Accessible Segments of the Fast Growing Hierarchy *

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Abstract. We examine two ways of "bootstrapping" segments of the fast growing hierarchy by autonomous generation. One method closes off at ε_0 with the provably recursive functionals of arithmetic, whereas the other exhausts the provably recursive functions of $\Pi_1^1 - CA_0$.

1 Introduction.

Autonomously generated hierarchies are constructed according to the principle: *proceed to level α if α is already coded or recognized at some earlier level.* The question addressed here is : how might this principle be applied in a subrecursive (rate of growth) context, and what is the effect? Clearly it will depend upon the way in which we choose to subrecursively code or recognize countable ordinals. Typically this could be done by means of some "natural" scale or hierarchy $\alpha \mapsto f_\alpha$ such that the growth-rate of function f_α reflects its rank α. However this is a delicate matter since (in contrast with generalized recursion theory, where ordinal comparison is a fundamental property) one cannot expect to compare and compute ordinals by sub-recursive functions without reference to some prior given notation system. Either one simply accepts this situation and does the best one can (and we will, later, in section 4); or alternatively one could shift the goalposts and try to reconsider the problem in a more amenable, generalized setting. One suitable place to look is the theory of type two recursive functionals where, as noted by Kleene (1958) and in stark contrast with the recursive functions, a good notation-free hierarchy already exists - by classifying total recursive functionals according to the ordinal heights of their trees of unsecured sequences (see also Wainer (1995)). This opens the possibility of coding and comparing recursive ordinals in terms of the majorization relationship between certain descent-recursive functionals which represent them.

* This paper presents newer material not surveyed in the author's conference lecture (much of which can already be found in ref.10), but expanding the same overall theme.

2 Ordinal Presentations and Scales.

By a *presentation* of a countable ordinal $\alpha > 0$ is meant a nested sequence $\alpha[0] \subset \alpha[1] \subset \alpha[2] \subset \ldots$ of finite subsets such that $\alpha = \cup_n \alpha[n]$. In addition we will require that (i) $0 \in \alpha[0]$, (ii) if $\beta + 1 \in \alpha[n]$ then $\beta \in \alpha[n]$, and also (iii) if $\beta \in \alpha[n]$ and $\beta + 1 < \alpha$ then $\beta + 1 \in \alpha[n+1]$.

Obviously a presentation of α induces a sub-presentation of each $\beta < \alpha$ by $\beta[n] = \alpha[n] \cap \beta$. It also induces for each n, the *n-predecessor* function $P_n(\beta) = \max \beta[n]$. Thus for each non-zero $\beta \leq \alpha$,

$$\beta[n] = \{0, \ldots, P_n^3(\beta), P_n^2(\beta), P_n(\beta)\} .$$

This minimal apparatus is all that's needed to construct "majorization hierarchies", or "scales" of increasing functions $\{f_\beta : \beta \leq \alpha\}$. For an alternative general approach, see Buchholz, Cichon and Weiermann (1994).

Definition 1. *Given a fixed presentation for ordinal α, we shall call any sequence of increasing functions $\{f_\beta : \beta \leq \alpha\}$ an α-scale if it satisfies the majorization condition:*

$$\gamma \in \beta[n] \Rightarrow f_\gamma(n) < f_\beta(n).$$

Note.
There is a crucial dependence on the chosen ordinal presentation : for given any strictly increasing function g we can define a presentation for ω by $\omega[n] = \{0, 1, 2, \ldots, g(n)-1\}$, so that any ω-scale must satisfy $f_\omega(n) \geq g(n)$. Thus "natural" hierarchies must depend upon "natural" choices of presentations.

