{ and "}B \subseteq \overset{\circ}{U}_n\text{"} \in p\}.$$

Claim: D_x is dense for each $x \in X \setminus A$ and E_x^n is dense for each $x \in A$ and $n \in \omega$.

proof:

To see that D_x is dense let $p \in \mathbb{P}$ be arbitrary. Choose n large enough so that $\overset{\circ}{U}_n$ is not mentioned in p, then $(p \cup \{``x \notin \overset{\circ}{U}_n"\}) \in \mathbb{P}$.

To see that E^n_x is dense let p be arbitrary and let $Y \subseteq X \setminus A$ be the set of elements of $X \setminus A$ mentioned by p. Since $x \in A$ and X is Hausdorff there exists $B \in \mathcal{B}$ with $B \cap Y = \emptyset$ and $x \in B$. Then $q = (p \cup \{``B \subseteq \overset{\circ}{U}_n"\}) \in \mathbb{P}$ and $q \in E^n_x$.

■

Since the cardinality of X is less than the continuum we can find a $\mathbb{P}$-filter G with the property that G meets each D_x for $x \in X \setminus A$ and each E^n_x for $x \in A$ and $n \in \omega$. Now define

$$U_n = \bigcup \{B : ``B \subseteq \overset{\circ}{U}_n" \in G\}.$$

Note that $A = \bigcap_{n \in \omega} U_n$ and so A is G_δ in X.

■

Spaces X in which every subset is G_δ are called *Q-sets.*

The following question was raised during an email correspondence with Zhou.

Question 5.2 *Suppose every set of reals of cardinality $\aleph_1$ is a Q-set. Then is $\mathfrak{p} > \omega_1$, i.e., is it true that for every family $\mathcal{F} \subseteq [\omega]^\omega$ of size ω_1 with the finite intersection property there exists an $X \in [\omega]^\omega$ with $X \subseteq^* Y$ for all $Y \in \mathcal{F}$?*

It is a theorem of Bell [11] that $\mathfrak{p}$ is the first cardinal for which MA for σ-centered forcing fails. Another result along this line due to Alan Taylor is that $\mathfrak{p}$ is the cardinality of the smallest set of reals which is not a γ-set, see Galvin and Miller [30].

Fleissner and Miller [23] show it is consistent to have a Q-set whose union with the rationals is not a Q-set.

For more information on Martin's Axiom see Fremlin [27]. For more on Q-sets, see Fleissner [24] [25], Miller [81] [85], Przymusinski [90], Judah and Shelah [45] [46], and Balogh [5].