3 Abstract Borel hierarchies

Suppose $F \subseteq P(X)$ is a family of sets. Most of the time we would like to think of F as a countable *field of sets* (i.e. closed under complements and finite intersections) and so analogous to the family of clopen subsets of some space.

We define the classes $\underset{\sim}{\Pi}^0_\alpha(F)$ analogously. Let $\underset{\sim}{\Pi}^0_0(F) = F$ and for every $\alpha > 0$ define $A \in \underset{\sim}{\Pi}^0_\alpha(F)$ iff there exists $B_n \in \underset{\sim}{\Pi}^0_{\beta_n}$ for some $\beta_n < \alpha$ such that

$$A = \bigcap_{n\in\omega} \sim B_n.$$

Define

- $\underset{\sim}{\Sigma}^0_\alpha(F) = \{\sim B : B \in \underset{\sim}{\Pi}^0_\alpha(F)\}$,
- $\underset{\sim}{\Delta}^0_\alpha(F) = \underset{\sim}{\Pi}^0_\alpha(F) \cap \underset{\sim}{\Sigma}^0_\alpha(F)$,
- $\text{Borel}(F) = \bigcup_{\alpha<\omega_1} \underset{\sim}{\Sigma}^0_\alpha(F)$, and
- let $\text{ord}(F)$ be the least α such that $\text{Borel}(F) = \underset{\sim}{\Sigma}^0_\alpha(F)$.

Theorem 3.1 *(Bing, Bledsoe, Mauldin [12]) Suppose $F \subseteq P(2^\omega)$ is a countable family such that* $\text{Borel}(2^\omega) \subseteq \text{Borel}(F)$. *Then* $\text{ord}(F) = \omega_1$.

Corollary 3.2 *Suppose X is any space containing a perfect set and $F \subseteq P(X)$ is a countable family such that* $\text{Borel}(X) \subseteq$ *Borel(F). Then* $\text{ord}(F) = \omega_1$.

proof:

Suppose $2^\omega \subseteq X$ and let $\hat{F} = \{A \cap 2^\omega : A \in F\}$. By Theorem 2.3 we have that $\text{Borel}(2^\omega) \subseteq \text{Borel}(\hat{F})$ and so by Theorem 3.1 we know $\text{ord}(\hat{F}) = \omega_1$. But this implies $\text{ord}(F) = \omega_1$.

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The proof of Theorem 3.1 is a generalization of Lebesgue's universal set argument. We need to prove the following two lemmas.

Lemma 3.3 *(Universal sets) Suppose $H \subseteq P(X)$ is countable and define*

$$R = \{A \times B : A \subseteq 2^\omega \text{ is clopen and } B \in H\}.$$

Then for every α with $1 \leq \alpha < \omega_1$ there exists $U \subseteq 2^\omega \times X$ with $U \in \underset{\sim}{\Pi}^0_\alpha(R)$ such that for every $A \in \underset{\sim}{\Pi}^0_\alpha(H)$ there exists $x \in 2^\omega$ with $A = U_x$.

proof:

This is proved exactly as Theorem 2.6, replacing the basis for X with H. Note that when we replace U_n by U_n^* it is necessary to prove by induction on β that for every set $A \in \underset{\sim}{\Pi}^0_\beta(R)$ and $n \in \omega$ that the set

$$A^* = \{(x, y) : (x_n, y) \in A\}$$

is also in $\underset{\sim}{\Pi}^0_\beta(R)$.

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Lemma 3.4 *Suppose $H \subseteq P(2^\omega)$, R is defined as in Lemma 3.3, and*

$$\text{Borel}(2^\omega) \subseteq \text{Borel}(H).$$

Then for every set $A \in \text{Borel}(R)$ the set $D = \{x : (x,x) \in A\}$ is in Borel(H).

proof:

If $A = B \times C$ where B is clopen and $C \in H$, then $D = B \cap C$ which is in Borel(H) by assumption. Note that

$$\{x : (x,x) \in \bigcap_n A_n\} = \bigcap_n \{x : (x,x) \in A_n\}$$

and

$$\{x : (x,x) \in \sim A\} = \sim \{x : (x,x) \in A\},$$

so the result follows by induction.

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Proof of Theorem 3.1:

Suppose $\text{Borel}(H) = \underset{\sim}{\Pi}^0_\alpha(H)$ and let $U \subseteq 2^\omega \times 2^\omega$ be universal for $\underset{\sim}{\Pi}^0_\alpha(H)$ given by Lemma 3.3. By Lemma 3.4 the set $D = \{x : (x,x) \in U\}$ is in Borel(H) and hence its complement is in $\text{Borel}(H) = \underset{\sim}{\Pi}^0_\alpha(H)$. Hence we get the same old contradiction: if $U_x = \sim D$, then $x \in D$ iff $x \notin D$.

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Theorem 3.5 *(Recław) If X is a second countable space and X can be mapped continuously onto the unit interval,* $[0,1]$, *then* $\text{ord}(X) = \omega_1$.

proof:

Let $f : X \to [0,1]$ be continuous and onto. Let $\mathcal{B}$ be a countable base for X and let $H = \{f(B) : B \in \mathcal{B}\}$. Since the preimage of an open subset of $[0,1]$ is open in X it is clear that $\text{Borel}([0,1]) \subseteq \text{Borel}(H)$. So by Corollary 3.2 it follows that $\text{ord}(H) = \omega_1$. But f maps the Borel hierarchy of X directly over to the hierarchy generated by H, so $\text{ord}(X) = \omega_1$.

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Note that if X is a discrete space of cardinality the continuum then there is a continuous map of X onto $[0,1]$ but $\text{ord}(X) = 1$.

The Cantor space 2^ω can be mapped continuously onto $[0,1]$ via the map

$$x \mapsto \sum_{n=0}^{\infty} \frac{x(n)}{2^{n+1}}.$$

This map is even one-to-one except at countably many points where it is two-to-one. It is also easy to see that $\mathbb{R}$ can be mapped continuously onto $[0,1]$ and ω^ω can be mapped onto 2^ω. It follows that in Theorem 3.5 we may replace $[0,1]$ by 2^ω, ω^ω, or $\mathbb{R}$.

Myrna Dzamonja points out that any completely regular space Y which contains a perfect set can be mapped onto $[0,1]$. This is true because if $P \subseteq Y$ is perfect, then there is a continuous map f from P onto $[0,1]$. But since Y is completely regular this map extends to Y.

Recław did not publish his result, but I did, see Miller [84] and [85].