CHAPTER 2

ABSTRACT LOGICS AS MODELS OF SENTENTIAL LOGICS

In this chapter we consider abstract logics as models of sentential logics. Abstract logics are suitable for modelling the metalogical properties that sentential logics can have; in this they differ notably from matrices. Our purpose is to single out for any sentential logic a class of abstract logics that exhibit some crucial metalogical properties of it. This leads us to distinguish two types of models for a sentential logic, the models "tout court" and the full models. The latter will be suitable for our purpose of modelling metalogical properties, an issue that will be dealt with specifically in the last section of this chapter, and also in Chapter 4.

2.1. Models and full models

We begin by using an abstract logic to define a logic on the algebra of formulas by the ordinary semantic procedure; using it the notion of model will be introduced.

DEFINITION 2.1. *If* $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ *is any abstract logic, the relation* $\models_{\mathbb{L}}$ *induced by* $\mathbb{L}$ *on the formula algebra is defined, for any* $\Gamma \cup \{\varphi\} \subseteq Fm$*, by:*

$$\Gamma \models_{\mathbb{L}} \varphi \iff \textit{for any } h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}) \,,\ h(\varphi) \in \mathrm{C}\big(h[\Gamma]\big) \,.$$

If L *is any class of abstract logics, then it induces on the formula algebra the relation* $\models_{\mathsf{L}} \ = \bigcap\{\models_{\mathbb{L}} : \mathbb{L} \in \mathsf{L}\}$.

PROPOSITION 2.2. *The relations* $\models_{\mathbb{L}}$ *and* $\models_{\mathsf{L}}$ *defined on the formula algebra* $\boldsymbol{Fm}$ *are structural consequence relations on this algebra.*

PROOF. It is easy to see that $\models_{\mathbb{L}}$ is a consequence relation, that is, that the operator defined as $\varphi \in \mathrm{Cn}_{\mathbb{L}}(\Gamma)$ iff $\Gamma \models_{\mathbb{L}} \varphi$ is a closure operator on $\boldsymbol{Fm}$. Actually, $\mathrm{Cn}_{\mathbb{L}}$ is the abstract logic on $\boldsymbol{Fm}$ projectively generated from $\mathbb{L}$ by the family of all homomorphisms $\mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$. By Theorem XII.2 of Brown and

Suszko [1973], it is structural. Moreover, the meet of any family of structural closure operators is also a structural closure operator. $\dashv$

PROPOSITION 2.3. *If there is a bilogical morphism between the abstract logics* $\mathbb{L}$ *and* $\mathbb{L}'$ *then* $\models_{\mathbb{L}} = \models_{\mathbb{L}'}$*; in particular,* $\models_{\mathbb{L}} = \models_{\mathbb{L}^*}$.

PROOF. Let $f : \boldsymbol{A} \to \boldsymbol{A}'$ be the epimorphism which is a bilogical morphism between $\mathbb{L}$ and $\mathbb{L}'$. Since for any $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$, $f \circ h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}')$, using 1.4.(ii) we get that $\models_{\mathbb{L}'} \subseteq \models_{\mathbb{L}}$. Conversely, given any $g \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}')$, since f is surjective, there is (by the Axiom of Choice) an $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ such that $f \circ h = g$; then if $\Gamma \models_{\mathbb{L}} \varphi$ we have $h(\varphi) \in \mathrm{C}\big(h[\Gamma]\big)$ which implies $f\big(h(\varphi)\big) \in f\big[\mathrm{C}\big(h[\Gamma]\big)\big]$ but since $f \circ \mathrm{C} = \mathrm{C}' \circ f$ by 1.4(iii), we obtain $g(\varphi) \in \mathrm{C}'\big(g[\Gamma]\big)$. This proves $\models_{\mathbb{L}} \subseteq \models_{\mathbb{L}'}$. $\dashv$

Now we introduce the general notion of an abstract logic being a model of a sentential logic.

DEFINITION 2.4. *An abstract logic* $\mathbb{L}$ *is a* ***model*** *of a sentential logic* $\mathcal{S}$ *when for any* $\Gamma \cup \{\varphi\} \subseteq Fm$, $\Gamma \vdash_{\mathcal{S}} \varphi$ *implies* $\Gamma \models_{\mathbb{L}} \varphi$*. The class of all models of* $\mathcal{S}$ *will be denoted by* $\mathsf{Mod}\mathcal{S}$.

A sentential logic $\mathcal{S}$ *is* ***complete*** *with respect to a class of abstract logics* L *when for any* $\Gamma \cup \{\varphi\} \subseteq Fm$, $\Gamma \vdash_{\mathcal{S}} \varphi$ *iff* $\Gamma \models_{\mathsf{L}} \varphi$.

From Proposition 2.3 follows at once:

PROPOSITION 2.5.

(1) *If there is a bilogical morphism between* $\mathbb{L}$ *and* $\mathbb{L}'$ *then* $\mathbb{L}$ *is a model of* $\mathcal{S}$ *iff* $\mathbb{L}'$ *is; in particular,* $\mathbb{L}$ *is a model of* $\mathcal{S}$ *iff* $\mathbb{L}^*$ *is.*
(2) *If* $\mathcal{S}$ *is complete with respect to a class* L *of abstract logics, then it is also complete with respect to the class* L^*. $\dashv$

The structurality of a sentential logic $\mathcal{S}$ implies that $\mathcal{S}$ is a model of itself, therefore so is its Lindenbaum-Tarski quotient $\mathcal{S}^* = \mathcal{S}/\widetilde{\boldsymbol{\Omega}}(\mathcal{S})$; thus we have:

PROPOSITION 2.6. *A sentential logic* $\mathcal{S}$ *is complete with respect to any class* L *of its models that includes either* $\mathcal{S}$ *or* $\mathcal{S}^*$*, and also with respect to the corresponding reduced class* L^**. In particular,* $\mathcal{S}$ *is complete with respect to the class of all its models, and also with respect to the class of all its reduced models.* $\dashv$

Since $h(\varphi) \in \mathrm{C}\big(h[\Gamma]\big)$ if and only if $h(\varphi) \in T$ for every $T \in \mathcal{C}$ such that $h[\Gamma] \subseteq T$, it results at once that:

PROPOSITION 2.7. *An abstract logic $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ is a model of a sentential logic $\mathcal{S}$ if and only if for every $T \in \mathcal{C}$, the matrix $\langle \boldsymbol{A}, T \rangle$ is a matrix for $\mathcal{S}$; that is, if and only if $\mathcal{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.* $\dashv$

Thus for every algebra $\boldsymbol{A}$, the whole family $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ determines a model of $\mathcal{S}$ on $\boldsymbol{A}$ having the finest closure system; therefore this model is the *weakest* model of $\mathcal{S}$ on $\boldsymbol{A}$, according to the ordering relation between abstract logics defined on page 18.

The notion of model we have just defined corresponds to the notion of a *generalized matrix of* a sentential logic, defined by Wójcicki as an arbitrary family of matrices over the same algebra in his [1969], [1973]. It is obvious that such a family is a generalized matrix for some $\mathcal{S}$ if and only if the abstract logic whose closure system is the one generated by the set of filters of the matrices in that family is a model of $\mathcal{S}$ in our sense, and conversely every model of $\mathcal{S}$ can be thought of as a generalized matrix for $\mathcal{S}$. The same notion of model, in the form of a closure operator rather than of a closure system, was put forward by Smiley in [1962].

In principle it might seem that this notion of model is finer than the usual one (a matrix), since each model possesses the same structure (a closure operator) which the sentential logic has; actually with its help one can express the notion of "being a model of a Gentzen-style rule" in a direct way (see Definition 4.3). However, we have seen that *any* family of matrices makes a model; due to this fact, models can be wildly different from what we intend them to model, and they might not exhibit some crucial metalogical properties of a sentential logic, like the Property of Disjunction or the Deduction Theorem, as discussed in Section 2.4 and in Chapter 4. For this reason we will define a more restricted kind of models.

DEFINITION 2.8. *If $\mathcal{S}$ is a sentential logic, then an abstract logic $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ is a **full model of** $\mathcal{S}$ iff $\mathbb{L}^*$ is equal to the abstract logic $\langle \boldsymbol{A}^*, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^* \rangle$, that is, iff the closure system of the reduction of $\mathbb{L}$ consists of all the $\mathcal{S}$-filters of the quotient algebra.*

The class of all full models of $\mathcal{S}$ will be denoted by $\mathsf{FMod}\mathcal{S}$, and the class of all reduced full models of $\mathcal{S}$ by $\mathsf{FMod}^*\mathcal{S}$; and for each algebra $\boldsymbol{A}$, the set of all full models of $\mathcal{S}$ on $\boldsymbol{A}$ will be denoted by $\mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}$.

We begin our study of full models by confirming that they are indeed models of the sentential logic, thus justifying the use of the term *model* in the name we have chosen for this notion. Moreover, we see that they inherit some properties of the sentential logic they model:

PROPOSITION 2.9. *Let $\mathbb{L}$ be a full model of a sentential logic $\mathcal{S}$. Then:*

(1) $\mathbb{L}$ *is a model of* $\mathcal{S}$.
(2) $\mathbb{L}$ *is finitary.*
(3) $\mathbb{L}$ *has theorems if and only if* $\mathcal{S}$ *has theorems.*

PROOF. If $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ is a full model of $\mathcal{S}$, then $\mathbb{L}^* = \langle \boldsymbol{A}^*, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^* \rangle$; but an abstract logic of this kind is always finitary (because the union of an upwards directed family of $\mathcal{S}$-filters is an $\mathcal{S}$-filter), and by 2.7 it is a model of $\mathcal{S}$; since the canonical projection from $\boldsymbol{A}$ onto $\boldsymbol{A}^*$ is a bilogical morphism, by 1.17 and 2.5, $\mathbb{L}$ itself will also be finitary and a model of $\mathcal{S}$, that is, (1) and (2) hold. If $\mathcal{S}$ does not have theorems then the empty set is the least $\mathcal{S}$-filter on any algebra, and thus it must be a closed set of any full model of $\mathcal{S}$. Conversely, if $\mathcal{S}$ has theorems then any $\mathcal{S}$-filter has to be non-empty, in particular the least closed set of any full model of $\mathcal{S}$. This proves (3). $\dashv$

It is not true that every model is a full model: see Section 5.1.1. Actually, an interesting problem is to find necessary and/or sufficient conditions for a model to be full which are at the same time logically interesting and useful for applications. In Sections 4.2 and 4.3 we solve this problem for two particular classes of sentential logics. Let us continue with elementary properties of full models of a sentential logic.

PROPOSITION 2.10. *For any algebra $\boldsymbol{A}$, the abstract logic $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \rangle$ is a full model of $\mathcal{S}$, and it is indeed the weakest full model of $\mathcal{S}$ on $\boldsymbol{A}$ (i.e., the one having the finest closure system)*[16].

PROOF. If we consider the reduction $\left\langle \boldsymbol{A}^*, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^* \right\rangle$ of $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \rangle$, then the canonical projection π is a bilogical morphism, so by 1.22 $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^* = \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^*$. As a consequence, $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \rangle$ is a full model of $\mathcal{S}$. And by 2.9 it is obviously the weakest one since it is simply the weakest model of $\mathcal{S}$. $\dashv$

In particular any sentential logic is a full model of itself, and it is actually the weakest one on $\boldsymbol{Fm}$.

PROPOSITION 2.11. *The class* $\mathbf{FMod}\mathcal{S}$ *is closed under bilogical morphisms: That is, if there is a bilogical morphism between two abstract logics $\mathbb{L}_1$ and $\mathbb{L}_2$ then $\mathbb{L}_1$ is a full model of $\mathcal{S}$ if and only if $\mathbb{L}_2$ is a full model of $\mathcal{S}$. In particular, an abstract logic $\mathbb{L}$ is a full model of $\mathcal{S}$ if and only if its reduction $\mathbb{L}^*$ is.*

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[16]The full models of the form $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \rangle$ for an arbitrary algebra $\boldsymbol{A}$ have been called ***basic full models*** of $\mathcal{S}$ in the later literature, beginning with Definition 2.10(i) in Font, Jansana, and Pigozzi [2001].

