EXISTENCE OF BOUNDED INVARIANT MEASURES IN ERGODIC THEORY

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1. Introduction

We present a survey of some of the recent work done on the problem of existence of bounded invariant measure for positive contractions defined on L^1-spaces.

2. Preliminaries

1. *Positive linear forms on L^∞-spaces.* Let $(E, \mathfrak{F}, \mu)$ be a fixed measure space (with μ σ-finite). Sets in $\mathfrak{F}$ and real measurable functions defined on $(E, \mathfrak{F})$ will always be considered up to μ-equivalence; hence, all equalities or inequalities between measurable sets or functions are to be taken in the almost sure sense with respect to μ.

We will denote by f, g (with or without subscripts) elements of the Banach space $L^1(E, \mathfrak{F}, \mu)$ and by h elements of the Banach space $L^\infty = L^\infty(E, \mathfrak{F}, \mu)$. The space L^∞ is the strong dual of L^1 for the bilinear form: $\langle f, h\rangle = \int_E fh\, d\mu$. Consideration of the strong dual of L^∞, in which L^1 is isometrically imbedded, has often been helpful in analysis. We here recall the following lemma from the theory of vectorial lattices, of which we sketch a proof out of completeness.

LEMMA 1. *Let λ be a positive linear form defined on L^∞; that is, let $\lambda \in (L^\infty)'_+$. Then there exists a largest element g in L^1_+ such that the form induced by it on L^∞ verifies $g \leq \lambda$. Moreover, the complement $G = \{g = 0\}$ of the support of g is the largest set in $\mathfrak{F}$ (up to equivalence) for which there exists an $h \in L^\infty_+$ such that $h > 0$ on G and $\lambda(h) = 0$; in particular, the following equivalences hold:*

(a) $g > 0$ *a.s.* $\Rightarrow \lambda(h) > 0$ *for every* $h \in L^\infty_+$, $h \neq 0$.

(b) $g = 0$ *a.s.* $\Rightarrow \lambda(h) = 0$ *for at least one* $h \in L^\infty$ *such that* $h > 0$ *a.s.*

PROOF. The class $\Lambda = \{f\colon f \in L^1_+, f \leq \lambda \text{ on } L^\infty_+\}$ is easily seen to be closed under least upper bounds and increasing limits; hence, $g = \sup \Lambda$ belongs to Λ, and is thus the largest element of Λ.

Given two linear forms ν_1, ν_2 on L^∞, it is known and easily checked that the formula $\nu(h) = \inf \{[\nu_1(u) + \nu_2(h - u)]; 0 \leq u \leq h\}$ where $h \in L^\infty_+$, defines on L^∞_+ a linear form ν on L^∞, which is the g.l.b. of ν_1 and ν_2. Now it follows from the maximality of g that 0 is the g.l.b. of $\lambda - g$ and f_0, where f_0 is an arbitrarily fixed strictly positive element of L^1 (which is considered here as a linear form on L^∞); hence, by what precedes, one has

$$\inf_{u:0\leq u\leq h} (\lambda(u) - \langle g, u\rangle + \langle f_0, h - u\rangle) = 0 \tag{1}$$

for every h in L^∞_+.

For $h = 1_G$ where $G = \{g = 0\}$, the term $\langle g, u\rangle$ always vanishes in the last formula; we have thus shown the existence of functions u_m $(m \geq 1)$ with the following properties:

$$0 \leq u_m \leq 1_G, \qquad \lambda(u_m) + \langle f_0, 1_G - u_m\rangle \leq 2^{-m}. \tag{2}$$

Then the $v_n = \inf_{m>n} u_m$ $(n \geq 1)$ verify

$$0 \leq v_n \leq 1_G, \qquad \lambda(v_n) = 0, \langle f_0, 1_G - v_n\rangle \leq \sum_{m>n} 2^{-m} = 2^{-n} \tag{3}$$

as follows from $v_n \leq u_m$ $(m > n)$ and $1_G - v_n \leq \sum_{m>n} (1_G - u_m)$. Finally, the function $h = \sum_{n\geq 1} 2^{-n}v_n$ belongs to L^∞_+ and verifies $\lambda(h) = 0$ since

$$\lambda(h) = \sum_{n\leq p} 2^{-n}\lambda(v_n) + \lambda\left(\sum_{n>p} 2^{-n}v_n\right) \leq 2^{-p}\lambda(1_G) \to 0 \qquad \text{as} \quad p \to \infty \tag{4}$$

because $\lambda(v_n) = 0$ and $\sum_{n>p} 2^{-n}v_n \leq 2^{-p}1_G$. Moreover, one has $h > 0$ on G, because by definition $\{h = 0\} = \bigcap_n \{v_n = 0\}$, and because

$$\int_{\{v_n=0\}G} f_0\, d\mu \leq \int f_0(1_G - v_n)\, d\mu \leq 2^{-n} \to 0 \qquad \text{as} \quad n \to \infty. \tag{5}$$

We have proved the existence of h in L^∞_+ such that $\lambda(h) = 0$ and $h > 0$ on G. Conversely, if $h \in L^\infty_+$ verifies $\lambda(h) = 0$, it follows from $0 \leq \int gh \leq \lambda(h)$ that $\{h > 0\} \subset G$, and this concludes the proof of the lemma.