Theorem 2. *Let $\Phi : (N \to N) \to (N \to N)$ be any operator taking increasing functions to increasing functions, which is monotone in the sense that for all increasing h_0, h_1 and all $k \in N$,*

$$\forall m \geq k\ (h_0(m) < h_1(m)) \Rightarrow \forall m \geq k\ (\Phi(h_0)(m) < \Phi(h_1)(m)).$$

Let f_0 be any increasing function such that $f_0(n) < \Phi(f_0)(n)$ for every n. Let α be equipped with a fixed presentation as above and for $0 < \beta \leq \alpha$ define

$$f_\beta(n) = \Phi(f_{P_n(\beta)})(n).$$

Then $\{f_\beta : \beta \leq \alpha\}$ is an α-scale.

Proof. By induction on β.

To see that each f_β is increasing note that for $m > n$,

$$P_n(\beta) \in \beta[n+1] = P_{n+1}(\beta)[n+1] \cup \{P_{n+1}(\beta)\} \subset P_{n+1}(\beta)[m] \cup \{P_{n+1}(\beta)\}$$

so by the induction hypothesis,$\forall m > n(f_{P_n(\beta)}(m) \leq f_{P_{n+1}(\beta)}(m))$. Therefore by the monotonicity of Φ we have for each n,

$$f_\beta(n) = \Phi(f_{P_n(\beta)})(n) \leq \Phi(f_{P_n(\beta)})(n+1) \leq \Phi(f_{P_{n+1}(\beta)})(n+1) = f_\beta(n+1).$$

Secondly, if $\gamma \in \beta[n]$ then $\gamma \in P_n(\beta)[n] \cup \{P_n(\beta)\}$, so by the induction hypothesis $f_\gamma(n) \leq f_{P_n(\beta)}(n)$. If $P_n(\beta) \neq 0$ then $P_n^2(\beta) \in P_n(\beta)[n]$, so again by the induction hypothesis and the monotonicity of Φ,

$$f_\gamma(n) \leq f_{P_n(\beta)}(n) = \Phi(f_{P_n^2(\beta)})(n) < \Phi(f_{P_n(\beta)})(n) = f_\beta(n).$$

If $P_n(\beta) = 0$ then $\gamma = 0$ and $f_0(n) < \Phi(f_0)(n) = f_\beta(n)$ by the assumption concerning f_0.

Example : G Scales.
Choose $\Phi_1(f)(n) = f(n) + 1$ in 2.2, to obtain the "Slow Growing" scale of functions $\{G_\beta : \beta \leq \alpha\}$ where

$$G_0(n) = 0 \ , \qquad G_\beta(n) = G_{P_n(\beta)}(n) + 1.$$

This is the *minimum* scale with respect to a chosen presentation, since $G_\beta(n) = \text{card } \beta[n]$.

Example : H Scales.
Choose $\Phi_2(f)(n) = f(n+1)$ in 2.2, to obtain the "Hardy" scale $\{H_\beta : \beta \leq \alpha\}$ where

$$H_0(n) = n \ , \qquad H_\beta(n) = H_{P_n(\beta)}(n+1).$$

Example : F Scales.
Choose $\Phi_3(f)(n) = f^{n+1}(n)$ in 2.2, to obtain the "Fast Growing" scale of functions $\{F_\beta : \beta \leq \alpha\}$ where

$$F_0(n) = n + 1, \qquad F_\beta(n) = F^{n+1}_{P_n(\beta)}(n).$$

Recall that f^k denotes the k-times iterate of f.
Remark. In addition to providing well known classifications of the provably recursive functions of arithmetical theories, the F-hierarchy has a more general recursion-theoretic significance in that it is the subrecursive analogue of the classical jump hierarchy. For an obvious formulation of a subrecursive (bounded) jump operator is $f \mapsto sj(f)$ where

$$\begin{aligned} sj(f)(e,x,y) &= \{e\}^f(x) && \text{if defined in } y \text{ steps} \\ &= 0 && \text{otherwise} \end{aligned}$$

and if f has at least exponential growth then $sj(f)$ has the same elementary-recursive degree as $f \oplus \Phi_3(f)$. Thus up to elementary degree, $F_{2+\beta}$ is the β-th iterate of the subrecursive jump. See Heaton and Wainer (1996) and, for a detailed exposition, Heaton (1997).