PROOF. If there is a bilogical morphism between $\mathbb{L}_1$ and $\mathbb{L}_2$ then $\mathbb{L}_1^*$ is (logically) isomorphic to $\mathbb{L}_2^*$. If one of them, say $\mathbb{L}_1$, is a full model of $\mathcal{S}$, then $\mathcal{C}_1^* = \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}_1^*$ and by 1.22 also $\mathcal{C}_2^* = \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}_2^*$; but since $\mathbb{L}_2^*$ is reduced, this implies that $\mathbb{L}_2$ is a full model of $\mathcal{S}$. ⊣

From Definition 2.8 and Propositions 2.10 and 2.11 it results at once:

COROLLARY 2.12. *An abstract logic $\mathbb{L}$ is a full model of $\mathcal{S}$ if and only if there is a bilogical morphism from $\mathbb{L}$ onto an abstract logic of the form $\langle \boldsymbol{B}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}\rangle$.* ⊣

COROLLARY 2.13. *The class* $\mathsf{FMod}\mathcal{S}$ *is the smallest class of abstract logics that contains all those of the form $\langle \boldsymbol{B}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}\rangle$ and is closed under bilogical morphisms (i.e., under the operations of taking images and inverse images by bilogical morphisms).* ⊣

We will use these facts very often, namely when we want to prove that some property holds for every full model: If the property is preserved under bilogical morphisms, then it is enough, and often simpler, to prove that it holds for every abstract logic of the form $\langle \boldsymbol{B}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}\rangle$. Each of these abstract logics is the finest full model on the corresponding algebra, and Corollary 2.13 tells us that all full models have this form, perhaps modulo a bilogical morphism; this may be seen as a justification of the use of the term *full* to describe the notion of full model.

Given an abstract logic $\mathbb{L}$, consider the projection of $\boldsymbol{A}$ onto $\boldsymbol{A}^* = \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}(\mathbb{L})$. It is a particular case of the situation described in Proposition 1.20, which tells us that the $\mathcal{S}$-filters on $\boldsymbol{A}^*$ are the result of reducing the $\mathcal{S}$-filters F on $\boldsymbol{A}$ such that $\widetilde{\boldsymbol{\Omega}}(\mathbb{L})$ is compatible with F, that is, such that $\widetilde{\boldsymbol{\Omega}}(\mathbb{L}) \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}}(F)$. Then we obtain the next characterization, which is particularly interesting for it offers another view of the "fullness" of full models: An abstract logic $\mathbb{L}$ is a full model of $\mathcal{S}$ if and only if its closure system consists of *all* the $\mathcal{S}$-filters that correspond to an $\mathcal{S}$-filter on the reduction $\boldsymbol{A}^*$ of $\boldsymbol{A}$ by $\widetilde{\boldsymbol{\Omega}}(\mathbb{L})$.

THEOREM 2.14. *An abstract logic $\mathbb{L} = \langle \boldsymbol{A}, \mathcal{C}\rangle$ is a full model of $\mathcal{S}$ if and only if $\mathcal{C} = \big\{F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathcal{C}) \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}}(F)\big\}$.*

PROOF. ($\Rightarrow$): If $\mathbb{L} = \langle \boldsymbol{A}, \mathcal{C}\rangle$ is a full model of $\mathcal{S}$ and $F \in \mathcal{C}$, then by Proposition 2.9 $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, and in general $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathcal{C}) \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}}(F)$, by Proposition 1.2. In order to prove the other inclusion assume that $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is such that $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathcal{C}) \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}}(F)$, that is, $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathcal{C})$ is compatible with F. By Proposition 1.20 there is some $G \in \mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathcal{C})\big)$ such that $F = \pi^{-1}[G]$, where π is the projection from $\boldsymbol{A}$ onto $\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathcal{C})$. Since π is a bilogical morphism from $\mathbb{L}$ onto $\mathbb{L}^*$, and $\mathcal{C}^* = \mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathcal{C})\big)$ because $\mathbb{L}$ is a full model of $\mathcal{S}$, it results that $F \in \mathcal{C}$.
($\Leftarrow$): Assume now that $\mathcal{C} = \big\{F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathcal{C}) \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}}(F)\big\}$. Using Proposition 1.20 again, we see that π is a bilogical morphism between $\mathbb{L} = \langle \boldsymbol{A}, \mathcal{C}\rangle$

and the abstract logic $\langle \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathcal{C}), \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathcal{C}))\rangle$. From this it follows that $\mathcal{C}^* = \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathcal{C}))$ and as a consequence $\mathbb{L}$ is a full model of $\mathcal{S}$. ⊣

PROPOSITION 2.15. *An abstract logic $\mathbb{L}$ is a full model of $\mathcal{S}$ if and only if there is a full model of $\mathcal{S}$, $\mathbb{L}_\kappa$, on a formula algebra $\boldsymbol{Fm}_\kappa$ of suitable cardinality, and there is some $\theta \in \mathrm{Con}\,\boldsymbol{Fm}_\kappa$ such that $\mathbb{L}$ is isomorphic to $\mathbb{L}_\kappa/\theta$. And $\mathbb{L}$ is a reduced full model of $\mathcal{S}$ iff $\mathbb{L}$ is isomorphic to the reduction of a full model of $\mathcal{S}$ on $\boldsymbol{Fm}_\kappa$.*

PROOF. Simply repeat the construction in the proof of Proposition 1.16 and apply Proposition 2.11. The second part follows from the first and Proposition 1.13. ⊣

2.2. $\mathcal{S}$-algebras

From the previous properties it follows that the reduced full models of $\mathcal{S}$ are exactly all those abstract logics of the form $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\rangle$ which are reduced. This observation suggests that we should highlight the algebras for which this situation happens:

DEFINITION 2.16. *If $\mathcal{S}$ is a sentential logic, then an algebra $\boldsymbol{A}$ is an **$\mathcal{S}$-algebra** if and only if the abstract logic $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\rangle$ is reduced, that is, iff it is the algebraic reduct of a reduced full model of $\mathcal{S}$.*

The class of all $\mathcal{S}$-algebras will be denoted by $\mathsf{Alg}\mathcal{S}$.

Thus the Lindenbaum-Tarski algebra $\boldsymbol{Fm}^* = \boldsymbol{Fm}/\widetilde{\boldsymbol{\Omega}}(\mathcal{S})$ is an $\mathcal{S}$-algebra as well. The term "$\mathcal{S}$-algebra" has already been used in the literature, in some algebraic approaches to smaller classes of sentential logics, to denote a class of algebras naturally associated with a sentential logic $\mathcal{S}$. Perhaps the most well-known case is Rasiowa [1974], where this term, introduced in Rasiowa and Sikorski [1953], is used for a class of logics of implicative character, the so-called *standard systems of implicative extensional propositional calculi*. In Czelakowski [1980] Proposition 8.5 it is proved that in all these cases Rasiowa's "$\mathcal{S}$-algebras" are the algebraic reducts of their reduced matrices; as we shall see, our Proposition 3.2 will confirm that the class we call $\mathcal{S}$-algebras coincides with the class she called by this name. The first extension of this terminology was performed in Czelakowski [1981] to *equivalential logics with an algebraic semantics*, a larger class of logics that also falls under the scope of Proposition 3.2.

From the definition and previous results on full models we immediately have:

PROPOSITION 2.17. *For any abstract logic* $\mathbb{L} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ *the following conditions are equivalent:*

(i) $\mathbb{L}$ *is a reduced full model of* $\mathcal{S}$.
(ii) $\mathbb{L}$ *is reduced and* $\mathcal{C} = \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.
(iii) $\boldsymbol{A}$ *is an* $\mathcal{S}$*-algebra and* $\mathcal{C} = \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. ⊣

PROPOSITION 2.18. *Let* $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ *be a full model of* $\mathcal{S}$. *Then the algebra* $\boldsymbol{A}^{*}$ *is an* $\mathcal{S}$*-algebra, and so* $\widetilde{\boldsymbol{\Omega}}(\mathbb{L}) \in \mathrm{Con}_{\mathbf{Alg}\mathcal{S}}\boldsymbol{A}$. ⊣

It may be interesting to observe that in order to obtain the class of $\mathcal{S}$-algebras one does not need the notion of full model; the notion of model is enough:

PROPOSITION 2.19. *For any sentential logic* $\mathcal{S}$*, the class of* $\mathcal{S}$*-algebras is the class of the algebraic reducts of all the reduced models of* $\mathcal{S}$.

PROOF. For any $\boldsymbol{A}$, the abstract logic $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \rangle$ is a model of $\mathcal{S}$, and if $\boldsymbol{A} \in \mathbf{Alg}\mathcal{S}$ then it is reduced. Conversely, if $\mathbb{L} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ is any model of $\mathcal{S}$ then $\mathcal{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ by 2.7, therefore $\widetilde{\boldsymbol{\Omega}}\big(\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \rangle\big) \subseteq \widetilde{\boldsymbol{\Omega}}(\mathbb{L})$; thus if $\mathbb{L}$ is reduced then so is $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \rangle$, and this means that $\boldsymbol{A} \in \mathbf{Alg}\mathcal{S}$. ⊣

PROPOSITION 2.20. *The class of* $\mathcal{S}$*-algebras is closed under isomorphisms.*

PROOF. If $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are two isomorphic algebras, then it is easy to prove, using 1.19, that the lattices $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}_1$ and $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}_2$ are also isomorphic by the induced mapping; therefore by 1.21 the abstract logics $\langle \boldsymbol{A}_1, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}_1 \rangle$ and $\langle \boldsymbol{A}_2, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}_2 \rangle$ are isomorphic abstract logics. Hence one of them is reduced iff the other one is. Therefore $\boldsymbol{A}_1$ is an $\mathcal{S}$-algebra iff $\boldsymbol{A}_2$ is. ⊣

Although it contains some redundancies, the next result is of interest since it has a general form corresponding to many of Verdú's results for particular $\mathcal{S}$, especially those in Font and Verdú [1988], [1989b], [1991] and those in Verdú [1978], [1979], [1987]. See Chapter 5 for the exact correspondence between 2.21 and these particular results; as we show there, using 2.21, these particular results give nice characterizations of full models of $\mathcal{S}$ in many cases where the $\mathcal{S}$-algebras and the $\mathcal{S}$-filters on them have already been characterized.

PROPOSITION 2.21. *For any abstract logic* $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ *the following conditions are equivalent:*

(i) $\mathbb{L}$ *is a full model of* $\mathcal{S}$.
(ii) $\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}(\mathbb{L})$ *is an* $\mathcal{S}$*-algebra and* $\mathcal{C}/\widetilde{\boldsymbol{\Omega}}(\mathbb{L}) = \mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}(\mathbb{L})\big)$.
(iii) *There is a bilogical morphism between the abstract logic* $\mathbb{L}$ *and an abstract logic* $\mathbb{L}' = \langle \boldsymbol{A}', \mathrm{C}' \rangle$ *such that* $\boldsymbol{A}'$ *is an* $\mathcal{S}$*-algebra and* $\mathcal{C}' = \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}'$. ⊣

Notice also that the characterizations of algebras in terms of closure operators contained in Verdú [1985], having the form "an algebra belongs to such-and-such class of algebras if and only if there is a closure operator on it satisfying some list of properties and being reduced", will become instances of the definition of $\mathcal{S}$-algebra for those $\mathcal{S}$ whose full models are characterized by that list of properties.

THEOREM 2.22 (Completeness Theorem). *Any sentential logic $\mathcal{S}$ is complete with respect to the following classes of abstract logics:*

(1) *The class of all full models of $\mathcal{S}$.*
(2) *The class of all abstract logics of the form $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\rangle$ for all algebras $\boldsymbol{A}$.*
(3) *The class of all reduced full models of $\mathcal{S}$, i.e., the class of all abstract logics of the form $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\rangle$ for all $\boldsymbol{A} \in \mathbf{Alg}\mathcal{S}$.*

PROOF. The three classes consist of models of $\mathcal{S}$, and $\mathcal{S}^*$ belongs to all of them. Therefore these classes satisfy the conditions of Proposition 2.6, so $\mathcal{S}$ is complete with respect to each one of them. ⊣

The usefulness of this result, especially its part (3), for a particular $\mathcal{S}$, depends on the characterizations we may have of the class $\mathbf{Alg}\mathcal{S}$ and of the operator of $\mathcal{S}$-filter-generation on the algebras of this class.

We have seen that the relationship between $\mathbf{Alg}\mathcal{S}$ and $\mathbf{FMod}\mathcal{S}$ is similar to the one existing between $\mathbf{Alg}^*\mathcal{S}$ and $\mathbf{Matr}\mathcal{S}$: in both cases the algebras are the algebraic reducts of the reduced models under consideration. Now we determine the precise relationship between the two classes of algebras.