2. *Conservative operators on L^1-spaces.* Let T be a positive linear operator defined on L^1; we suppose that T has norm ≤ 1 (that is, a contraction) or, what is equivalent, that its dual operator T^* defined on L^∞ verifies $T^*1 \leq 1$.

If $P = \{P(x, F); x \in E, F \in \mathfrak{F}\}$ is a transition function defined on $(E, \mathfrak{F})$, the formula

$$\int_F Tf\, d\mu = \int_E fP(\cdot, F)\, d\mu, \qquad (f \in L^1, F \in \mathfrak{F}) \tag{6}$$

defines (with the aid of the Radon-Nikodym theorem) a positive linear operator T of norm 1 on L^1, provided only that the measure $\int \mu(dx)P(x, \cdot)$ is absolutely continuous with respect to μ. For the Markovian random sequence $\{X_n, n \geq 0\}$ of initial μ-density f, $(f \geq 0, \int f\, d\mu = 1)$, and transition probability P, sums of the form $\sum_{n\in M} T^n f$ where M is a subset of the set $N = \{0, 1, 2, \cdots\}$ of positive integers, can be interpreted as densities: indeed, $\int_F \sum_M T^n f$ is the expected number of times n such that $n \in M$ and $X_n \in F$. This well-known fact gives probabilistic meaning to some of the conditions of the sequel.

The operator T is said to be *conservative* if one of the following equivalent conditions is satisfied:

(a) $\sum_{n\geq 0} T^n f_0 = \infty$, a.s., where f_0 is an arbitrarily fixed element of L^1 such that $f_0 > 0$, a.s.;

(b) for any $h \in L^\infty_+$, the condition $\sum_{n\geq 0} T^{*n}h < \infty$ a.s. implies that $h = 0$;

(b′) for any $F \in \mathfrak{F}$, the condition $\sum_{n\geq 0} T^{*n}{}_{\wedge F} < \infty$ a.s. implies that $F = \phi$ a.s.

(Once it has been deduced from Hopf's maximal ergodic lemma that (a) does not depend on f_0, the equivalence of these conditions is easily proven by an argument similar to that of section 6 of the proof of theorem 1 below.)

The operator T is said to be *dissipative* if one of the following equivalent conditions is satisfied:

(a) $\sum_{n\geq 0} T^n f_0 < \infty$ a.s., with f_0 as above;

(b) $\sum_{n\geq 0} T^{*n}h \in L^\infty$ holds for at least one $h \in L^\infty_+$ such that $h > 0$ a.s.

The preceding conditions are to be compared with those of theorems 1 and 2 below.

3. *Banach limits.* A Banach limit L is by definition a positive linear form defined on $\ell^\infty(N)$, which is normalized and invariant under translation, that is, which verifies $L(\{1\}) = 1$ and $L(\{x_{n+1}, n \in N\}) = L(\{x_n, n \in N\})$. Here $\ell^\infty(N)$ denotes as usual the Banach space of bounded sequences $\{x_n, n \in N\}$ of real numbers provided with the norm $\|\{x_n\}\| = \sup_N |x_n|$. The following classical lemma proves the existence of Banach limits as a corollary and gives the value of $\sup_L L(\{x_n\})$ as found by L. Sucheston [12] by another method.

LEMMA 2. *If Λ is a subvectorial space of $\ell^\infty(N)$ containing $\{1\}$, any linear form L defined on Λ and positive (in the sense that it takes nonnegative values on $\Delta \cap \ell^\infty_+(N)$), can be extended to a linear positive form on $\ell^\infty(N)$. Moreover, for any fixed $\{x_n\} \in \ell^\infty(N)$, one has*

$$(7)\qquad \sup_{\tilde{L}} \tilde{L}(\{x_n\}) = \inf\,[L(\{y_n\});\qquad \{y_n\} \in \Lambda \quad\text{and}\quad y_n \geq x_n\ (n \in N)]$$

where $\tilde{L}$ ranges in the first member over all positive linear extensions of L to $\ell^\infty(N)$.

PROOF. The set of all linear positive forms defined on subvectorial spaces of $\ell^\infty(N)$ and extending L is provided with an order by: $L' \subset L''$, if L'' is defined and equal to L' on the domain of definition of L'; this order is clearly inductive. Let us show that any element maximal for this order is necessarily defined on the whole space $\ell^\infty(N)$.

If L' is a positive linear form defined on a vectorial subspace Λ' of $\ell^\infty(N)$ which contains $\{1\}$, and if for a given sequence $\{x_n\} \in \ell^\infty(N)$, $\{y'_n\}$ (resp. $\{y''_n\}$) is a sequence in Λ' such that $y'_n \geq x_n$ $(n \in N)$ (resp. $x_n \geq y''_n$ $(n \in N)$), then $L'(\{y'_n\}) \geq L'(\{y''_n\})$ because $\{y'_n - y''_n\} \in \Lambda' \cap \ell^\infty_+(N)$. Hence, it is possible to choose a real number c such that

$$(8)\qquad \inf L'(\{y'_n\}) \geq c \geq \sup L'(\{y''_n\}),$$

where $\{y'_n\}$ (resp. $\{y''_n\}$) ranges among the sequences of Λ' such that $y'_n \geq x_n$ for all n (resp. $y''_n \leq x_n$ for all n). The formula

$$(9)\qquad L''(\{y_n + ax_n\}) = L'(\{y_n\}) + ac, \qquad (\{y_n\} \in \Lambda', a \in R)$$

then defines a positive linear extension of L' to the subspace generated by Λ'

and $\{x_n\}$. And since $\{x_n\}$ can be arbitrarily chosen in $\ell^\infty(N)$, Λ' can only be maximal if it is defined on the whole space $\ell^\infty(N)$.