The Bootstrapping Problem.
The problem addressed here is to make some sense of the following principle of autonomous inductive generation along the F-hierarchy:

$$\begin{cases} \tau_0 &= \ 0 \\ \tau_{n+1} &= \ \text{the ordinal "recognized" by } F_{\tau_n}. \end{cases}$$

Thus (i) what could "recognition" mean ? (ii) is τ_{n+1} uniquely defined ? and (iii) what is the closure ordinal $\tau = \sup \tau_n$? We give two solutions.

3 Recognition by Descent Functionals.

The predecessor functions P_n associated with a given presentation of an ordinal α allow a natural reformulation of Friedman's descent functional:

$$f \mapsto \text{ least } k : P_{f(k-1)}P_{f(k-2)} \cdots P_{f(1)}P_{f(0)}(\alpha) = 0$$

which "witnesses" the well-foundedness of α (see Friedman and Sheard (1995)). In our present context it will be convenient to adopt the more restrictive formulation where f is replaced by its successive iterates thus:

$$D_\alpha(f)(n) = \text{ least } k : P_{f^{k-1}(n)}P_{f^{k-2}(n)} \cdots P_{f(n)}P_n(\alpha) = 0.$$

Not only this, but we shall even go a stage further and use, instead of D_α, the functional

$$H_\alpha(f)(n) = f^k(n) \text{ where } k = D_\alpha(f)(n).$$

Note that this is just a functional version of the H-scale defined in section 2 since it satisfies the same recursive definition:

$$H_0(f)(n) = n\ , \qquad H_\beta(f)(n) = H_{P_n(\beta)}(f)(f(n))$$

but with the successor function replaced by an arbitrary (strictly increasing) function f. Similarly, a functional version of the F-scale is

$$F_0(f)(n) = f(n)\ , \qquad F_\beta(f)(n) = F_{P_n(\beta)}(f)^{n+1}(n).$$

The reason for using H rather than D is that it connects neatly with the fast-growing F.

Definition 3. *Given presentations for α and β, define a presentation for $\alpha + \beta$ by:*

$$\alpha + \beta[n] \ = \ \alpha[n] \cup \{\alpha + \gamma : \gamma \in \beta[n]\}$$

and a presentation for ω^β as follows:

$$\omega^\beta[n] \ = \ \{\omega^{\gamma_1}.m_1 + \omega^{\gamma_2}.m_2 + \cdots + \omega^{\gamma_{k-1}}.m_{k-1} + \omega^{\gamma_k}.m_k : m_i \le n\} \cup \{0\}$$

where $\beta[n] = \{\gamma_k, \gamma_{k-1}, \ldots, \gamma_2, \gamma_1\}_<$, i.e. $\gamma_i = P_n^i(\beta)$ for each $i = 1, \ldots, k$ and k is the least such that $\gamma_k = 0$.

Lemma 4. *Given fixed presentations for α and β, and with the induced presentations for $\alpha + \beta$ and ω^β defined as above, we have for all f,*

(i) $H_{\alpha+\beta}(f) = H_\alpha(f) \circ H_\beta(f)$

(ii) $H_{\omega^\beta}(f) = F_\beta(f)$.

Proof. By inductions on β.