THEOREM 2.23. *For any sentential logic $\mathcal{S}$, the class $\mathbf{Alg}\mathcal{S}$ is the class of all subdirect products of algebras in the class $\mathbf{Alg}^*\mathcal{S}$; in symbols: $\mathbf{Alg}\mathcal{S} = \mathbf{P}_{\mathrm{SD}}\mathbf{Alg}^*\mathcal{S}$.*

PROOF. If $\boldsymbol{A} \in \mathbf{Alg}\mathcal{S}$, we have that $Id_A = \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) = \bigcap\{\boldsymbol{\Omega}_{\boldsymbol{A}}(F) : F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\}$. In this situation we know that $\boldsymbol{A}$ is a subdirect product of the quotients $\{\boldsymbol{A}/\boldsymbol{\Omega}_{\boldsymbol{A}}(F) : F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\}$, and it is always true that $\boldsymbol{A}/\boldsymbol{\Omega}_{\boldsymbol{A}}(F) \in \mathbf{Alg}^*\mathcal{S}$ when $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Conversely, let $\boldsymbol{A}$ be a subdirect product of a family $\{\boldsymbol{A}_i : i \in I\} \subseteq \mathbf{Alg}^*\mathcal{S}$; thus for each $i \in I$ there is some $F_i \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}_i$ such that $\boldsymbol{\Omega}_{\boldsymbol{A}_i}(F_i) = Id_{A_i}$. Now consider the closure system $\mathcal{C}$ generated on $\boldsymbol{A}$ by the family of subsets $\{\pi_i^{-1}[F_i] : i \in I\}$, where π_i is the canonical epimorphism from $\boldsymbol{A}$ onto $\boldsymbol{A}_i$. The abstract logic $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is obviously a model of $\mathcal{S}$, and it is reduced: If $\langle a, b\rangle \in \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathcal{C}) = \bigcap\{\boldsymbol{\Omega}_{\boldsymbol{A}}(T) : T \in \mathcal{C}\}$ then for each $i \in I$, $\langle a, b\rangle \in \boldsymbol{\Omega}_{\boldsymbol{A}}\big(\pi_i^{-1}[F_i]\big) = \pi_i^{-1}\big[\boldsymbol{\Omega}_{\boldsymbol{A}_i}(F_i)\big] = \pi_i^{-1}[Id_{A_i}] = \ker \pi_i$, that is, $\pi_i(a) = \pi_i(b)$ for all $i \in I$, which implies $a = b$. We have proved that $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is a reduced model of $\mathcal{S}$. By Proposition 2.19, $\boldsymbol{A} \in \mathbf{Alg}\mathcal{S}$. ⊣

COROLLARY 2.24. *For any sentential logic $\mathcal{S}$, $\mathbf{Alg}^*\mathcal{S} \subseteq \mathbf{Alg}\mathcal{S}$; and $\mathbf{Alg}^*\mathcal{S} = \mathbf{Alg}\mathcal{S}$ if and only if the class $\mathbf{Alg}^*\mathcal{S}$ is closed under subdirect products.* $\dashv$

Among the many consequences of Theorem 2.23 is that the class $\mathbf{Alg}\mathcal{S}$ is always closed under subdirect products; hence it is also closed under direct products. Since quasivarieties are always closed under subdirect products, it may be interesting to record the following:

COROLLARY 2.25. *If the class $\mathbf{Alg}^*\mathcal{S}$ is a quasivariety, then $\mathbf{Alg}^*\mathcal{S} = \mathbf{Alg}\mathcal{S}$. In particular, this holds when $\mathbf{Alg}^*\mathcal{S}$ is a variety.* $\dashv$

This covers many of the common sentential logics, whose associated classes of algebras are quasivarieties or varieties. Moreover, in Chapter 3 we prove that for all protoalgebraic sentential logics, a rather wide class, the equality $\mathbf{Alg}^*\mathcal{S} = \mathbf{Alg}\mathcal{S}$ also holds, even if this class is not a variety or a quasi-variety; the logic LJ of the "last judgement" invented by Herrmann [1993b] is an example where this class is not even elementary. In addition, the converse of Corollary 2.25 is not true, that is, $\mathbf{Alg}\mathcal{S}$ can be a quasivariety, or even a variety, without being equal to $\mathbf{Alg}^*\mathcal{S}$; again the $\{\wedge, \vee\}$-fragment of classical logic is an example, see Section 5.1.1.

Another consequence of Theorem 2.23 is that, even if they are different, these two classes generate the same quasivariety, and a fortiori the same variety:

PROPOSITION 2.26. *For each sentential logic $\mathcal{S}$, the classes of algebras $\mathbf{Alg}\mathcal{S}$ and $\mathbf{Alg}^*\mathcal{S}$ generate the same quasivariety and the same variety; this variety is the class $\mathbf{K}_{\mathcal{S}}$, that is, the variety generated by the Lindenbaum-Tarski algebra $\boldsymbol{Fm}^*$.*

PROOF. From the result in Theorem 2.23 it follows that the quasivariety generated by $\mathbf{Alg}\mathcal{S}$ is included in the quasivariety generated by $\mathbf{Alg}^*\mathcal{S}$; but since by Corollary 2.24 $\mathbf{Alg}^*\mathcal{S} \subseteq \mathbf{Alg}\mathcal{S}$, the opposite inclusion also holds, and the two quasivarieties are equal. As a consequence, the varieties generated by them also coincide, and by Proposition 1.23 this variety is $\mathbf{K}_{\mathcal{S}}$. $\dashv$

This result adds further support, from within our theory, to the common idea that if one insists on associating a variety with every sentential logic in a uniform way, then the variety $\mathbf{K}_{\mathcal{S}}$ generated by the Lindenbaum-Tarski algebra is the most natural one; but we have already mentioned that there are cases where there is no point in doing so.

PROPOSITION 2.27. *If $\mathcal{S}$ and $\mathcal{S}'$ are two sentential logics, and $\mathcal{S}'$ is stronger than $\mathcal{S}$, then $\mathbf{Alg}\mathcal{S}' \subseteq \mathbf{Alg}\mathcal{S}$ and $\mathbf{Alg}^*\mathcal{S}' \subseteq \mathbf{Alg}^*\mathcal{S}$.*

PROOF. If $\mathcal{S} \leqslant \mathcal{S}'$ then for any $\boldsymbol{A}$, $\mathcal{F}i_{\mathcal{S}'}\boldsymbol{A} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, and this directly implies $\mathsf{Alg}^*\mathcal{S}' \subseteq \mathsf{Alg}^*\mathcal{S}$. Then, using Theorem 2.23, this implies $\mathsf{Alg}\mathcal{S}' \subseteq \mathsf{Alg}\mathcal{S}$. $\dashv$

So far we have associated *three classes of algebras* with an arbitrary sentential logic: $\mathsf{Alg}^*\mathcal{S}$, $\mathsf{Alg}\mathcal{S}$, $\mathsf{K}_{\mathcal{S}}$. We have seen that $\mathsf{Alg}^*\mathcal{S} \subseteq \mathsf{Alg}\mathcal{S} \subseteq \mathsf{K}_{\mathcal{S}}$; the two extreme ones have already been considered in the literature, but sometimes $\mathsf{Alg}^*\mathcal{S}$ is too small and $\mathsf{Alg}^*\mathcal{S} \subsetneq \mathsf{Alg}\mathcal{S} = \mathsf{K}_{\mathcal{S}}$, and sometimes $\mathsf{K}_{\mathcal{S}}$ is too large and $\mathsf{Alg}^*\mathcal{S} = \mathsf{Alg}\mathcal{S} \subsetneq \mathsf{K}_{\mathcal{S}}$, as in several examples we have already mentioned. It is then natural to ask the following question:

OPEN PROBLEM. *Is there a sentential logic $\mathcal{S}$ such that the three classes of algebras are all different, that is, such that $\mathsf{Alg}^*\mathcal{S} \subsetneq \mathsf{Alg}\mathcal{S} \subsetneq \mathsf{K}_{\mathcal{S}}$?*[17]

By Proposition 3.2, a logic with this property cannot be protoalgebraic, and by Corollary 2.25 the first two classes cannot be quasivarieties. Moreover, by the results we will find in Chapter 4, such a logic cannot be selfextensional and at the same time satisfy the Property of Conjunction, or the Deduction Theorem.

2.3. The lattice of full models over an algebra

In this section we will prove that the ordered set $\langle \mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}, \leqslant\rangle$ and the ordered set $\langle \mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}, \subseteq\rangle$ are isomorphic through the Tarski operator (Theorem 2.30) and that the second one is a complete lattice (Theorem 2.31); as a consequence the set of all full models of $\mathcal{S}$ over an algebra will also become a complete lattice. We begin by introducing a construction which will turn out to be inverse to the Tarski operator, and which has an interest of its own.

DEFINITION 2.28. *Let $\boldsymbol{A}$ be any algebra. For any $\theta \in \mathrm{Con}\,\boldsymbol{A}$, we denote by $\widetilde{\mathbf{H}}_{\boldsymbol{A}}(\theta) = \langle \boldsymbol{A}, \mathrm{C}_\theta\rangle$ the abstract logic projectively generated on $\boldsymbol{A}$ from the abstract logic $\left\langle \boldsymbol{A}/\theta, \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)\right\rangle$ by the canonical projection π of $\boldsymbol{A}$ onto $\boldsymbol{A}/\theta$.*

Note that with this definition π becomes a bilogical morphism between $\widetilde{\mathbf{H}}_{\boldsymbol{A}}(\theta)$ and the abstract logic$\langle \boldsymbol{A}/\theta, \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)\rangle$. Now we record some general properties of this construction.

LEMMA 2.29. *For any $\theta \in \mathrm{Con}\,\boldsymbol{A}$, it holds that $\theta \in \mathrm{Con}\,\widetilde{\mathbf{H}}_{\boldsymbol{A}}(\theta)$. Moreover, it holds that $\widetilde{\mathbf{H}}_{\boldsymbol{A}}(\theta)/\theta = \langle \boldsymbol{A}/\theta, \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)\rangle$, that $\widetilde{\mathbf{H}}_{\boldsymbol{A}}(\theta) \in \mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}$ and that the mapping $\theta \mapsto \widetilde{\mathbf{H}}_{\boldsymbol{A}}(\theta)$ is order-preserving: If $\theta_1, \theta_2 \in \mathrm{Con}\,\boldsymbol{A}$ are such that $\theta_1 \subseteq \theta_2$ then $\widetilde{\mathbf{H}}_{\boldsymbol{A}}(\theta_1) \leqslant \widetilde{\mathbf{H}}_{\boldsymbol{A}}(\theta_2)$.*

[17] This problem was solved in the affirmative in Bou [2001] in the context of the study of certain subintuitionistic logics, and, independently, in Babyonyshev [2003] by an ad-hoc construcion.

PROOF. If $\langle a,b\rangle \in \theta$ then $\pi(a)=\pi(b)$ and thus $\mathrm{Fi}_{\mathcal{S}}^{\boldsymbol{A}/\theta}\big(\pi(a)\big)=\mathrm{Fi}_{\mathcal{S}}^{\boldsymbol{A}/\theta}\big(\pi(b)\big)$, therefore $\pi^{-1}\big[\mathrm{Fi}_{\mathcal{S}}^{\boldsymbol{A}/\theta}\big(\pi(a)\big)\big] = \pi^{-1}\big[\mathrm{Fi}_{\mathcal{S}}^{\boldsymbol{A}/\theta}\big(\pi(b)\big)\big]$. But by construction we know that $\mathrm{C}_\theta = \pi^{-1}\circ \mathrm{Fi}_{\mathcal{S}}^{\boldsymbol{A}/\theta}\circ\pi$; therefore we get $\mathrm{C}_\theta(a)=\mathrm{C}_\theta(b)$. Thus we have proved that $\theta \in \operatorname{Con}\widetilde{\mathbf{H}}_{\boldsymbol{A}}(\theta)$. The second part of the statement comes directly from the construction. Moreover, since $\big\langle \boldsymbol{A}/\theta, \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)\big\rangle$ is always a full model of $\mathcal{S}$, by 2.11 $\widetilde{\mathbf{H}}_{\boldsymbol{A}}(\theta)$ is also a full model of $\mathcal{S}$. To prove the last part of the Lemma, take $\theta_1,\theta_2 \in \operatorname{Con}\boldsymbol{A}$ and consider the natural projections $\pi_1 : \boldsymbol{A}\to\boldsymbol{A}/\theta_1$ and $\pi_2 : \boldsymbol{A}\to\boldsymbol{A}/\theta_2$. If moreover $\theta_1\subseteq\theta_2$ then the mapping $j(a/\theta_1)=a/\theta_2$ is an epimorphism from $\boldsymbol{A}/\theta_1$ onto $\boldsymbol{A}/\theta_2$, and the following diagram

$$\begin{tikzcd} \boldsymbol{A} \arrow[r, "\pi_1"] \arrow[dr, "\pi_2"'] & \boldsymbol{A}/\theta_1 \arrow[d, "j"] \\ & \boldsymbol{A}/\theta_2 \end{tikzcd}$$

commutes. As a consequence, if $Z\in\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta_2)$ then $j^{-1}[Z]\in\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta_1)$. Due to this, the closure system of $\widetilde{\mathbf{H}}_{\boldsymbol{A}}(\theta_2)$, which is projectively generated from $\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta_2)$ by π_2, satisfies

$$\begin{aligned}\mathcal{C}_{\theta_2} &= \{\pi_2^{-1}[Z] : Z\in\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta_2)\} = \\ &= \big\{\pi_1^{-1}\big[j^{-1}[Z]\big] : Z\in\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta_2)\big\}\subseteq \\ &\subseteq \big\{\pi_1^{-1}[X] : X\in\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta_1)\big\} = \mathcal{C}_{\theta_1},\end{aligned}$$

that is, it is contained in the closure system of $\widetilde{\mathbf{H}}_{\boldsymbol{A}}(\theta_1)$, which is projectively generated from $\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta_1)$ by π_1. Therefore, $\widetilde{\mathbf{H}}_{\boldsymbol{A}}(\theta_1)\leqslant\widetilde{\mathbf{H}}_{\boldsymbol{A}}(\theta_2)$ as was to be shown. ⊣

We will now prove[18] that, when restricted to $\operatorname{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$, this mapping is exactly the inverse of the Tarski operator, and that both mappings are order-isomorphisms.