This proves the first part of the lemma, and the second part is easily derived from the preceding argument.

COROLLARY. *Banach limits exist, and moreover, for every* $\{x_n\} \in \ell^\infty(N)$; *the following limit exists*

$$\lim_{p\to\infty} \sup_{n\geq 0} \frac{1}{p} \sum_{m=0}^{p-1} x_{m+n} \tag{10}$$

and is equal to $\sup_L L(\{x_n\})$ *where* L *ranges over all Banach limits.*

PROOF. Let Λ be the subvectorial space of $\ell^\infty(N)$ generated by $\{1\}$ and by $\{y_{n+1} - y_n, n \in N\}$, where $\{y_n\}$ ranges over $\ell^\infty(N)$. Define L on Λ by $L(\{c + y_{n+1} - y_n\}) = c$. Since for every $c \in R$ and every $\{y_n\} \in \ell^\infty(N)$, the inequality $c + y_{n+1} - y_n \geq 0$ $(n \in N)$ implies that $c \geq 0$ because of

$$0 \leq \frac{1}{n} \sum_{m=0}^{n-1} (c + y_{m+1} - y_m) = c + \frac{1}{n}(y_n - y_0) \to c \qquad \text{as} \quad n \to \infty, \tag{11}$$

the preceding definition of L is unambiguous (if $c + y_{n+1} - y_n = 0$ $(n \in N)$, then $c = 0$), and L is a positive linear form defined on Λ.

The lemma proves the existence of Banach limits because these are exactly the positive linear extensions of L to $\ell^\infty(N)$. It also shows that

$$\sup_L L(\{x_n\}) = \inf\,[c\colon c + y_{n+1} - y_n \geq x_n \ (n \in N)] \tag{12}$$

where c ranges over R and $\{y_n\}$ over $\ell^\infty(N)$. Let I be the infimum of the 2d member; it can be evaluated as follows.

First it follows from $x_n \leq c + y_{n+1} - y_n$ by letting $x_n^{(p)} = (1/p) \sum_{m=0}^{p-1} x_{m+n}$ that

$$x_n^{(p)} \leq c + \frac{1}{p}(y_{n+p} - y_n) \leq c + \frac{2}{p} \|\{y_n\}\|; \tag{13}$$

hence that, using the definition of I,

$$\lim_{p\to\infty} \sup \sup_n x_n^{(p)} \leq I. \tag{14}$$

On the other hand, since $x_n - x_n^{(p)}$ is of the form $\{y_{n+1} - y_n\}$ for a $\{y_n\}$ in $\ell^\infty(N)$, it follows from

$$x_n \leq \sup_\ell x_\ell^{(p)} + (x_n - x_n^{(p)}) \tag{15}$$

that the inequality $I \leq \sup_n x_n^{(p)}$ holds for every $p \geq 1$. Hence, $I = \lim_p \sup_n x_n^{(p)}$.

3. Existence of invariant measures

The main part of the following theorem was proved in [2] by Hajian and Kakutani in the particular case where the operator T is induced by a measurable and nonsingular transformation of the space $(E, \mathfrak{F}, \mu)$. It was then extended in [7] and [11], whereas its proof was at the same time simplified by the introduction of Banach limits ([12]; see also [1]).

THEOREM 1. *For any positive linear contraction T of a space $L^1(E, \mathfrak{F}, \mu)$, the following conditions are equivalent:*

(a) *there exists $g \in L^1$ such that $Tg = g$ and $g > 0$, a.s.;*

(b_n) *for any $h \in L^\infty_+$, the equality* $\liminf_{n\to\infty} \langle T^n f_0, h\rangle = 0$ *implies that $h = 0$ (here and in the following, f_0 denotes an arbitrary but fixed element of L^1 such that $f_0 > 0$, a.s.);*

(b_s) *for any $F \in \mathfrak{F}$, the equality* $\lim_{p\to\infty} \sup_n 1/p \sum_{m=0}^{n-1} \langle T^{m+n} f_0, 1_F\rangle = 0$ *implies that $F = \phi$;*

(c_n) *for any $h \in L^\infty_+$, the a.s. convergence $\sum_i T^{*n_i}h < \infty$ for an infinite sequence $0 \leq n_0 < n_1 < \cdots$ of integers implies that $h = 0$;*

(c_s) *for any $F \in \mathfrak{F}$, the a.s. inequality $\sum_i T^{*n_i}1_F \leq 1 + \epsilon$ for an infinite sequence $0 = n_0 < n_1 < \cdots$ of integers starting with $n_0 = 0$ implies that $F = \phi$ (here ϵ denotes an arbitrarily fixed strictly positive real number);*

(d) *$\sum_i T^{n_i} f_0 = \infty$ holds a.s. for every infinite sequence $0 \leq n_0 \leq n_1 < \cdots$ of integers.*

The preceding conditions imply that T is conservative. If T is conservative, then these conditions are still equivalent to the following:

(e) *for every $h \in L^\infty$ such that $h > 0$, a.s., one has $\sum_i T^{*n_i}h = \infty$, a.s. for every infinite sequence $0 \leq n_0 < n_1 < \cdots$ of integers;*