Firstly, if $\beta = 0$ we have for all $n \in N$,

$$H_{\alpha+0}(f)(n) = H_\alpha(f)(n) = H_\alpha(f)(H_0(f)(n)),$$

and if $\beta > 0$ then since $P_n(\alpha+\beta) = \alpha + P_n(\beta)$, we have

$$H_{\alpha+\beta}(f)(n) = H_{\alpha+P_n(\beta)}(f)(f(n))$$

and so by the induction hypothesis,

$$H_{\alpha+\beta}(f)(n) = H_\alpha(f)(H_{P_n(\beta)}(f)(f(n))) = H_\alpha(f)(H_\beta(f)(n)).$$

Secondly, if $\beta = 0$ then $P_n(\omega^\beta) = 0$ so

$$H_{\omega^0}(f)(n) = H_0(f)(f(n)) = f(n) = F_0(f)(n).$$

If $\beta > 0$ let $\beta[n] = \{\gamma_k, \gamma_{k-1}, \ldots, \gamma_2, \gamma_1\}_<$ where $\gamma_i = P_n^i(\beta)$ for each $i = 1, \ldots, k$ and k is minimal such that $\gamma_k = 0$. Then by definition of $\omega^\beta[n]$,

$$P_n(\omega^\beta) = \omega^{\gamma_1}.n + \omega^{\gamma_2}.n + \ldots + \omega^{\gamma_{k-1}}.n + \omega^{\gamma_k}.n$$

so by the induction hypothesis and repeated applications of part (i) we have

$$H_{\omega^\gamma.n}(f) = F_\gamma(f)^n$$

and hence

$$\begin{aligned}
H_{\omega^\beta}(f)(n) &= F_{\gamma_1}(f)^n \circ F_{\gamma_2}(f)^n \circ \cdots \circ F_{\gamma_{k-1}}(f)^n \circ F_{\gamma_k}(f)^n \circ f(n) \\
&= F_{\gamma_1}(f)^n \circ F_{\gamma_2}(f)^n \circ \cdots \circ F_{\gamma_{k-1}}(f)^n \circ F_{\gamma_k}(f)^{n+1}(n) \\
&= F_{\gamma_1}(f)^n \circ F_{\gamma_2}(f)^n \circ \cdots \circ F_{\gamma_{k-1}}(f)^n \circ F_{\gamma_{k-1}}(f)(n) \\
&= F_{\gamma_1}(f)^n \circ F_{\gamma_2}(f)^n \circ \cdots \circ F_{\gamma_{k-1}}(f)^{n+1}(n) \\
&= F_{\gamma_1}(f)^n \circ F_{\gamma_2}(f)^n \circ \cdots \circ F_{\gamma_{k-2}}(f)^{n+1}(n) \\
&= \ldots\ldots\ldots\ldots \\
&= F_{\gamma_1}(f)^n \circ F_{\gamma_2}(f)^{n+1}(n) \\
&= F_{\gamma_1}(f)^n \circ F_{\gamma_1}(f)(n) \\
&= F_{\gamma_1}(f)^{n+1}(n) \text{ where } \gamma_1 = P_n(\beta) \\
&= F_\beta(f)(n).
\end{aligned}$$

The next result shows how the ordering between (set-theoretic) ordinals is reflected by the majorization relation between descent functionals.

Lemma 5. *Let $\{\alpha[n] : n \in N\}$ and $\{\alpha'[[n]] : n \in N\}$ be any two chosen presentations of ordinals α and α' where $\alpha \le \alpha'$. Let g be any strictly increasing function such that for every $n, \alpha[n] \subset \alpha'[[g(n)]]$. Then*

$$H_\alpha(g) \le H'_{\alpha'}(g) \circ g$$

where $H_\alpha(g), H'_{\alpha'}(g)$ denote the H-scales determined by the respective presentations for α and α'.

Proof. The existence of such functions g with the stated properties is immediate, provided $\alpha \leq \alpha'$, though of course g may not be recursive (it depends on how "effective" the given presentations of α and α' are, and they needn't even be recursive ordinals).