THEOREM 2.30 (The Isomorphism Theorem). *For any algebra $\boldsymbol{A}$, the Tarski operator $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}$ is an order-isomorphism between the ordered sets $\langle\mathcal{FM}od_{\mathcal{S}}\boldsymbol{A},\leqslant\rangle$ and $\langle\operatorname{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A},\subseteq\rangle$; and the mapping $\widetilde{\mathbf{H}}_{\boldsymbol{A}}$ is its inverse.*

PROOF. As we have already observed in Proposition 2.18, if $\mathbb{L}\in\mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}$ then $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathbb{L})\in\operatorname{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$. Dually, in Lemma 2.29 we have seen that if $\theta\in$

[18] For an essentially different proof, see Font, Jansana, and Pigozzi [2006].

$\mathrm{Con}_{\mathbf{Alg}\mathcal{S}}\boldsymbol{A}$ then $\widetilde{\mathbf{H}}_{\boldsymbol{A}}(\theta) \in \mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}$. So both mappings are well-defined. Now we will see that they are bijections.

We first prove that $\widetilde{\mathbf{H}}_{\boldsymbol{A}}\big(\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathbb{L})\big) = \mathbb{L}$ assuming that $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C}\rangle \in \mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}$: If $\mathbb{L}$ is a full model of $\mathcal{S}$, we have already seen in 2.18 that $\boldsymbol{A}^*$ is an $\mathcal{S}$-algebra, and that $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathbb{L}) \in \mathrm{Con}_{\mathbf{Alg}\mathcal{S}}\boldsymbol{A}$; moreover, $\mathcal{C}$ is projectively generated from its reduction $\mathcal{C}^* = \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^*$ by the canonical projection from $\boldsymbol{A}$ onto $\boldsymbol{A}^* = \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathbb{L})$. By Definition 2.28, this is exactly $\widetilde{\mathbf{H}}_{\boldsymbol{A}}\big(\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathbb{L})\big)$, therefore $\widetilde{\mathbf{H}}_{\boldsymbol{A}}\big(\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathbb{L})\big) = \mathbb{L}$.

Let now $\theta \in \mathrm{Con}_{\mathbf{Alg}\mathcal{S}}\boldsymbol{A}$; we now prove that $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}\big(\widetilde{\mathbf{H}}_{\boldsymbol{A}}(\theta)\big) = \theta$: Observe that $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}/\theta}\big(\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)\big) = Id_{A/\theta}$ precisely because $\theta \in \mathrm{Con}_{\mathbf{Alg}\mathcal{S}}\boldsymbol{A}$. Then, using Proposition 1.7, we have

$$\begin{aligned}\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}\big(\widetilde{\mathbf{H}}_{\boldsymbol{A}}(\theta)\big) &= \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}\big(\pi^{-1}\big[\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)\big]\big)\\ &= \pi^{-1}\big[\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}/\theta}\big(\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)\big)\big]\\ &= \pi^{-1}[Id_{A/\theta}] = \theta\end{aligned}$$

Putting together the results of the last two paragraphs, we obtain that both mappings are bijections. Since by its own definition $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}$ is order-preserving, and by Lemma 2.29 $\widetilde{\mathbf{H}}_{\boldsymbol{A}}$ is also order-preserving, we conclude that they are order-isomorphisms between the two ordered sets. This ends the proof of the theorem. $\dashv$

Independently of this result, we can determine that the set $\mathrm{Con}_{\mathbf{Alg}\mathcal{S}}\boldsymbol{A}$ involved in the Isomorphism Theorem, ordered under $\subseteq$, has a lattice structure.

THEOREM 2.31. *For any algebra $\boldsymbol{A}$, the ordered set $\langle \mathrm{Con}_{\mathbf{Alg}\mathcal{S}}\boldsymbol{A}, \subseteq\rangle$ is a complete lattice, where* inf *is intersection.*

PROOF. Let $\{\theta_i : i \in I\}$ be a non-empty family of elements of $\mathrm{Con}_{\mathbf{Alg}\mathcal{S}}\boldsymbol{A}$, and put $\theta = \bigcap\{\theta_i : i \in I\}$; we will prove that $\theta \in \mathrm{Con}_{\mathbf{Alg}\mathcal{S}}\boldsymbol{A}$. First of all we observe that for any $a \in A$, $a/\theta = \bigcap\{a/\theta_i : i \in I\}$, and consider, for each $i \in I$, the mapping $h_i : \boldsymbol{A}/\theta \to \boldsymbol{A}/\theta_i$ defined by $h_i(a/\theta) = a/\theta_i$, which is an epimorphism. By assumption, for every $i \in I$, the abstract logic $\mathbb{L}_i = \big\langle \boldsymbol{A}/\theta_i, \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta_i)\big\rangle$ is reduced, and we have to show that $\mathbb{L} = \big\langle \boldsymbol{A}/\theta, \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)\big\rangle$ is reduced. Since $h_i^{-1}\big[\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta_i)\big] \subseteq \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)$ by Proposition 1.7 we have

$$\begin{aligned}\widetilde{\boldsymbol{\Omega}}(\mathbb{L}) &\subseteq \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}/\theta}\big(h_i^{-1}\big[\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta_i)\big]\big) =\\ &= h_i^{-1}\big[\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}/\theta_i}\big(\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta_i)\big)\big] = h_i^{-1}[Id_{A/\theta_i}]\end{aligned}$$

because $\mathbb{L}_i$ is reduced. Therefore if $\langle a/\theta, b/\theta\rangle \in \widetilde{\boldsymbol{\Omega}}(\mathbb{L})$ then $a/\theta_i = b/\theta_i$ for each $i \in I$, and as a consequence $a/\theta = b/\theta$. This proves that $\mathbb{L}$ is reduced, that is, that $\theta \in \mathrm{Con}_{\mathbf{Alg}\mathcal{S}}\boldsymbol{A}$. Thus $\mathrm{Con}_{\mathbf{Alg}\mathcal{S}}\boldsymbol{A}$ is closed under intersections of non-empty families. On the other hand, if $\boldsymbol{A}$ is trivial (1-element) then either

$\mathcal{F}i_{\mathcal{S}}\boldsymbol{A} = \{A\}$, if $\mathcal{S}$ has theorems, or $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A} = \{\emptyset, A\}$, if $\mathcal{S}$ doesn't; in either case the abstract logic $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\rangle$ is reduced, and hence it is a full model of $\mathcal{S}$, which shows that $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$. As a consequence, for an arbitrary $\boldsymbol{A}$, the universal congruence $A \times A \in \mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$, which concludes the proof that the ordered set $\mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$ is a complete lattice. ⊣

Since Theorem 2.30 establishes an order-isomorphism, we get immediately:

COROLLARY 2.32. *For any $\boldsymbol{A}$, the ordered set $\langle \mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}, \leqslant\rangle$ is a complete lattice, and the Tarski operator is a lattice isomorphism between $\langle \mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}, \leqslant\rangle$ and $\langle \mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}, \subseteq\rangle$.* ⊣

Note that, although $\mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}$ is a subset of the complete lattice of all abstract logics over $\boldsymbol{A}$, it need not be a sublattice; indeed, we do not have nice characterizations of the lattice operations in $\langle \mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}\,, \leqslant\rangle$. The only thing we can say is that, as a consequence of the preceding results, given any collection $\{\mathbb{L}_i : i \in I\}$ of full models of $\mathcal{S}$ on the same algebra $\boldsymbol{A}$, its infimum in the lattice of full models of $\mathcal{S}$ can be obtained as the abstract logic projectively generated from $\langle \boldsymbol{A}/\theta, \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)\rangle$ by the canonical projection of $\boldsymbol{A}$ onto $\boldsymbol{A}/\theta$, where $\theta = \bigcap\{\, \widetilde{\boldsymbol{\Omega}}(\mathbb{L}_i) : i \in I\}$.

PROPOSITION 2.33. *Let $\mathbb{L}_1$ and $\mathbb{L}_2$ be two full models of $\mathcal{S}$, and let h be a bilogical morphism between them. Then the mapping $\mathcal{C} \mapsto \{h[X] : X \in \mathcal{C}\}$ is an isomorphism between the lattice of all full models of $\mathcal{S}$ on $\boldsymbol{A}_1$ extending $\mathbb{L}_1$ and the lattice of all full models of $\mathcal{S}$ on $\boldsymbol{A}_2$ extending $\mathbb{L}_2$. And also the principal ideals of $\mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}_1$ and of $\mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}_2$ determined respectively by $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}_1}(\mathbb{L}_1)$ and by $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}_2}(\mathbb{L}_2)$ are isomorphic.*

PROOF. In Corollary 1.6 we have seen that the mapping $\mathcal{C} \mapsto \hat{h}(\mathcal{C}) = \{h[X] : X \in \mathcal{C}\}$ is an isomorphism between the lattices of all abstract logics on $\boldsymbol{A}_1$ extending $\mathbb{L}_1$ and of all abstract logics on $\boldsymbol{A}_2$ extending $\mathbb{L}_2$. But this mapping establishes in each case a bilogical morphism between the two abstract logics whose closure systems are $\mathcal{C}$ and $\hat{h}(\mathcal{C})$, and by Proposition 2.11 one of these is a full model of $\mathcal{S}$ if and only if the other one is. ⊣

And as a particular case we have:

COROLLARY 2.34. *If $\boldsymbol{A}, \boldsymbol{B}$ are algebras and $h : \boldsymbol{A} \to \boldsymbol{B}$ is an epimorphism satisfying any of the equivalent conditions appearing in Proposition 1.21, then h induces an isomorphism between the complete lattices $\mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}$ and $\mathcal{FM}od_{\mathcal{S}}\boldsymbol{B}$; and also the lattices $\mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$ and $\mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{B}$ are isomorphic.*

PROOF. This is the conjunction of 1.21 and 2.33 taking into account that, by 2.10, the abstract logic $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\rangle$ is the weakest full model of $\mathcal{S}$ on $\boldsymbol{A}$. ⊣

Finally we will use the language of categories to express the fact that, in some sense, using $\mathcal{S}$-algebras is "equivalent" to using reduced full models of $\mathcal{S}$, and that the process of reduction $\mathbb{L} \longmapsto \mathbb{L}^*$ has a good behaviour when considered globally, as a relationship between two categories of abstract logics.