(e′) *for every sequence $\{F_k, k \geq 1\}$ of measurable subsets of E such that $E = \bigcup_k F_k$, one has $\bigcup_k \{\sum_i T^{*n_i}1_{F_k} = \infty\} = E$ for every infinite sequence $0 \leq n_0 < n_1 < \cdots$ of integers;*

(f) *for any $f \in L^1_+$, the a.s. convergence $\sum_i T^{n_i}f < \infty$ for an infinite sequence $0 \leq n_0 < n_1 < \cdots$ of integers implies that $f = 0$.*

REMARK. In case T is induced by a measurable non-singular transformation θ of $(E, \mathfrak{F}, \mu)$, that is, when $T^*h = h_0\theta$ $(h \in L^\infty)$, the condition (c_s) may be restated as follows (if ϵ is chosen < 1): there exists no set $F \in \mathfrak{F}$, nonnegligible, such that the $\theta^{-n_i}(F)$ are mutually disjoint for an infinite sequence $0 = n_0 < n_1 < n_2 < \cdots$ of integers (namely, there exists no weakly wandering set in the sense of [2]).

PROOF OF THEOREM 1. The proof is long and will be divided in eight parts; however, after the remark of alinea 1, only the reasoning of alinea 2 and 4 are not "immediate."

1. The following remark makes the implication $a \Rightarrow (b_n)$ obvious and will be also used in the sequel. For any fixed $h \in L^\infty_+$, the condition $\liminf \langle T^n f_0, h\rangle = 0$ where f_0 is a fixed strictly positive element of L^1, implies that

$$\liminf_{n\to\infty} \langle T^n f, h\rangle = 0 \tag{16}$$

for every $f \in L^1_+$.

Indeed, the general inequality $f \leq af_0 + (f - af_0)^+$ implies that

$$\langle T^n f, h\rangle \leq a\langle T^n f_0, h\rangle + \|(f - af_0)^+\|_1 \|h\|_\infty, \qquad (a \in R) \tag{17}$$

because T^n is a contraction. Letting $n \to \infty$, one gets the desired result because $(f - af_0)^+ \downarrow 0$, a.s. and in L^1, as $a \to \infty$, since f_0 is strictly positive.

From this fact follows that the validity of $\liminf \langle T^n f_0, h\rangle = 0$ for a fixed $h \in L^\infty_+$ is independent of the strictly positive f_0 chosen in L^1. Hence, condition $(\mathrm{b_n})$ does not depend on the chosen f_0 and is implied by condition (a), as is readily seen by taking $f_0 = g$.

2. If L denotes a Banach limit (see preliminaries), the formula

$$\lambda(h) = L(\{\langle T^n f_0, h\rangle, n \in N\}), \qquad (h \in L^\infty) \tag{18}$$

defines a positive linear form on L^∞ such that $\lambda(T^*h) = \lambda(h)$ for every $h \in L^\infty$. This invariance indeed follows from the invariance of L under translation and the fact that $\langle T^n f_0, T^*h\rangle = \langle T^{n+1}f_0, h\rangle$. The largest element g in L^1_+ bounded above by λ (see lemma 1 of preliminaries) is then invariant under T. Indeed, on one hand,

$$\langle Tg, h\rangle = \langle g, T^*h\rangle \leq \lambda(T^*h) = \lambda(h) \tag{19}$$

holds for every $h \in L^\infty_+$ by the definitions and shows that $Tg \leq g$; on the other hand, it follows from

$$\lambda(T^*1) = \lambda(1), \qquad (\lambda - g)(T^*1) \leq (\lambda - g)(1) \tag{20}$$

(the inequality holds because $\lambda - g \geq 0$ and $T^*1 \leq 1$), that

$$\langle Tg, 1\rangle = \langle g, T^*1\rangle \geq \langle g, 1\rangle. \tag{21}$$

Hence $Tg = g$.

Suppose that $(\mathrm{b_n})$ holds; then $\lambda(h) \geq \liminf_{n\to\infty} \langle T^n f_0, h\rangle > 0$ holds for every $h \in L^\infty_+$, $h \neq 0$. By lemma 1, it follows that $g > 0$ a.s. and the implication $(\mathrm{b_n}) \Rightarrow (\mathrm{a})$ is so proved.

3. The use of Banach limits, as in the preceding alinea, also gives an easy proof of the implication $(\mathrm{b_s}) \Rightarrow (\mathrm{c_s})$.

If $F \in \mathfrak{F}$ verifies

$$\textstyle\sum_i T^{*n_i}1_F \in L^\infty \tag{22}$$

for an infinite sequence $0 \leq n_0 < n_1 < \cdots$ of integers, then for any form λ obtained from a Banach limit L, as in alinea 2, one has for every integer $j \geq 1$,

$$\lambda(\textstyle\sum T^{*n_i}1_F) \geq \left(\sum\limits_{i<j} T^{*n_i}1_F\right) = j\lambda(1_F), \tag{23}$$

and since the first member is finite and independent of j, $\lambda(1_F) = 0$. On the other hand, one has by the preliminaries (section 3),

$$\sup_\lambda \lambda(1_F) = \sup_L L(\{\langle T^n f_0, 1_F\rangle\}) = \lim_{p\to\infty} \sup_n \frac{1}{p}\sum_{m=0}^{p-1} \langle T^{m+n}f_0, 1_F\rangle. \tag{24}$$

Thus if F verifies the hypothesis of the beginning, this last member is 0, and if $(\mathrm{b_s})$ holds, F must then be a.s. equal to ϕ; that is, condition $(\mathrm{c_s})$ is implied by $(\mathrm{b_s})$.