We first show, by induction on γ, that for all n,

$$\gamma \in \alpha[n] \Rightarrow H_\gamma(g)(n) \leq H'_\gamma(g)(g(n)).$$

This holds trivially if $\gamma = 0$, and otherwise we have by the induction hypothesis, if $\gamma \in \alpha[n]$,

$$H_\gamma(g)(n) = H_{P_n(\gamma)}(g)(g(n)) \leq H'_{P_n(\gamma)}(g)(g^2(n)).$$

But $P_n(\gamma) \in \gamma[n] \subset \gamma[[g(n)]]$ so letting β denote the maximum element of $\gamma[[g(n)]]$ we have $P_n(\gamma) \leq \beta$ and hence

$$H_\gamma(g)(n) \leq H'_{P_n(\gamma)}(g)(g^2(n)) \leq H'_\beta(g)(g^2(n)) = H'_\gamma(g)(g(n)).$$

Now applying this with $\gamma = P_n(\alpha) \in \alpha'[[g(n)]]$ we obtain

$$H_\alpha(g)(n) = H_{P_n(\alpha)}(g)(g(n)) \leq H'_{P_n(\alpha)}(g)(g^2(n)) \leq H'_{\alpha'}(g)(g(n))$$

as required.

Theorem 6. *Let H_α and $H'_{\alpha'}$ be the descent functionals determined by arbitrary but fixed presentations of their respective ordinals α and α'. Then if α' is a limit we have*

$$\alpha < \alpha' \Leftrightarrow \exists g \ \exists k \ \forall f \ (g <_k f \rightarrow H_\alpha(f) \circ f <_k H'_{\alpha'}(f))$$

where g and f range over (strictly) increasing functions on N, $k \in N$, and $g <_k f$ means that $g(n) < f(n)$ for all $n \geq k$.

Proof. Suppose $\alpha < \alpha'$ where α' is a limit. Choose k and g so that, with the notation of Lemma 3.3, $\alpha + 2 \in \alpha'[[k]]$ and for every $n, \alpha[n] \subset \alpha'[[g(n)]]$. Then if $g <_k f$ we also have $\alpha[n] \subset \alpha'[[f(n)]]$ for $n \geq k$, so by the Lemma we obtain, for each $n \geq k$,

$$H_\alpha(f)(f(n)) \leq H'_\alpha(f)(f^2(n)) = H'_{\alpha+2}(f)(n) < H'_{\alpha'}(f)(n)$$

the final inequality coming about because of 2.2.

Conversely if $\alpha' \leq \alpha$ then, given any g and k we can certainly find a strictly increasing f such that $g <_k f$ and $\alpha'[[n]] \subset \alpha[f(n)]$ for every n. But then by Lemma 3.3 with α, α' reversed, we have

$$H'_{\alpha'}(f) \leq H_\alpha(f) \circ f.$$

This completes the proof.

Bootstrapping the Fast Growing Functionals.

Define a sequence of ordinal presentations τ_n as follows by induction on n,

$$\begin{cases} \tau_0 &= 0 \\ \tau_{n+1} &= \text{that ordinal } \beta \text{such that for some presentation of it, } F_{\tau_n} = H_\beta . \end{cases}$$

By 3.2, an immediate candidate for β is ω^{τ_n} with the "standard" presentation described earlier. Its uniqueness is ensured by 3.3. Thus $\tau_{n+1} = \omega^{\tau_n}$.

Theorem 7. $\tau_{n+1} = \omega^{\omega^{\cdot^{\cdot^{\omega}}}}$ *with exponential stack of height n, and hence* $\sup \tau_n = \varepsilon_0$.

The proof theory of Peano Arithmetic shows that each F_{τ_n} is a provably recursive functional of PA in the sense that $F_{\tau_n}(f)$ is a provably recursive function of $PA + \forall x(f(x) < f(x+1))$; and conversely every provably recursive functional of PA is dominated by some F_{τ_n}.

4 Recognition by Minimum Scales.

In this section we choose as our notion of "ordinal recognition" the following: a fast-growing function F recognizes α if $F = G_\alpha$, the topmost element of the minimum (slow growing) α-scale $\{G_\beta : \beta \leq \alpha\}$.