THEOREM 2.35. *The algebraic category of the $\mathcal{S}$-algebras together with homomorphisms is isomorphic to the category whose objects are the reduced full models of $\mathcal{S}$, and whose arrows are the logical morphisms between its objects.*

PROOF. It is trivial to check that the class of abstract logics mentioned in the statement is really a category, since the identity mapping is a logical morphism, the composition of two logical morphisms is a logical morphism, and this composition is associative. To see that the category of $\mathcal{S}$-algebras is isomorphic to the category of reduced full models of $\mathcal{S}$ it is enough to consider the functor defined on objects by $\boldsymbol{A} \longmapsto \langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\rangle$, and defined on arrows by the identity: We know that if $\boldsymbol{A} \in \mathbf{Alg}\mathcal{S}$ then $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\rangle \in \mathbf{FMod}^*\mathcal{S}$, and that every reduced full model of $\mathcal{S}$ is of this form, so this is a bijection between objects; and since for every $h \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$, $h^{-1}[F] \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, h is a logical morphism between $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\rangle$ and $\langle \boldsymbol{B}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}\rangle$, thus clearly this is a functor at the arrows level, and this finishes the proof that this functor is an isomorphism between the two categories. $\dashv$

The category of reduced full models of $\mathcal{S}$ considered in 2.35 is trivially a full subcategory of the category whose objects are all full models of $\mathcal{S}$ with logical morphisms as arrows. But if we only use surjective arrows then we obtain a more precise relationship between both categories of abstract logics:

THEOREM 2.36. *The category $\mathfrak{L}^*$ of reduced full models of $\mathcal{S}$ with surjective logical morphisms is a full reflective subcategory of the category $\mathfrak{L}$ of all full models of $\mathcal{S}$ with surjective logical morphisms; and the reflector is the functor associated with the process of "reduction": $\mathbb{L} \longmapsto \mathbb{L}^*$.*

PROOF. $\mathfrak{L}^*$ is trivially a full subcategory of $\mathfrak{L}$. In order to check that the process of reduction $\mathbb{L} \mapsto \langle \mathbb{L}^*, \pi_{\mathbb{L}}\rangle$ (where $\pi_{\mathbb{L}} : \mathbb{L} \to \mathbb{L}^*$ is the canonical projection) gives the announced reflector, it is enough to check (see Balbes and Dwinger [1974] I.18.2, for instance) that for an arbitrary surjective logical morphism $f : \mathbb{L} \to \mathbb{L}'$ between an $\mathbb{L} \in \mathbf{FMod}\mathcal{S}$ and an $\mathbb{L}' \in \mathbf{FMod}^*\mathcal{S}$ there is a unique surjective logical morphism $f^* : \mathbb{L}^* \to \mathbb{L}'$ such that $f^* \circ \pi_{\mathbb{L}} = f$. Since $\pi_{\mathbb{L}}$ is a bilogical morphism, we can use Proposition 1.15 if we prove that $\ker f \supseteq \ker \pi_{\mathbb{L}} = \widetilde{\boldsymbol{\Omega}}(\mathbb{L})$. For this, consider the logic $\mathbb{L}_0$ projectively generated from $\mathbb{L}'$ by f; since f is an epimorphism, it becomes a bilogical morphism between $\mathbb{L}_0$ and $\mathbb{L}'$, and since by assumption $f^{-1}[T] \in \mathcal{C}$ for all $T \in \mathcal{C}'$, it results that $\mathbb{L} \leqslant \mathbb{L}_0$. Now, using Proposition 1.7 and that $\mathbb{L}'$ is reduced, we have that $\ker \pi_{\mathbb{L}} = \widetilde{\boldsymbol{\Omega}}(\mathbb{L}) \subseteq \widetilde{\boldsymbol{\Omega}}(\mathbb{L}_0) = f^{-1}\big[\widetilde{\boldsymbol{\Omega}}(\mathbb{L}')\big] = f^{-1}[Id_{A'}] = \ker f$. Then we can use Proposition 1.15 to obtain a unique logical morphism $f^* : \mathbb{L}^* \to \mathbb{L}'$ with $f^* \circ \pi_{\mathbb{L}} = f$; and this equality implies it is surjective. $\dashv$

In Theorem 2.44 we will find a better result for a restricted class of sentential logics, where this reflector will reflect all logical morphisms, and not just the surjective ones.

2.4. Full models and metalogical properties

In this section we will see how some typical metalogical properties are inherited by full models of a sentential logic, while others may require additional assumptions. We have already noted in Proposition 2.9 that every full model of $\mathcal{S}$ inherits some of the basic properties of a sentential logic $\mathcal{S}$: those of being finitary, of having theorems and of not having theorems. Clearly the second of these properties is inherited by arbitrary models, while it is easy to see that the first and the third one are not.

In general, the metalogical properties under consideration must be such that it makes sense to ask whether an arbitrary abstract logic satisfies them. That is, they must be properties of the closure operator $\mathrm{Cn}_{\mathcal{S}}$ associated with $\vdash_{\mathcal{S}}$ and of its relationship with the algebraic structure of the underlying algebra. Most of them can be expressed in the form of a Gentzen-style rule for the derivability relation $\vdash_{\mathcal{S}}$. In order to obtain a useful degree of precision we give the following definition of Gentzen-style rule. A ***sequent*** will be a pair $\langle \Gamma, \varphi \rangle$, written $\Gamma \vdash \varphi$, where Γ is a finite set of formulas and φ is a formula. A ***Gentzen-style rule*** is a pair which consists of a finite set $\{\Gamma_i \vdash \varphi_i : i < k\}$ of sequents and a sequent $\Gamma \vdash \varphi$, which follows from the set according to the rule; the rule is often writen symbolycally in the "fraction" form

$$\frac{\{\Gamma_i \vdash \varphi_i : i < k\}}{\Gamma \vdash \varphi}\,, \tag{2.7}$$

and one says that a sentential logic $\mathcal{S}$ ***satisfies the Gentzen-style rule*** represented in (2.7) whenever for any substitution σ the following implication holds:

If for all $i < k$, $\sigma[\Gamma_i] \vdash_{\mathcal{S}} \sigma(\varphi_i)$ holds, then $\sigma[\Gamma] \vdash_{\mathcal{S}} \sigma(\varphi)$ holds. (2.8)

In practice Gentzen-style rules are often described by using *schemes* that group together rules having a common form. For example, the expression

$$\frac{\Gamma, \psi_1 \vdash \varphi \qquad \Gamma, \psi_2 \vdash \varphi}{\Gamma, \psi_1 \vee \psi_2 \vdash \varphi} \tag{2.9}$$

has to be understood as varying over all finite sets of formulas Γ and all formulas φ, ψ_1, ψ_2. Strictly speaking it describes an infinite set of Gentzen-style rules which is closed under substitution instances (i.e. if it contains a rule, it contains the rule we obtain by applying an arbitrary substitution to all its formulas). In this way one does not need to use substitutions when characterizing the sentential logics that satisfy it, and we have that the sentence "(2.9) is a (Gentzen-style) rule of $\mathcal{S}$" actually means that for all finite Γ and all φ, ψ_1, ψ_2, if $\Gamma, \psi_1 \vdash_{\mathcal{S}} \varphi$ and $\Gamma, \psi_2 \vdash_{\mathcal{S}} \varphi$ then $\Gamma, \psi_1 \vee \psi_2 \vdash_{\mathcal{S}} \varphi$, as the rule scheme suggests.

Several of the properties considered in this section are of this kind, and one of the ways of further formalizing these issues in a general setting is by the use of Gentzen systems; we do this in Chapter 4.

By contrast, by a ***Hilbert-style rule*** of $\mathcal{S}$ we mean any sequent $\Gamma \vdash \varphi$ such that $\Gamma \vdash_{\mathcal{S}} \varphi$ holds. It is clear that both Hilbert-style and Gentzen-style rules can be formulated for an abstract logic $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ by substituting the $\vdash_{\mathcal{S}}$ relation by the closure operator C of $\mathbb{L}$, in an obvious way. Hence, they are metalogical properties of a sentential logic suitable to be investigated in the sense explained above. Note that, actually, an abstract logic is a model of $\mathcal{S}$ iff it satisfies all Hilbert-style rules of $\mathcal{S}$.

The congruence property

Recall that, for an arbitrary closure operator C, we denote by C^T the closure operator whose closure system is $\mathcal{C}^T = \{T' \in \mathcal{C} : T \subseteq T'\}$. We introduce an equivalence relation and a mapping naturally associated with any closure operator:

DEFINITION 2.37. *Let* C *be a closure operator on a set* A. *Then the* ***Frege relation*** *of* C *is:*

$$\boldsymbol{\Lambda}(\mathrm{C}) = \big\{ \langle a, b \rangle \in A \times A : \mathrm{C}(a) = \mathrm{C}(b) \big\}.$$

The ***Frege operator*** *is the mapping* $\boldsymbol{\Lambda}_{\mathrm{C}} : F \subseteq A \longmapsto \boldsymbol{\Lambda}_{\mathrm{C}}(F) = \boldsymbol{\Lambda}(\mathrm{C}^F)$.

If $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ *is an abstract logic, it will be convenient to use the notations* $\boldsymbol{\Lambda}(\mathbb{L})$ *and* $\boldsymbol{\Lambda}_{\mathbb{L}}$ *instead of* $\boldsymbol{\Lambda}(\mathrm{C})$ *and* $\boldsymbol{\Lambda}_{\mathrm{C}}$ *respectively.*

Note that $\boldsymbol{\Lambda}(\mathrm{C}) = \boldsymbol{\Lambda}_{\mathrm{C}}\big(\mathrm{C}(\emptyset)\big)$, and that $\boldsymbol{\Lambda}_{\mathrm{C}}$ is always order-preserving: if $F \subseteq G$ then $\boldsymbol{\Lambda}_{\mathrm{C}}(F) \subseteq \boldsymbol{\Lambda}_{\mathrm{C}}(G)$. Moreover:

PROPOSITION 2.38. *A closure operator* C *on a set* A *is finitary iff the Frege operator* $\boldsymbol{\Lambda}_{\mathrm{C}}$ *preserves unions of directed families of subsets of* A*; that is, for any directed family* $\mathcal{D}$ *of subsets of* A, $\boldsymbol{\Lambda}_{\mathrm{C}}(\bigcup\mathcal{D}) = \bigcup\{\boldsymbol{\Lambda}_{\mathrm{C}}(F) : F \in \mathcal{D}\}$.

PROOF. Let $\mathcal{D}$ be any directed family of subsets of A. Since $\boldsymbol{\Lambda}_{\mathrm{C}}$ is always order-preserving, we have $\boldsymbol{\Lambda}_{\mathrm{C}}(\bigcup\mathcal{D}) \supseteq \bigcup\{\boldsymbol{\Lambda}_{\mathrm{C}}(F) : F \in \mathcal{D}\}$. To prove the converse inclusion, suppose that $\langle a, b\rangle \in \boldsymbol{\Lambda}_{\mathrm{C}}(\bigcup\mathcal{D})$, that is, $\mathrm{C}(\bigcup\mathcal{D}, a) = \mathrm{C}(\bigcup\mathcal{D}, b)$. The finitarity of C implies that there are $c_1, \ldots, c_n \in \bigcup\mathcal{D}$ such that

$$\mathrm{C}(c_1, \ldots, c_n, a) = \mathrm{C}(c_1, \ldots, c_n, b),$$

but since $\mathcal{D}$ is directed there is some $F \in \mathcal{D}$ such that all $c_i \in F$, which implies $\mathrm{C}(F, a) = \mathrm{C}(F, b)$, that is, $\langle a, b\rangle \in \boldsymbol{\Lambda}_{\mathrm{C}}(F)$, and therefore $\langle a, b\rangle \in \bigcup\{\boldsymbol{\Lambda}_{\mathrm{C}}(F) : F \in \mathcal{D}\}$. This proves that $\boldsymbol{\Lambda}_{\mathrm{C}}(\bigcup\mathcal{D}) = \bigcup\{\boldsymbol{\Lambda}_{\mathrm{C}}(F) : F \in \mathcal{D}\}$, and thus that $\boldsymbol{\Lambda}_{\mathrm{C}}$ preserves unions of directed families of subsets of A. Conversely, take any non-empty $X \subseteq A$ and put $\mathcal{D} = \{F \subseteq X : F \text{ is finite}\}$; this is a directed family with $\bigcup\mathcal{D} = X$. If $a \in \mathrm{C}(X)$ then for any $b \in X$ it holds that $\mathrm{C}(X, a) = \mathrm{C}(X, b) = \mathrm{C}(X)$ so in particular $\langle a, b\rangle \in \boldsymbol{\Lambda}_{\mathrm{C}}(X) = \boldsymbol{\Lambda}_{\mathrm{C}}(\bigcup\mathcal{D}) = \bigcup\{\boldsymbol{\Lambda}_{\mathrm{C}}(F) : F \in \mathcal{D}\}$ by assumption. So there is a finite $F \subseteq X$ with $\langle a, b\rangle \in \boldsymbol{\Lambda}_{\mathrm{C}}(F)$, that is, $\mathrm{C}(F, a) = \mathrm{C}(F, b)$ which implies $a \in \mathrm{C}(F, b)$. Since $F \cup \{b\}$ is a finite subset of X, this proves that C is finitary. ⊣

For any closure operator C, the Frege relation $\boldsymbol{\Lambda}(\mathrm{C})$ is trivially an equivalence relation. If $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C}\rangle$ is an abstract logic, then in general $\boldsymbol{\Lambda}(\mathbb{L})$ is not a congruence of $\boldsymbol{A}$; actually $\operatorname{Con}\mathbb{L} = \big\{\theta \in \operatorname{Con}\boldsymbol{A} : \theta \subseteq \boldsymbol{\Lambda}(\mathbb{L})\big\}$, and $\widetilde{\boldsymbol{\Omega}}(\mathbb{L}) = \max\operatorname{Con}\mathbb{L}$ is precisely the greatest logical congruence of $\mathbb{L}$ included in $\boldsymbol{\Lambda}(\mathbb{L})$.