4. Since the implication $(\mathrm{b_n}) \Rightarrow (\mathrm{b_s})$ is clear, the proof of the implication $(\mathrm{c_s}) \Rightarrow (\mathrm{b_n})$ will establish the equivalence of $(\mathrm{b_n})$, $(\mathrm{b_s})$, and $(\mathrm{c_s})$. This proof rests on the following generalization of a lemma of [2] given in [11].

LEMMA 3. *If for an $h \in L^\infty$ such that $0 \leq h \leq 1$, one has*

$$(25) \qquad \liminf_{n\to\infty} \langle T^n f_0, h\rangle = 0,$$

then there exists for each $\delta > 0$ an element $h_\delta \in L^\infty_+$ such that $h_\delta \leq h$, $\langle f_0, h - h_\delta\rangle \leq \delta$ and $\sum_i T^{*n_i} h_\delta \leq 1$ for a suitably chosen infinite sequence $0 = n_0 < n_1 < \cdots$ of integers (starting at $n_0 = 0$). Hence for every $F \in \mathfrak{F}$ such that

$$(26) \qquad \liminf_{n\to\infty} \langle T^n f_0, 1_F\rangle = 0,$$

there exists for every $\epsilon, \epsilon' > 0$ a subset $F_{\epsilon,\epsilon'}$ of F such that $\langle f_0, 1_F - 1_{F_{\epsilon,\epsilon'}}\rangle \leq \epsilon'$ and $\sum_i T^{*n_i} 1_{F_{\epsilon,\epsilon'}} \leq 1 + \epsilon$ for a suitably chosen infinite sequence $0 = n_0 < n_1 < \cdots$ of integers.

PROOF OF LEMMA. Given an infinite sequence $0 = n_0 < n_1 < \cdots$ of integers we let

$$(27) \qquad h' = \Big(h - \sum_{0\leq i\leq j}\sum (T^*)^{n_{j+1}-n_i} h\Big)^+.$$

Obviously $0 \leq h' \leq h$ and $h' \in L^\infty$.

The sequence $\{n_i\}$ can be chosen so that $\langle f_0, h - h'\rangle \leq \delta$ for a given $\delta > 0$. Indeed, it follows from

$$(28) \qquad h - h' \leq \sum_{j\geq 0}\sum_{i=0}^{j} (T^*)^{n_{j+1}-n_i} h = \sum_{j\geq 0} (T^*)^{n_{j+1}-n_j} \sum_{i=0}^{j} (T^*)^{n_j - n_i} h$$

that

$$(29) \qquad \langle f_0, h - h'\rangle \leq \sum_{j\geq 0} \langle T^{n_{j+1}-n_j} f^{(j)}, h\rangle$$

where we have let

$$(30) \qquad f^{(j)} = \sum_{i=0}^{j} T^{n_j - n_i} f_0$$

when $j \geq 0$. Hence, the hypothesis $\liminf_{n\to\infty} \langle T^n f_0, h\rangle = 0$ made on h, where one may substitute f_0 by $f^{(j)}$ by the remark of alinea 1, makes it possible to choose the n_{j+1} by recurrence on j from $n_0 = 0$, so that

$$(31) \qquad \langle T^{n_{j+1}-n_j} f^{(j)}, h\rangle \leq \delta 2^{-(j+1)},$$

because $f^{(j)}$ only depends on $n_0, \cdots, n_j$.

The following inequality holds for every integer $i \geq 0$ and every integer $k \geq 0$, as will be proved by recurrence on k,

$$(32) \qquad \sum_{j=i}^{i+k} (T^*)^{n_j - n_i} h' \leq 1.$$

Taking $i = 0$ and letting $k \to \infty$, we obtain that

$$(33) \qquad \textstyle\sum_j (T^*)^{n_j} h' \leq 1;$$

namely, that h' has the properties stated for h_δ in the lemma. The above inequality is true for $k = 0$ since $h' \leq h \leq 1$ and $(T^*)^n 1 \leq 1$ for every n. Assuming that the inequality is true for every $i \geq 0$ and for the value $k - 1$ of the recurrence parameter, we deduce from

$$(34) \qquad \sum_{j=i}^{i+k} (T^*)^{n_j - n_i} h' = h' + (T^*)^{n_{i+1} - n_i} \left(\sum_{j=i+1}^{(i+1)+k-1} (T^*)^{n_j - n_{i+1}} h' \right) \leq h' + (T^*)^{n_{i+1} - n_i} 1$$

that on the set $\{h' = 0\}$, the first member is bounded above by 1. On the other hand, we have that on $\{h' > 0\}$,

$$(35) \qquad h' = h - \sum_{0 \leq i \leq j} \sum (T^*)^{n_{j+1} - n_i} h,$$

and thus that

(36)

$$\sum_{j=i}^{i+k} (T^*)^{n_j - n_i} h' = h' + \sum_{j=i}^{i+k-1} (T^*)^{n_{j+1} - n_i} h' \leq h' + \sum_{j=i}^{i+k-1} (T^*)^{n_{j+1} - n_i} h \leq h \leq 1.$$

The recurrence is established.