As already noted, these function-hierarchies depend crucially upon the chosen presentation of α. In contrast with the results of section 3 which concern type-2 *functionals*, we cannot expect the majorization relation between *functions* to preserve the ordering between set-theoretic ordinals without restricting our attention to a fixed initial segment α with a chosen presentation, for only then do we have

$$\beta_1 < \beta_2 \leq \alpha \;\Leftrightarrow\; \exists k \; (G_{\beta_1} <_k G_{\beta_2}).$$

This "intensionality" is the major stumbling block in subrecursion theory. Why should one ordinal presentation be chosen in preference to another? - presumably because it is in some sense more "natural", but although we seem able to agree about the naturalness of some presentations when we see them, no clear definition of the concept is at hand.

All we do know, immediately from the foregoing, is that given arbitrary ordinal presentations $\{\alpha[n] : n \in N\}$ and $\{\alpha'[[n]] : n \in N\}$ where $\alpha \leq \alpha'$, there must be a function $g : N \to N$ such that for every $n, \alpha[n] \subset \alpha'[[g(n)]]$ and hence

$$\forall \beta \leq \alpha \; \exists k \; (G_\beta \leq_k G'_\beta \circ g \leq_k G'_{\alpha'} \circ g)$$

where G, G' are the slow growing hierarchies determined by the respective presentations. Now experience does seem to show that "natural" presentations possess a certain subrecursive stability in the sense that the function g above will usually turn out to be elementary or primitive recursive, so their respective hierarchies will be subrecursively comparable.

With this in mind we proceed to construct a particular ordinal presentation which will turn out to be just large enough for our purposes. The construction is based on generalizations of the fast growing hierarchy, operating on the higher "tree-ordinal" classes $\Omega_0, \Omega_1, \Omega_2, \ldots$ given by the following iterated inductive definition.

Definition 8. *$\Omega_0 = N$ and for $k > 0$, Ω_k is generated by the inductive clauses*

(i) $0 \in \Omega_k$,

(ii) *if* $\alpha \in \Omega_k$ *then* $\alpha + 1 := \alpha \cup \{\alpha\} \in \Omega_k$,

(iii) *if* $\alpha : \Omega_i \to \Omega_k$ *for some* $i < k$ *then* $\alpha \in \Omega_k$.

Notation.
Henceforth, lower case greek letters denote tree ordinals, not set-theoretic ordinals, although of course each tree ordinal α has a set-theoretic ordinal height, denoted $|\alpha|$. For each countable ordinal there are many different tree ordinals $\alpha \in \Omega_1$ with that same height. We shall usually reserve λ to denote "limit" tree ordinals arising by clause (iii) above, and call i its "type". Furthermore, we sometimes write, more suggestively, $\lambda = \sup_{\Omega_i} \lambda_\zeta$ where λ_ζ denotes the value of the function λ at argument $\zeta \in \Omega_i$. The sub-tree partial ordering $\prec$ is defined as the transitive closure of (i) $\alpha \prec \alpha + 1$ and (ii) $\lambda_\zeta \prec \lambda$ for every $\zeta \in \Omega_i$.

Definition 9. *For $\alpha \in \Omega_1$ and $n \in N$ define $\alpha[n]$ as follows:*

$$0[n] = \emptyset, \qquad \alpha + 1[n] = \alpha[n] \cup \{\alpha\}, \qquad \lambda[n] = \lambda_n[n].$$

Then call α structured if it satisfies the following conditions

(i) $\forall n \; (\alpha[n] \subset \alpha[n+1])$

(ii) $\forall \beta \prec \alpha \; \exists n \; (\beta \in \alpha[n])$

(iii) $\forall \beta \, \forall n \; (\beta \in \alpha[n] \wedge \beta + 1 \prec \alpha \to \beta + 1 \in \alpha[n+1])$.

Thus if α is structured, $\{\beta : \beta \prec \alpha\}$ is well-ordered by $\prec$ and a presentation of $|\alpha|$ is induced by $|\alpha|[n] = \{|\beta| : \beta \in \alpha[n]\}$.