DEFINITION 2.39. *We say that an abstract logic* $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C}\rangle$ *has the* ***congruence property*** *when* $\boldsymbol{\Lambda}(\mathbb{L}) \in \operatorname{Con}\boldsymbol{A}$, *that is, when* $\boldsymbol{\Lambda}(\mathbb{L}) = \widetilde{\boldsymbol{\Omega}}(\mathbb{L})$.

Note that a reduced abstract logic $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C}\rangle$ has the congruence property if and only if for all $a, b \in A$, $\mathrm{C}(a) = \mathrm{C}(b)$ implies $a = b$; that is, when $\mathrm{C}(a) = \mathrm{C}(b)$ holds exactly when $a = b$. By this, we see that properties of the underlying algebra can be expressed as properties of the closure operator, which can thus be extended to greater classes of abstract logics (for instance, if they are preserved under bilogical morphisms).

PROPOSITION 2.40. *The congruence property is preserved by bilogical morphisms. That is, if there is a bilogical morphism between two abstract logics then one of them has the congruence property if and only if the other one has it.*

PROOF. Suppose that h is a bilogical morphism between the abstract logics $\mathbb{L}$ and $\mathbb{L}'$. By Proposition 1.7 we have $\widetilde{\boldsymbol{\Omega}}(\mathbb{L}) = h^{-1}\big[\widetilde{\boldsymbol{\Omega}}(\mathbb{L}')\big]$. Now if $\mathbb{L}'$

has the congruence property and $\langle a, b\rangle \in \boldsymbol{\Lambda}(\mathrm{C})$, it follows that $\big\langle h(a), h(b)\big\rangle \in \boldsymbol{\Lambda}(\mathrm{C}') = \widetilde{\boldsymbol{\Omega}}(\mathbb{L}')$ and so $\langle a, b\rangle \in \widetilde{\boldsymbol{\Omega}}(\mathbb{L})$; therefore $\boldsymbol{\Lambda}(C) = \widetilde{\boldsymbol{\Omega}}(\mathbb{L})$ which means that $\mathbb{L}$ has the congruence property. Conversely, suppose that $\mathbb{L}$ does have it, and that $\langle a', b'\rangle \in \boldsymbol{\Lambda}(\mathrm{C}')$. Let $a' = h(a)$ and $b' = h(b)$ for some $a, b \in A$. Then $h\big[\mathrm{C}(a)\big] = \mathrm{C}'\big(h(a)\big) = \mathrm{C}'\big(h(b)\big) = h\big[\mathrm{C}(b)\big]$, and by Proposition 1.5 $\mathrm{C}(a) = h^{-1}\big[h\big[\mathrm{C}(a)\big]\big] = h^{-1}\big[h\big[\mathrm{C}(b)\big]\big] = \mathrm{C}(b)$, that is, $\langle a, b\rangle \in \boldsymbol{\Lambda}(C) = \widetilde{\boldsymbol{\Omega}}(\mathbb{L}) = h^{-1}\big[\widetilde{\boldsymbol{\Omega}}(\mathbb{L}')\big]$ which yields $\langle a', b'\rangle \in \widetilde{\boldsymbol{\Omega}}(\mathbb{L}')$. This proves that $\mathbb{L}'$ has the congruence property. $\dashv$

These definitions apply obviously to sentential logics; the Frege relation is then just interderivability, while for any theory Γ, the relation $\boldsymbol{\Lambda}_{\mathcal{S}}(\Gamma)$ is the interderivability relation modulo the theory Γ. With regard to the behaviour of these two relations, we define two kinds of sentential logics of particular interest:

DEFINITION 2.41. *A sentential logic $\mathcal{S}$ is **selfextensional** when, considered as an abstract logic, it has the congruence property, that is, when $\boldsymbol{\Lambda}(\mathcal{S}) = \widetilde{\boldsymbol{\Omega}}(\mathcal{S})$.*

*A sentential logic $\mathcal{S}$ is **strongly selfextensional**[19] when all its full models have the congruence property, that is, when for any $\mathbb{L} \in \mathbf{FMod}\mathcal{S}$, $\widetilde{\boldsymbol{\Omega}}(\mathbb{L}) = \boldsymbol{\Lambda}(\mathbb{L})$.*

The notion of a selfextensional logic has been introduced and studied by Wójcicki (see Section 5.6 of his [1988]). A sentential logic $\mathcal{S}$ is selfextensional if and only if it satisfies the following metalogical property:

$$\text{If } \varphi_i \dashv\vdash_{\mathcal{S}} \psi_i \text{ for all } i < n, \text{ then } \varpi\varphi_0 \ldots \varphi_{n-1} \vdash_{\mathcal{S}} \varpi\psi_0 \ldots \psi_{n-1}$$

for each basic operation ϖ of the similarity type, where n is the arity of the operation. A sentential logic is strongly selfextensional when this property is inherited, in the obvious sense, by all its full models. In view of Proposition 2.40 and Corollary 2.12, we observe:

PROPOSITION 2.42. *A sentential logic $\mathcal{S}$ is strongly selfextensional iff every abstract logic of the form $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\rangle$ has the congruence property.* $\dashv$

Thus the congruence property is hard-wired inside strongly selfextensional logics, since by taking all filters on any algebra we always obtain it. It is clear that any strongly selfextensional logic is also selfextensional.

OPEN PROBLEM. *Is every selfextensional logic strongly selfextensional ?*

[19]This property is defined by requiring that all full models satisfy the congruence property, which defines selfextensionality for sentential logics. Accordingly, in later publications the more descriptive term ***fully selfextensional*** has been adopted, beginning with Definition 16 in Font [2003b].

In Chapter 4 we answer this affirmatively for two very large classes of logics, those with Conjunction and those with the Deduction Theorem, but a general answer is not known[20].

Now we give two properties showing that sentential logics in these classes have a nice behaviour.

PROPOSITION 2.43. *Let $\mathcal{S}$ be any selfextensional sentential logic. Then an equation $\varphi \approx \psi$ is valid in the variety $\mathsf{K}_{\mathcal{S}}$ if and only if $\varphi \dashv\vdash_{\mathcal{S}} \psi$.*

PROOF. By (1.6), an equation $\varphi \approx \psi$ holds in $\mathsf{K}_{\mathcal{S}}$ iff $\gamma(\varphi, \vec{q}) \dashv\vdash_{\mathcal{S}} \gamma(\psi, \vec{q})$ for any $\gamma(p, \vec{q}) \in Fm$. By taking $\gamma(p, \vec{q}) = p$ we obtain one of the implications, which holds in general. Conversely, if $\varphi \dashv\vdash_{\mathcal{S}} \psi$ and $\mathcal{S}$ is selfextensional, then the congruence property implies the replacement property, that is, that for any $\gamma(p, \vec{q}) \in Fm$, $\gamma(\varphi, \vec{q}) \dashv\vdash_{\mathcal{S}} \gamma(\psi, \vec{q})$, and this tells us that $\varphi \approx \psi$ holds in $\mathsf{K}_{\mathcal{S}}$. ⊣

The next result is the improvement of Theorem 2.36 we announced before.

THEOREM 2.44. *If $\mathcal{S}$ is a strongly selfextensional sentential logic, then the category $\mathfrak{L}_{+}^{*}$ of all reduced full models of $\mathcal{S}$ with all logical morphisms is a full reflective subcategory of the category $\mathfrak{L}_{+}$ of all full models of $\mathcal{S}$ with all logical morphisms; and the reflector is the functor associated with the process of "reduction": $\mathbb{L} \longmapsto \mathbb{L}^{*}$.*

PROOF. The proof follows the lines of the proof of Theorem 2.36 except for the proof of the central point. $\mathfrak{L}_{+}^{*}$ is trivially a full subcategory of $\mathfrak{L}_{+}$. In order to check that the process of reduction $\mathbb{L} \mapsto \langle \mathbb{L}^{*}, \pi_{\mathbb{L}} \rangle$ (where $\pi_{\mathbb{L}} : \mathbb{L} \to \mathbb{L}^{*}$ is the canonical projection) gives the announced reflector, it is enough to check (see Balbes and Dwinger [1974] I.18.2 for instance) that for any logical morphism $f : \mathbb{L} \to \mathbb{L}'$ between an $\mathbb{L} \in \mathsf{FMod}\mathcal{S}$ and an $\mathbb{L}' \in \mathsf{FMod}^{*}\mathcal{S}$ there is a unique logical morphism $f^{*} : \mathbb{L}^{*} \to \mathbb{L}'$ such that $f^{*} \circ \pi_{\mathbb{L}} = f$. Since $\pi_{\mathbb{L}}$ is a bilogical morphism, we can use Proposition 1.15 if we prove that $\ker f \supseteq \ker \pi_{\mathbb{L}} = \widetilde{\boldsymbol{\Omega}}(\mathbb{L})$. Let $a, b \in A$ with $\langle a, b \rangle \in \widetilde{\boldsymbol{\Omega}}(\mathbb{L})$; since $\mathcal{S}$ is strongly selfextensional, $\mathbb{L}$ has the congruence property, so we have $\langle a, b \rangle \in \boldsymbol{\Lambda}(\mathbb{L})$, that is, for any $T \in \mathcal{C}_{\mathbb{L}}$, $a \in T$ iff $b \in T$. Since $\mathbb{L}'$ is a reduced full model of $\mathcal{S}$, $\mathcal{C}'_{\mathbb{L}} = \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}'$, and since f is a logical morphism, this implies that for any $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}'$, $f^{-1}[F] \in \mathcal{C}_{\mathbb{L}}$, and so $f(a) \in F$ iff $f(b) \in F$, that is, $\big\langle f(a), f(b) \big\rangle \in \boldsymbol{\Lambda}(\mathbb{L}') = \widetilde{\boldsymbol{\Omega}}(\mathbb{L}') = Id_{A'}$ again because $\mathbb{L}'$ has the congruence property and is reduced. Thus $f(a) = f(b)$ which proves $\langle a, b \rangle \in \ker f$. Then by Proposition 1.15 there is a unique logical morphism $f^{*} : \mathbb{L}^{*} \to \mathbb{L}'$ with $f^{*} \circ \pi_{\mathbb{L}} = f$ as was desired. ⊣

[20] The general question was answered negatively in Babyonyshev [2003] by providing an example of a selfextensional logic that is not strongly selfextensional.

The Property of Conjunction

DEFINITION 2.45. *Let $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ be an abstract logic of some similarity type, and let $\wedge$ be a binary operation symbol, either primitive or defined by a term. We say that $\mathbb{L}$ has the **Property of Conjunction (PC)** with respect to $\wedge$ when for any $a, b \in A$,*

$$\mathrm{C}(a\,, b) = \mathrm{C}(a \wedge b)\,. \tag{PC}$$

In the literature it is also said that an abstract logic $\mathbb{L}$ is *conjunctive* or that the binary term $\wedge$ is a *Conjunction for* $\mathbb{L}$ when $\mathbb{L}$ has the PC with respect to $\wedge$. Normally we will omit the reference "with respect to $\wedge$" since the operation involved will be clear from context. The following observations are straightforward and/or well-known:

1. $\mathbb{L}$ has the PC iff for any $T \in \mathcal{C}$ and any $a, b \in A\,,\ a \wedge b \in T$ iff $a \in T$ and $b \in T$.
2. The Property of Conjunction is preserved under bilogical morphisms (see Font and Verdú [1991], Proposition 4.1). In particular, $\mathbb{L}$ has the PC iff $\mathbb{L}^*$ has the PC.
3. If $\mathbb{L}$ has the PC with respect to $\wedge$ then $\boldsymbol{\varLambda}(\mathbb{L})$ is a congruence with respect to $\wedge$, and for every $F \subseteq A\,,\ \boldsymbol{\varLambda}_{\mathbb{L}}(F)$ is also a congruence with respect to $\wedge$.
4. A sentential logic $\mathcal{S}$ has the PC iff the following rules hold for $\mathcal{S}$:
$$\varphi \wedge \psi \vdash \psi \ \ , \ \ \varphi \wedge \psi \vdash \varphi \quad \text{and} \quad \{\varphi, \psi\} \vdash \varphi \wedge \psi\,.$$
5. If a sentential logic $\mathcal{S}$ has the PC then all its models also have the PC (with respect to the same operation). In particular, all its full models have the PC.