Letting $h = 1_F$ in the preceding result and $\delta = \epsilon\epsilon'/1 + \epsilon$,

$$(37) \qquad F_{\epsilon,\epsilon'} = \{h_\delta > 1/(1 + \epsilon)\}$$

one obtains from

$$(38) \qquad 1_{F_{\epsilon,\epsilon'}} \leq (1 + \epsilon) h_\delta \quad \text{that} \quad \sum {}_i T^{*n_i} 1_{F_{\epsilon,\epsilon'}} \leq 1 + \epsilon$$

and from

$$(39) \qquad 1_F - 1_{F_{\epsilon,\epsilon'}} \leq 1 + \epsilon/\epsilon(h - h_\delta) \quad \text{that} \quad \langle f_0, 1_F - 1_{F_{\epsilon,\epsilon'}} \rangle \leq \frac{1 + \epsilon}{\epsilon} \delta = \epsilon'.$$

This concludes the proof of the lemma.

It is easy to deduce the implication $(c_s) \Rightarrow (b_n)$ from the preceding lemma. Indeed, if $h \in L_+^\infty$ verifies $\liminf \langle T^n f_0, h \rangle = 0$, then 1_F verifies a similar relation if $F = \{h > a\}$ and a is a strictly positive real number. The sets $F_{\epsilon,\epsilon'}$ constructed from F as above are negligible if (c_s) is valid; hence, $\langle f_0, 1_F \rangle \leq \epsilon$ for every $\epsilon > 0$, and F is itself negligible. Finally, h is 0, since a was arbitrary.

5. To conclude the proof of the first part of the theorem, we show that $(b_n) \Rightarrow (d) \Rightarrow (c_n) \Rightarrow (b_n)$.

If $0 \leq n_0 < n_1 < \cdots$ is an infinite sequence of integers, we let

$$(40) \qquad h = \xi(1 + \sum T^{n_i} f_0)^{-1}$$

where ξ is a fixed strictly positive element of $L^1 \cap L^\infty$ and with the convention that $(+\infty)^{-1} = 0$. Then $0 \leq h \leq \xi$ so that $h \in L_+^\infty$ and $h(\sum_i T^{n_i} f_0) \leq \xi$, a.s. (with the convention $0.\infty = 0$) so that $\sum_i \langle T^{n_i} f_0, h \rangle < \infty$; hence,

$$(41) \qquad \liminf_{n \to \infty} \langle T^n f_0, h \rangle = 0,$$

and if (b_n) is satisfied, h must be 0; that is, $\sum T^{n_i} f_0 = +\infty$, a.s. This shows that $(b_n) \Rightarrow (d)$.

If $h \in L^\infty_+$ verifies $\sum_i T^{*n_i}h < \infty$, a.s. for an infinite sequence

(42) $$0 \leq n_0 < n_1 < \cdots$$

of integers, let $f = \xi(1 + \sum T^{*n_i}h)^{-1}$. Then $f > 0$, a.s. and $f \leq \xi$ so that $f \in L^1_+$; from $f(\sum_i T^{*n_i}h) \leq \xi$ follows that $\int (\sum T^{n_i}f)h\, d\mu < \infty$. But if (d) is verified, $\sum T^{n_i}f = \infty$, a.s. so that h must be 0; hence (d) $\Rightarrow$ ($\mathrm{c_n}$).

Finally, if $h \in L^\infty_+$ verifies $\liminf \langle T^n f_0, h\rangle = 0$, select an infinite sequence $0 \leq n_0 < n_1 < \cdots$ such that $\langle T^{n_i}f_0, h\rangle \leq 2^{-i}$. Then

(43) $$\int f_0(\sum T^{*n_i}h)\, d\mu = \sum \langle T^{n_i}f_0, h\rangle < \infty,$$

so that

(44) $$\textstyle\sum_i T^{*n_i}h < \infty, \text{ a.s.}$$

If ($\mathrm{c_n}$) is verified, it implies that $h = 0$; hence ($\mathrm{c_n}$) $\Rightarrow$ ($\mathrm{b_n}$).

6. The existence of a strictly positive invariant element g in L^1 immediately implies that T is conservative since $\sum_{n\geq 0} T^n g = \sum_{n\geq 0} g = \infty$; it also implies the validity of condition (e).

Indeed, the formula $T'f = g\cdot T^*(f/g)$ where $f \in L^1$ is such that $f/g \in L^\infty$, defines a positive linear contraction T' of L^1 on the dense subspace

(45) $$\{f\colon f \in L^1, f/g \in L^\infty\}$$

of L^1; T' is indeed linear and positive on this subspace, and since it verifies these

(46) $$\int T'f\, d\mu = \langle g, T^*(f/g)\rangle = \langle Tg, f/g\rangle = \int f\, d\mu,$$

it can be extended by continuity to the whole of L^1. Moreover, g is T'-invariant since $T^*1 = 1$. Hence, condition (d) of the theorem is verified by T', and this implies that condition (e) is verified by T. Indeed, if $h \in L^\infty$ is strictly positive, so is gh in L^1 and

(47) $$g\Big(\sum_i T^{*n_i}h\Big) = \sum_i T'^{n_i}(gh) = \infty$$

holds a.s. for every infinite sequence $0 \leq n_0 < n_1 < \cdots$ of integers.