There are various ways to ensure the structuredness of $\alpha \in \Omega_1$. One sufficient condition (see e.g. Wainer (1990), Kadota (1993)) is

$$\forall \lambda \preceq \alpha \; \forall n \; (\lambda_n \in \lambda[n+1]).$$

The question then is how to construct large structured tree ordinals in Ω_1. This leads us into the realm of "collapsing functions" as used in proof theoretical ordinal analysis, but there is a fairly obvious procedure to follow in our present context. Since the fast growing F-hierarchy uses countable ordinal presentations as indices to name large numbers, a straight generalization of it should enable us to name large tree ordinals in Ω_k, using indices from Ω_{k+1}. The named elements of Ω_k may then be used to index large elements of Ω_{k-1}, then large elements of Ω_{k-2} etcetera until we finish up naming large elements of Ω_1.

Definition 10. *For each $k \in N$ define the function*

$$\varphi^{(k)} : \Omega_{k+1} \times \Omega_k \to \Omega_k$$

by recursion over Ω_{k+1} as follows:

$$\begin{array}{lll} \varphi^{(k)}(0,\beta) & = & \beta + 1 \\ \varphi^{(k)}(\alpha+1,\beta) = & & \varphi^{(k)}(\alpha,-)^{1+\beta}(\beta) \\ \varphi^{(k)}(\lambda,\beta) & = & \sup_{\Omega_i} \varphi^{(k)}(\lambda_\zeta,\beta) \quad \textit{if type } (\lambda) = i < k \\ \varphi^{(k)}(\lambda,\beta) & = & \varphi^{(k)}(\lambda_\beta,\beta) \quad \textit{if type } (\lambda) = k. \end{array}$$

Note.
At level $k = 0$ we have, for all structured $\alpha \in \Omega_1$ and all $n \in N$,

$$\varphi^{(0)}(\alpha, n) = F_\alpha(n)$$

where F_α is defined with respect to the presentation induced by α.

Definition 11. *For each k let $\omega_k \in \Omega_{k+1}$ be the identity on Ω_k. Define $\tau = \sup \tau_k \in \Omega_1$ where $\tau_0 = 0$ and for each $k > 0$,*

$$\tau_k = \varphi^{(1)}(\varphi^{(2)}(\ldots \varphi^{(k-1)}(\varphi^{(k)}(0,\omega_{k-1}),\omega_{k-2})\ldots,\omega_1),\omega_0).$$

By Kadota (1993), τ is structured.
By Wainer (1989), $G_{\tau_{k+1}} = F_{\tau_k}$.
This establishes our second bootstrapping principle.

Theorem 12. *For each $k \in N$,*

$$\tau_{k+1} = \textit{ that } \beta \prec \tau \textit{ such that } F_{\tau_k} = G_\beta.$$

With some careful technical modification one can carry through the ordinal analysis of the theories ID_k by Buchholz (1987) using, instead of his collapsing functions $D : \Omega_{k+1} \to \Omega_1$, the functions

$$\alpha \mapsto \varphi^{(1)}(\varphi^{(2)}(\ldots \varphi^{(k)}(\alpha,\omega_{k-1})\ldots,\omega_1),\omega_0).$$

His analysis computes the proof theoretic ordinal of ID_k as $D(\varepsilon_{\omega_k+1})$ which, in terms of φ, is equivalent to substituting $\alpha := \varphi^{(k+1)}(\varphi^{(k+2)}(0,\omega_{k+1}),\omega_k)$ in the above expression. This gives τ_{k+2} precisely. Thus $|\tau|$ is the proof-theoretic ordinal of $ID_{<\omega}$ and, again by Buchholz's analysis, the functions elementary-recursive in the sequence $\{F_\beta : \beta \prec \tau\}$ are exactly those provably recursive in the theory $\Pi^1_1 - CA_0$.

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