Moreover we can prove:

PROPOSITION 2.46. *Let $\mathcal{S}$ be a sentential logic with the PC. Then every finitary model of $\mathcal{S}$ (having no theorems if $\mathcal{S}$ does not) which satisfies the congruence property is a full model of $\mathcal{S}$.*

PROOF. Suppose that $\mathbb{L}$ is a finitary model for $\mathcal{S}$, that is, $\mathcal{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, such that $\emptyset \in \mathcal{C}$ iff $\mathcal{S}$ does not have theorems, and with the congruence property. We must prove that $\mathcal{C}^* = \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^*$. If $F \in \mathcal{C}^*$ then also $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^*$ by Proposition 1.19, since $\pi^{-1}[F] \in \mathcal{C}$. Conversely, let $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^*$. If $F = \emptyset$ then $\mathcal{S}$ cannot have theorems, and by assumption $\emptyset \in \mathcal{C}$ so also $\emptyset \in \mathcal{C}^*$. If $F \neq \emptyset$ then, by finitarity of $\mathbb{L}^*$, for any $a \in \mathrm{C}^*(F)$ there are $a_1, \ldots, a_n \in F$ such that $a \in \mathrm{C}^*(a_1, \ldots, a_n)$. But $\mathbb{L}$ has the PC because it is a model of $\mathcal{S}$, so $\mathbb{L}^*$ also has it, therefore $a \in \mathrm{C}^*\big(a_1 \wedge (\ldots \wedge a_n)\big)$ and this implies $\mathrm{C}^*\big(a \wedge \big(a_1 \wedge (\ldots \wedge a_n)\big)\big) = \mathrm{C}^*\big(a_1 \wedge (\ldots \wedge a_n)\big)$. But since $\mathbb{L}$ has the congruence property, by 2.40 $\mathbb{L}^*$ also has it, and since it is reduced, we conclude that $a \wedge \big(a_1 \wedge (\ldots \wedge a_n)\big) = a_1 \wedge (\ldots \wedge a_n)$. Now F is an $\mathcal{S}$-filter and $\mathcal{S}$ has the PC; this implies that $a_1 \wedge (\ldots \wedge a_n) \in F$, therefore also $a \wedge \big(a_1 \wedge (\ldots \wedge a_n)\big) \in F$, and from this it follows that $a \in F$. This proves that $\mathrm{C}^*(F) = F$, that is, $F \in \mathcal{C}^*$. This completes the proof that $\mathcal{C}^* = \mathcal{F}i_{\mathcal{S}} \boldsymbol{A}^*$, and so $\mathbb{L}$ is a full model of $\mathcal{S}$. $\dashv$

In Section 4.2 we will prove the converse of this result for selfextensional logics: every full model of a selfextensional logic with the PC has the congruence property, and therefore such a logic is strongly selfextensional. Thus we see that the PC is a very strong property: it makes the congruence property (for *all* the connectives of the language) to be inherited from the logic by all its full models.

The Deduction-Detachment Theorem

We will consider here only the more classical version of the Deduction Theorem, that is, the one concerning only a binary connective, either primitive or defined by a single term; more general versions, including weaker "Deduction Theorems", have been dealt with in Blok and Pigozzi [1991], [1989b], Czelakowski [1985], [1986] and Czelakowski and Dziobiak [1991].

Strictly speaking, the name of *Deduction Theorem* is usually applied to just the implication

$$\Gamma, \varphi \vdash_{\mathcal{S}} \psi \implies \Gamma \vdash_{\mathcal{S}} \varphi \rightarrow \psi\,, \tag{DT}$$

while the converse one receives the name of *Modus Ponens* (MP) or *Detachment*; we will follow this distinction, since the metalogical status of both properties is very different: while the MP is equivalent to a Hilbert-style rule, and so is inherited by all models of a sentential logic, this is not the case of the DT; the latter is, however, inherited by all full models.

DEFINITION 2.47. *Let $\rightarrow$ be a binary operation symbol, either primitive or defined by a term, and let $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ be an abstract logic. We say that $\mathbb{L}$ satisfies, with respect to $\rightarrow$, the:*

(1) ***Modus Ponens (MP)*** *when for any $a, b \in A$ and any $X \subseteq A$,*

$$a \rightarrow b \in \mathrm{C}(X) \ \textit{implies}\ \ b \in \mathrm{C}(X, a)\,. \tag{MP}$$

(2) ***Deduction Theorem (DT)*** *when for any $a, b \in A$ and any $X \subseteq A$,*

$$b \in \mathrm{C}(X, a) \ \textit{implies}\ \ a \rightarrow b \in \mathrm{C}(X)\,. \tag{DT}$$

(3) ***Deduction-Detachment Theorem (DDT)*** *when it satisfies the MP and the DT.*

We will usually omit the reference "with respect to $\rightarrow$" since only one such operation will be considered. The following observations are straightforward or well-known:

1. If an abstract logic $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ satisfies the DDT then it has theorems, namely for any $a \in A$, $\mathrm{C}(a \rightarrow a) = \mathrm{C}(\emptyset)$. Some particular theorems of such abstract logics will be used, specially in the case of sentential logics; we highlight the following, for all $a, b, c \in A$:

$$a \rightarrow a$$
$$a \rightarrow (b \rightarrow a)$$
$$\big(a \rightarrow \big(b \rightarrow c\big)\big) \rightarrow \big((a \rightarrow b) \rightarrow (a \rightarrow c)\big)$$

2. As a consequence of 1, if a sentential logic $\mathcal{S}$ has the DDT then every $\mathcal{S}$-filter is non-empty, and thus every model of it, as well as every full model, has theorems.
3. An abstract logic $\mathbb{L}$ has the MP iff for every $a, b \in A$, $b \in \mathrm{C}(a, a \rightarrow b)$, and also iff for every closed set $T \in \mathcal{C}$, if $a \in T$ and $a \rightarrow b \in T$ then $b \in T$; informally we refer to this property as being *closed under the MP*.
4. A sentential logic $\mathcal{S}$ has the MP if and only if the following is a rule of $\mathcal{S}$: $\{\varphi, \varphi \rightarrow \psi\} \vdash_{\mathcal{S}} \psi$. As a consequence, each $\mathcal{S}$-filter is closed under the MP and every model of $\mathcal{S}$ (and hence every full model) also has the MP.
5. The DDT is preserved under bilogical morphisms (see the Corollary to Proposition 6 of Verdú [1987]). In particular, $\mathbb{L}$ has the DDT iff $\mathbb{L}^*$ has the DDT. Actually this holds separately for the MP and for the DT.
6. If $\mathbb{L}$ has the DDT then for any $F \subseteq A$, the Frege relation $\boldsymbol{\Lambda}_{\mathbb{L}}(F)$ is a congruence with respect to $\rightarrow$.

Thus the MP is inherited by all models of a sentential logic satisfying it. Next we see that the DT (and hence the DDT) is inherited by full models, a fact that is essentially contained in Theorem 2.2 of Czelakowski [1985].

THEOREM 2.48. *If $\mathcal{S}$ has the DDT then every full model of $\mathcal{S}$ has the DDT.*

PROOF. Assume that $\mathcal{S}$ has the DDT, that is, the MP and the DT. By Corollary 2.12, it will be enough to prove that every abstract logic of the form $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \rangle$ has the DDT. As we have already noticed, every $\mathcal{S}$-filter is closed under the MP, thus $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \rangle$ has the MP. Now we have to prove that for all $X \subseteq A$ and all $a, b \in A$, if $b \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X, a)$ then $a \rightarrow b \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X)$. We use the characterization of $\mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X, a)$ given in Lemma 1.18: $\mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X, a) = \bigcup\{X_n : n \in \omega\}$, where the X_n are defined as in 1.18, starting with $X_0 = X \cup \{a\}$. Then we prove by induction on n that if $b \in X_n$ then $a \rightarrow b \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X)$: Assume that $n = 0$ and $b \in X_0 =$

$X \cup \{a\}$; if $b \in X$ then since $b \to (a \to b) \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X)$ also $a \to b \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X)$, and if $b = a$ then $a \to b = a \to a \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X)$. Now assume $n \geqslant 1$ and the thesis true for n, and let $b \in X_{n+1}$: there are a finite $\Gamma \subseteq Fm$ and $\varphi \in Fm$ such that $\Gamma \vdash_{\mathcal{S}} \varphi$ and there is $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ such that $h[\Gamma] \subseteq X_n$ and $h(\varphi) = b$. If $\Gamma = \emptyset$ then trivially $b \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X)$. Now assume $\Gamma = \{\psi_1, \dots, \psi_k\}$. Let $q \in Var$ be any variable not appearing in $\psi_1, \dots \psi_k, \varphi$; using the DDT and its consequences for $\vdash_{\mathcal{S}}$ we obtain $\{q \to \psi_i : i = 1, \dots, k\} \vdash_{\mathcal{S}} q \to \varphi$. Define $h' \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ such that $h'(p) = h(p)$ if $p \neq q$ while $h'(q) = a$. By the inductive hypothesis $h'(q) \to h'(\psi_i) = a \to h(\psi_i) \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X)$ for $i = 1, \dots, k$, therefore $a \to b = a \to h(\varphi) = h'(q) \to h'(\varphi) \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X)$. This finishes the inductive proof. Therefore $a \to b \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X)$. ⊣

In Section 4.3 we will use the DT to find a characterization of full models among the class of all (finitary) models; in the meantime we can prove a partial result we will need there:

PROPOSITION 2.49. *Let $\mathcal{S}$ be a sentential logic with the DDT. If $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ is a finitary model of $\mathcal{S}$ with the DT and with the congruence property, then $\mathbb{L}$ is a full model of $\mathcal{S}$.*

PROOF. Suppose that $\mathbb{L}$ is a finitary model of $\mathcal{S}$ with the DT and the congruence property. We have that $\mathcal{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, and we must prove that $\mathcal{C}^* = \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^*$. If $F \in \mathcal{C}^*$ then also $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^*$ by Proposition 1.19, since $\pi^{-1}[F] \in \mathcal{C}$. Conversely, let $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^*$. Since $\mathcal{S}$ has the DDT, $F \neq \emptyset$, so by finitarity of $\mathbb{L}^*$, for any $a \in \mathrm{C}^*(F)$ there are $a_1, \dots, a_n \in F$ such that $a \in \mathrm{C}^*(a_1, \dots, a_n)$. But $\mathbb{L}$ has the DDT by assumption, so $\mathbb{L}^*$ also has it, therefore $a_1 \to \big(\dots (a_n \to a) \dots\big) \in \mathrm{C}^*(\emptyset) = \mathrm{C}^*(a \to a)$ and this implies $\mathrm{C}^*\big(a_1 \to \big(\dots (a_n \to a) \dots\big)\big) = \mathrm{C}^*(a \to a)$. But since $\mathbb{L}$ has the congruence property, by 2.40 $\mathbb{L}^*$ also has it, and since it is reduced, we conclude that $a_1 \to \big(\dots (a_n \to a) \dots\big) = a \to a \in F$. Since F is an $\mathcal{S}$-filter and $\mathcal{S}$ has the MP, this implies that $a \in F$. This proves that $\mathrm{C}^*(F) = F$, that is, $F \in \mathcal{C}^*$. This completes the proof that $\mathcal{C}^* = \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^*$, that is, $\mathbb{L}$ is a full model of $\mathcal{S}$. ⊣

As we will prove in Corollary 4.30, if $\mathcal{S}$ is selfextensional then the converse of this property also holds.