7. We show next that (e) $\Rightarrow$ ($\mathrm{c_s}$) if T is conservative.

If the set F is such that $\sum_i T^{*n_i}1_F \in L^\infty$ for an infinite sequence $0 \leq n_0 < n_1 < \cdots$ of integers, then $h = \sum_{n\geq 0} 2^{-n}T^{*n}1_F$ is an element of L^∞_+ such that:

(48) $$\sum_i T^{*n_i}h = \sum_n 2^{-n}\, T^{*n}\Big(\sum_i T^{*n_i}1_F\Big) \in L^\infty;$$

moreover, the set

(49) $$H = \{h > 0\} = \bigcup_{n\geq 0} \{T^{*n}1_F > 0\}$$

is, that $\sum T^{*n_i}h < \infty$ on $\{f > 0\}$; hence if (e) holds, f must be 0, that is, condition (f) holds. Conversely, if (f) holds and $h \in L^\infty$ is strictly positive, then $f = \xi(1 + \sum T^{*n_i}h)^{-1}$ belongs to L^1_+ and verifies

(50) $$\int (\sum T^{n_i}f)h\, d\mu = \int f(\sum T^{*n_i}h)\, d\mu \leq \int \xi\, d\mu < \infty.$$

Therefore, $\sum_i T^{n_i}f < \infty$, a.s. and f must be 0, that is, $\sum_i T^{*n_i}h = \infty$, a.s.

4. Strong conservativeness

The following theorem is a counterpart to theorem 1.

THEOREM 2. *For any positive linear contraction* T *of a space* $L^1(\epsilon, \mathfrak{F}, \mu)$, *the following conditions are equivalent:*

(a) *the only* $g \in L^1_+$ *such that* $Tg = g$ *is* 0;

(b_n) *there exists an element* $h \in L^\infty$ *such that* $h > 0$, *a.s. and*

$$\liminf_{n\to\infty} \langle T^n f_0, h\rangle = 0 \tag{51}$$

(f_0 *denotes an arbitrarily fixed element of* L^1 *such that* $f_0 > 0$, *a.s.*);

(b_s) *there exists an element* $h \in L^\infty$ *such that* $h > 0$, *a.s. and*

$$\lim_{p\to\infty} \sup_n 1/p \sum_{m=0}^{p-1} \langle T^{m+n} f_0, h\rangle = 0; \tag{52}$$

(c) *there exists an element* $h \in L^\infty$ *such that* $h > 0$, *a.s. and* $\sum_i T^{*n_i}h < \infty$, *a.s. for a suitably chosen infinite sequence* $0 \le n_0 < n_1 < \cdots$ *of integers;*

(d) $\sum_i T^{n_i} f_0 < \infty$ *holds a.s. for at least an infinite sequence* $0 \le n_0 < n_1 < \cdots$ *of integers.*

PROOF OF THEOREM 2. (1) To prove the implication (a) $\Rightarrow$ (b_n), consider the construction in alinea 2 of the proof of theorem 1 of an invariant $g \in L^1$ starting from a Banach limit L. Since $g = 0$ by (a), lemma 1 of the preliminaries shows the existence of a strictly positive $h \in L^\infty$ such that $\lambda(h) = 0$. Then (b_n) follows from the inequality $0 \le \liminf_{n\to\infty} \langle T^n f_0, h\rangle \le \lambda(h)$.

Conversely, (b_n) $\Rightarrow$ (a). The condition $\liminf_{n\to\infty} \langle T^n f_0, h\rangle = 0$ indeed implies by a previous remark that $\liminf_{n\to\infty} \langle T^n f, h\rangle = 0$ for any $f \in L^1_+$, hence, that $\langle g, h\rangle = 0$ if g is invariant. Since $h > 0$, a.s., this shows that 0 is the only invariant element in L^1_+.

(2) To show that (b_n) implies (c) and (d), choose an infinite sequence $0 \le n_0 < n_1 < \cdots$ of integers such that $\langle T^{n_i} f_0, h\rangle \le 2^{-i}$. Then

$$\int f_0\left(\sum T^{*n_i}h\right) d\mu = \int \left(\sum T^{n_i} f_0\right) h \, d\mu \le \sum 2^{-i} < \infty \tag{53}$$

implies that $\sum T^{n_i} f_0 < \infty$ a.s. since $h > 0$ a.s., resp. that $\sum T^{*n_i} h < \infty$ a.s. since $f_0 > 0$ a.s.

Conversely, (c) $\Rightarrow$ (b_n) and (d) $\Rightarrow$ (b_n), for letting, as in alinea 5,

$$f_0 = \xi\left(1 + \sum T^{*n_i} h\right)^{-1} \tag{54}$$

in the first case and $h = \xi(1 + \sum T^{n_i} f_0)$ in the second case, one obtains that

$$0 \le \liminf_{n\to\infty} \langle T^n f_0, h\rangle \le \lim_i \langle T^{n_i} f_0, h\rangle = 0 \tag{55}$$

since $\sum_i \langle T^{n_i} f_0, h\rangle < \infty$ holds in both cases. This proves the implications above, because (b_n) does not depend on the f_0 selected, as was previously noted.

(3) It is clear that (b_s) $\Rightarrow$ (b_n). Conversely, if (b_n) holds, it is possible by lemma 3 to construct for each $\delta > 0$ an element $h_\delta \in L^\infty$ such that $0 \le h_\delta \le h$, $\langle f_0, h - h_\delta\rangle \le \delta$, and that $\sum_i T^{*n_i} h_\delta \in L^\infty$ for a suitably chosen infinite sequence

$0 \leq n_0 < n_1 < \cdots$ of integers. Then $\lambda(h_\delta) = 0$ holds whatever Banach limit L has been chosen to define λ, and it follows from the corollary to lemma 2 that

$$\lim_{n\to\infty} \sup_n \frac{1}{p} \sum_{m=0}^{p-1} \langle T^{m+n} f_0, h_\delta \rangle = 0. \tag{56}$$

Letting $h' = \sum 2^{-p} h_{2-p}$, one obtains an element $h' \in L^\infty_+$ such that

$$\lim_{p\to\infty} \sup_n \frac{1}{p} \sum_{m=0}^{p-1} \langle T^{m+n} f_0, h' \rangle = 0, \tag{57}$$

which is, moreover, strictly positive since $\{h' > 0\} = \bigcup_p \{h_{2-p} > 0\}$ and

$$\int_{\{h_2-p=0\}} f_0 h \, d\mu \leq \int f_0(h - h_{2-p}) \, d\mu \leq 2^{-p} \to 0 \qquad \text{as} \quad p \uparrow \infty. \tag{58}$$

Thus h' satisfies condition (b_s).

We propose to call the set defined in the following theorem the *strongly conservative* set associated to T.

THEOREM 3. *For any positive linear contraction T of a space $L^1(E, \mathfrak{F}, \mu)$, there exists a measurable subset $\tilde{C}$ of E (defined up to an equivalence), which is characterized by each of the following properties, the third one being valid only if T is conservative.*

(a) Every T-invariant element $g \in L^1$ is carried by $\tilde{C}$, namely, $\{g \neq 0\} \subset \tilde{C}$. Conversely, there exists a T-invariant element $\tilde{g} \in L^1_+$ such that $\{\tilde{g} > 0\} = \tilde{C}$.

(b) For any infinite sequence $0 \leq n_0 < n_1 < \cdots$ of integers, one has $\sum_i T^{n_i} f_0 = \infty$ on $\tilde{C}$, and there exists, conversely, an infinite sequence

$$0 \leq \tilde{n}_0 < \tilde{n}_1 < \cdots \tag{59}$$

such that $\{\sum_i T^{\tilde{n}_i} f = \infty\} = \tilde{C}$ (f_0 denotes a strictly positive, arbitrarily fixed element of L^1).

(c) For every strictly positive $h \in L^\infty$ and every infinite sequence

$$0 \leq n_0 < n_1 < \cdots \tag{60}$$

of integers, one has $\sum T^{*n_i} h = \infty$ on $\tilde{C}$. Conversely, there exists a strictly positive $\tilde{h} \in L^\infty$ and an infinite sequence $0 < \tilde{n}_0 < \tilde{n}_1 < \cdots$ of integers such that $\{\sum T^{*\tilde{n}_i} \tilde{h} = \infty\} = \tilde{C}$.

Moreover, $\tilde{C}$ is an invariant subset of the conservative part C of T.

PROOF OF THEOREM 3. Let G denote the set of all T-invariant g in L^1_+ and consider the essential supremum of the carriers $\{g > 0\}$ $(g \in G)$. Let $\tilde{C}$ be this set. By a general property of essential suprema, there exists a sequence $\{g_n\}$ in G such that $C = \bigcup \{g_n > 0\}$. Letting $\tilde{g} = \sum_n \|g_n\|^{-1} 2^{-n} g_n$, we obtain an element of G such that $\{\tilde{g} > 0\} = \tilde{C}$. Since $Tg = g$ $(g \in L^1)$ implies $T|g| = |g|$, one has $\{g \neq 0\} = \{|g| > 0\} \subset \tilde{C}$ for every T-invariant g in L^1. The existence and uniqueness of a set $\tilde{C}$ with property (a) is thus proved.

Moreover, since $C = \{g > 0\} = \{\sum_n T^n \tilde{g} = \infty\}$, the set C is an invariant subset of C (see [10]).

Applying theorem 1 to the restriction of T to $\tilde{C}$, which is a contraction of

$L^1[\tilde{C}, \tilde{C} \cap \mathfrak{F}, \mu(\tilde{C} \cap \cdot)]$, with the restriction of $\tilde{g}$ to $\tilde{C}$ as invariant strictly positive element, we obtain that $\sum T^{n_i} f_0 = \infty$ on $\tilde{C}$ for every infinite sequence $0 \leq n_0 < n_1 < \cdots$ of integers provided that f_0 belongs to L^1_+ and is strictly positive on $\tilde{C}$ (remark that the invariance of $\tilde{C}$ implies that the powers of the restriction of T to $\tilde{C}$ are the restrictions to $\tilde{C}$ of the powers of T). When applying theorem 2 to the restriction of T to $E - \tilde{C}$, we obtain the existence of an infinite sequence $0 \leq \tilde{n}_0 < \tilde{n}_1 < \cdots$ of integers such that $\sum_i T^{\tilde{n}_i} f_0 < \infty$ holds on $E - \tilde{C}$. This suffices to establish property (b).

When T is conservative, a reasoning similar to the preceding, but using condition (e) of theorem 1 and condition (c) of theorem 2, establishes the validity of property (c) of theorem 3 and concludes its proof.

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