The Property of Disjunction

This property, which should not be confused with the so-called "Disjunction Property" of some intermediate logics (stating that if $\vdash_{\mathcal{S}} \varphi \vee \psi$ then $\vdash_{\mathcal{S}} \varphi$ or $\vdash_{\mathcal{S}} \psi$), corresponds to the method of *Proof by Cases* of traditional logic; in the literature it is also said that a logic $\mathbb{L}$ is *disjunctive* when it satisfies this property, see Czelakowski [1984]:

DEFINITION 2.50. *An abstract logic $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ satisfies the* ***Property of Disjunction (PDI)*** *with respect to a binary operation symbol $\vee$, either primitive or defined by a term, when, for any $X \subseteq A$, $a, b \in A$,*

$$\mathrm{C}(X, a \vee b) = \mathrm{C}(X, a) \cap \mathrm{C}(X, b). \tag{PDI}$$

Some easy or well-known consequences are:

1. A sentential logic $\mathcal{S}$ satisfies the PDI iff the following rules hold: The two Hilbert-style rules: $\varphi \vdash_{\mathcal{S}} \varphi \vee \psi$, $\varphi \vdash_{\mathcal{S}} \psi \vee \varphi$ and the Gentzen-style rule:
$$\frac{\Gamma, \psi_1 \vdash \varphi \qquad \Gamma, \psi_2 \vdash \varphi}{\Gamma, \psi_1 \vee \psi_2 \vdash \varphi}.$$
2. If a sentential logic $\mathcal{S}$ satisfies the PDI then the following Hilbert-style rules also hold: $\varphi \vee \psi \dashv\vdash_{\mathcal{S}} \psi \vee \varphi$ and $\varphi \dashv\vdash_{\mathcal{S}} \varphi \vee \varphi$.
3. The PDI is preserved under bilogical morphisms. In particular, $\mathbb{L}$ satisfies the PDI iff $\mathbb{L}^*$ satisfies it. See Font and Verdú [1991], Proposition 4.1.
4. If $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ satisfies the PDI then an easy inductive argument shows that for any $a_1, \ldots, a_n, b \in A$ and any $X \subseteq A$,
$$\mathrm{C}(X, a_1 \vee b, \ldots, a_n \vee b) = \mathrm{C}(X, a_1, \ldots, a_n) \cap \mathrm{C}(X, b).$$

LEMMA 2.51. *Let $\mathcal{S}$ be a sentential logic satisfying the PDI and assume that $\psi_1, \ldots, \psi_n \vdash_{\mathcal{S}} \varphi$. Then for any ξ , $\psi_1 \vee \xi, \ldots, \psi_n \vee \xi \vdash_{\mathcal{S}} \varphi \vee \xi$.*

PROOF. From the generalization of the PDI mentioned in item 4 above we can obtain, as a particular case, that for any $\psi_1, \ldots, \psi_n, \xi \in Fm$, $\mathrm{Cn}_{\mathcal{S}}(\psi_1 \vee \xi, \ldots, \psi_n \vee \xi) = \mathrm{Cn}_{\mathcal{S}}(\psi_1, \ldots, \psi_n) \cap \mathrm{Cn}_{\mathcal{S}}(\xi)$. Now, $\varphi \in \mathrm{Cn}_{\mathcal{S}}(\psi_1, \ldots, \psi_n)$ by assumption, and obviously $\xi \in \mathrm{Cn}_{\mathcal{S}}(\xi)$. But the PDI implies that $\varphi \vee \xi \in \mathrm{Cn}_{\mathcal{S}}(\varphi) \cap \mathrm{Cn}_{\mathcal{S}}(\xi)$. Therefore we obtain $\varphi \vee \xi \in \mathrm{Cn}_{\mathcal{S}}(\psi_1 \vee \xi, \ldots, \psi_n \vee \xi)$ as desired. $\dashv$

Next we see that the PDI is inherited by full models; the essential part of the proof is also mentioned in Czelakowski [1984].

THEOREM 2.52. *If $\mathcal{S}$ is a sentential logic with the PDI then every full model of $\mathcal{S}$ satisfies the PDI as well.*

PROOF. By Corollary 2.12 it will be enough to prove that, for any $\boldsymbol{A}$ and any $X \cup \{a, b\} \subseteq A$, $\mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X, a \vee b) = \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X, a) \cap \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X, b)$. From the Hilbert-style rules mentioned in item 2 above it follows that $\mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X, a \vee b) \subseteq \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X, a) \cap$

$\mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X, b)$. In order to establish the reverse inclusion we first prove that for any $a, b, c \in A$,

$$c \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X, a) \text{ implies } c \vee b \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X, a \vee b). \qquad (*)$$

For this consider the characterization $\mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X, a) = \bigcup\{X_n : n \in \omega\}$ of Lemma 1.18. Let us prove by induction on n that if $c \in X_n$ then $c \vee b \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X, a \vee b)$. Since $X_0 = X \cup \{a\}$, the case $n = 0$ is trivial. Assuming the property is true for n, let $c \in X_{n+1}$: this means that there are $\varphi, \psi_1, \dots \psi_k \in Fm$ such that $\psi_1, \dots \psi_k \vdash_{\mathcal{S}} \varphi$ and there is $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ with $h(\psi_i) \in X_n$ and $h(\varphi) = c$. Now choose some variable q not appearing in these formulas, and modify h at q in order to obtain $h' \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ such that $h'(q) = b$ and $h'(\psi_i) = h(\psi_i)$. By the induction hypothesis $h'(\psi_i \vee q) = h'(\psi_i) \vee b \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X, a \vee b)$, and since by Lemma 2.51 $\psi_1 \vee q, \dots \psi_k \vee q \vdash_{\mathcal{S}} \varphi \vee q$, it follows that $c \vee b = h'(\varphi \vee q) \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X, a \vee b)$. Thus $(*)$ is proved and using it we can now prove the remaining part of the PDI: Take any $c \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X, a) \cap \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X, b)$. From $c \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X, a)$ it follows $c \vee b \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X, a \vee b)$, and from $c \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X, b)$ it follows $c \vee c \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X, b \vee c)$. Since $c \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(c \vee c)$ and $b \vee c \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(c \vee b)$ we conclude that $c \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(X, a \vee b)$, as had to be proved. ⊣

The fact that not every model of $\mathcal{S}$ inherits the PDI is shown in Section 5.1.1 by a simple example. The Property of Disjunction can be generalized by using a finite set of terms instead of a single term.

The two forms of Reductio ad Absurdum

Now we consider the forms of *Reductio ad Absurdum* that hold in Intuitionistic Logic and in Classical Logic:

DEFINITION 2.53. *Let $\neg$ be a unary operation symbol, either primitive or defined by a term. An abstract logic $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ satisfies the **Property of Intuitionistic Reductio ad Absurdum (PIRA)** with respect to $\neg$ when for any $X \subseteq A$ and any $a \in A$,*

$$\neg a \in \mathrm{C}(X) \iff \mathrm{C}(X, a) = A;$$

*and it satisfies the **Property of Reductio ad Absurdum (PRA)** with respect to $\neg$ when for any $X \subseteq A$ and any $a \in A$,*

$$a \in \mathrm{C}(X) \iff \mathrm{C}(X, \neg a) = A.$$

It is easy to see that an abstract logic satisfies the PRA if and only if it satisfies both the PIRA and that $a \in \mathrm{C}(\neg\neg a)$. Speaking of sentential logics, this last property is a Hilbert-style rule, which is inherited by all models; hence the problem of inheritance of the PRA by full models reduces to that of the PIRA.

The PIRA is not inherited in general by all full models of a sentential logic having it: Take as an example the $\neg$-fragment $\mathcal{S}$ of intuitionistic logic: In Porębska and Wroński [1975] it is proved that this fragment is characterized precisely by the PIRA (it is the weakest sentential logic having it, when the language has just negation), and it does not have theorems. Now every one-element set $A = \{a\}$ provides us with a counterexample, since we must have $\neg a = a$: Clearly $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A} = \{\emptyset, A\}$ and the abstract logic $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\rangle$, which is a full model of $\mathcal{S}$, does not satisfy the PIRA: $\mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(a) = A$ but $\neg a \notin \mathrm{Fi}^{\boldsymbol{A}}_{\mathcal{S}}(\emptyset) = \emptyset$.

The difficulties revealed by the analysis of the general case of this problem tell us that negation is a difficult connective to deal with alone. But one of the main results of Section 4.2 will enable us to prove that if $\mathcal{S}$ is a selfextensional logic with the PC and the PIRA then every full model of $\mathcal{S}$ has the PC and the PIRA; and in the case where conjunction and negation are the only connectives of the language we will be able to remove the assumption of selfextensionality; see Propositions 4.34 and 4.35. At this moment we can treat the case of the DDT and the PIRA together. Actually, in the presence of the DDT, the PIRA is equivalent to a very simple requirement.

If $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C}\rangle$ is an abstract logic, we say that an element $\bot \in A$ is an ***inconsistent element*** when $\mathrm{C}(\bot) = A$; authors in the field of *paraconsistent logics* sometimes prefer to call such elements *trivial*. Then:

LEMMA 2.54. *Let $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C}\rangle$ be an abstract logic with the DDT with respect to a binary operation symbol $\rightarrow$. Then $\mathbb{L}$ satisfies the PIRA with respect to some unary operation symbol $\neg$ if and only if $\mathbb{L}$ has an inconsistent element $\bot$. Moreover, in this situation, $\mathrm{C}(\neg a) = \mathrm{C}(a \rightarrow \bot)$ for any $a \in A$.*

PROOF. It is trivial to check (using the DDT) that, if $\mathbb{L}$ satisfies the PIRA with respect to $\neg$ then for any $a \in A$ the element $\neg(a \rightarrow a)$ is inconsistent, and that if $\bot$ is an inconsistent element, then $\mathbb{L}$ satisfies the PIRA with respect to the operation $\neg a = a \rightarrow \bot$. In general, if $\bot$ is inconsistent, from the MP it follows that $\mathrm{C}(a, a \rightarrow \bot) = A$, and therefore by the PIRA $\neg a \in \mathrm{C}(a \rightarrow \bot)$; since $\neg a \in \mathrm{C}(\neg a)$, we have that $\bot \in \mathrm{C}(a, \neg a)$, and by the DDT this implies $a \rightarrow \bot \in \mathrm{C}(\neg a)$; therefore we have shown that $\mathrm{C}(\neg a) = \mathrm{C}(a \rightarrow \bot)$. $\dashv$

Since having an inconsistent element is a property clearly inherited by any model, it follows from Theorem 2.48 and the previous lemma:

COROLLARY 2.55. *If a sentential logic satisfies the DDT and the PIRA then all its full models satisfy them.* ⊣

Some rules of introduction of modality

One of the strongest metalogical properties of *normal modal logics* is the so-called ***Rule of Necessitation***. In its *strong form* it is:

$$\varphi \vdash \Box\varphi,$$

and like all Hilbert-style rules, it is inherited by every model; so it is not especially interesting to consider it here. The same rule has a *weak form*, which has also been considered in the literature:

$$\text{If } \vdash \varphi \text{ then } \vdash \Box\varphi.$$

However, this Gentzen-style rule is but a particular case of the rule more commonly taken in many Gentzen-style formulations of systems of modal logic as a rule for introduction of the necessity operator, see Zeman [1973],

$$(\mathrm{I}\Box) \quad \frac{\Gamma \vdash \varphi}{\Box\Gamma \vdash \Box\varphi},$$

where $\Box\Gamma = \{\Box\gamma : \gamma \in \Gamma\}$. Actually, the same rule holds for the possibility operator $\Diamond$ in the place of $\Box$, and also for a number of other unary operators of modal character (temporal, dynamic, etc.), and even for double negation $\neg\neg$, which in some logics has been shown to have a modal behaviour, see Došen [1986]. Accordingly, let $\#$ be an arbitrary unary operation symbol, either primitive or defined by a term; we say that an abstract logic $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C}\rangle$ is ***closed under introduction of*** $\#$ when for any $X \subseteq A$, $\#\mathrm{C}(X) \subseteq \mathrm{C}(\#X)$, that is, when $a \in \mathrm{C}(X)$ implies $\#a \in \mathrm{C}(\#X)$. Then:

PROPOSITION 2.56. *If $\mathcal{S}$ is a sentential logic closed under introduction of a unary connective $\#$ then all its full models are also closed under introduction of the same connective.*

PROOF. In Jansana [1995] it has been proved that the property of being closed under introduction of a unary connective is preserved under bilogical morphisms. Therefore, as usual, it will be enough to prove, for any $\boldsymbol{A}$, any $X \subseteq A$ and any $a \in A$, that if $a \in \mathrm{Fi}_{\mathcal{S}}^{\boldsymbol{A}}(X)$ then $\#a \in \mathrm{Fi}_{\mathcal{S}}^{\boldsymbol{A}}(\#X)$. Put $\mathrm{Fi}_{\mathcal{S}}^{\boldsymbol{A}}(X) = \bigcup\{X_n : n \in \omega\}$ as in Proposition 1.18, and prove by induction that if $a \in X_n$ then $\#a \in \mathrm{Fi}_{\mathcal{S}}^{\boldsymbol{A}}(\#X)$. Since $X_0 = X$, the case $n = 0$ is trivial. If $a \in X_{n+1}$ then for some formulas $\psi_1, \dots, \psi_k \vdash_{\mathcal{S}} \varphi$ and there is an homomorphism h such that $h(\psi_i) \in X_n$ and $h(\varphi) = a$. By induction $h(\#\psi_i) = \#h(\psi_i) \in \mathrm{Fi}_{\mathcal{S}}^{\boldsymbol{A}}(\#X)$, and by introduction of $\#$ for $\mathcal{S}$ we have $\#\psi_1, \dots, \#\psi_k \vdash_{\mathcal{S}} \#\varphi$, therefore $\#a = \#h(\varphi) = h(\#\varphi) \in$

$\mathrm{Fi}_{\mathcal{S}}^{\boldsymbol{A}}(X)$. Therefore this holds for every n, and thus $\#\mathrm{Fi}_{\mathcal{S}}^{\boldsymbol{A}}(X) \subseteq \mathrm{Fi}_{\mathcal{S}}^{\boldsymbol{A}}(\#X)$, that is, the abstract logic $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}^{\boldsymbol{A}} \rangle$ is closed under introduction of $\#$. $\dashv$