CLASSIFICATION OF STATES
FOR OPERATORS

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1. Summary

We characterize the sets of positive states and null states for nonsingular Markov processes and, more generally, for positive contractions in L_1. The set P of positive states is an invariant set and carries all finite invariant measures which are absolutely continuous with respect to a given measure μ, the initial distribution. The Cesaro averages of the "probabilities of being in B at time n" converge to a positive limit for any subset B of P with $\mu(B) > 0$. The set N of null states is a countable union of sets X_i with the property that the Cesaro averages of the "probabilities of being in X_i" tend to 0 for each X_i. We further generalize Hopf's decomposition of the state space into a conservative and dissipative part by introducing monotonically decreasing weights, obtaining the positive part P as a special "weighted conservative part" with divergent sum of weights. As an application we derive an ergodic theorem with appropriate weighted averages under conditions which do not imply the usual ergodic theorem (corollary 2).

Different characterizations of the decomposition into P and N have been described by Mrs. Dowker [7] (for point mappings) and by Neveu [20]. (See also Neveu's paper of this Berkeley Symposium. I noticed the decomposition independently, but later than Neveu. Also A. Hajian and Y. Ito have some related (so far unpublished) results, which overlap with Neveu's present paper and are based on his paper [20].) I am indebted to Professors D. Freedman, Y. Ito, and W. Pruitt for some references.

2. Introduction

Let $(X, \mathfrak{F}, \mu)$ be a measure space with $\mu(X) = 1$. All sets and functions introduced are assumed to be measurable. Sets as well as functions are identified if they coincide almost everywhere. Let T be a positive contraction in $L_1 = L_1(X, \mathfrak{F}, \mu)$, that is, a linear operator in L_1 with $Tf \geq 0$ for all $0 \leq f \in L_1$, and with $\|T\| = \sup_{\|f\|=1} \|Tf\| \leq 1$. By the Radon-Nikodym theorem, L_1 is isomorphic to the Banach space Φ of all signed measures φ which are absolutely con-

tinuous with respect to μ: $\varphi \ll \mu$. The contraction T induces in Φ an isomorphic operator Λ defined by

$$(1) \qquad (\Lambda\varphi)(A) = \int_A T(d\varphi/d\mu)\, d\mu, \qquad (\varphi \in \Phi)$$

Conversely, Λ may be given first and T defined by $\varphi_f(A) = \int_A f\, d\mu$ and $Tf = d\varphi_f/d\mu$. This is the case if Λ is given by a stochastic kernel $P(x, A)$ by the relation

$$(2) \qquad (\Lambda\varphi)(A) = \int_X P(x, A)\, d\varphi, \qquad (\varphi \in \Phi, A \in \mathfrak{F})$$

where $P(x, A)$ is *nonsingular*, that is, $\mu(A) = 0$ implies $P(x, A) = 0$.

Let A^c be the complement of A. Functions f with $Tf = f$ and measures φ with $\Lambda\varphi = \varphi$ are called *invariant*. A set $I \in \mathfrak{F}$ is called *invariant* if $Tf = 0$ on I^c for any $f \geq 0$ with $f = 0$ on I^c, or, equivalently, if $\Lambda\varphi(I^c) = 0$ for any $\varphi \geq 0$ with $\varphi(I^c) = 0$.

Our main result will be derived from the following generalization of a theorem of Y. Ito [15], which was obtained independently by Dean and Sucheston [6] and by Neveu [20].

THEOREM A. *The following conditions are equivalent:*
   (i) *there exists a strictly positive invariant function $\bar{f} \in L_1$;*
   (ii) $\inf_n \Lambda^n\mu(A) > 0$ *for all $A \in \mathfrak{F}$ with $\mu(A) > 0$;*
   (iii) $\lim_{n\to\infty} \{\sup_j n^{-1} \sum_{i=0}^{n-1} \Lambda^{i+j} \mu(A)\} > 0$ *for all $A \in \mathfrak{F}$ with $\mu(A) > 0$.*

Note that (i) is equivalent to the existence of a finite invariant measure $\varphi \geq 0$ with $\mu \ll \varphi \ll \mu$. For references concerning the existence of invariant measures see [22], [15], [10], [16]. Some more conditions are described in a paper by Hajian and Ito [9].

The following theorem is essentially due to Hopf [12].

THEOREM B. *The space X is the disjoint union of two uniquely determined sets C and D, respectively the conservative part and the dissipative part of X, such that*
   (i) *for every $f \geq 0$, $\sum_{k=0}^{\infty} T^k f$ converges on D;*
   (ii) *for every $f \geq 0$, $\sum_{k=0}^{\infty} T^k f$ diverges on $\{x: \sum_{k=0}^{\infty} T^k f > 0\} \cap C$;*
   (iii) C *is invariant.*

Chacon and Ornstein [4] proved the following theorem, which was conjectured by Hopf. This author also gave a simplified proof later [12], [13].

THEOREM C. *For every $f \in L_1$ and $0 \leq p \in L_1$, the limit*

$$(3) \qquad \lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} T^k f}{\sum_{k=0}^{n-1} T^k p} = h(f, p)$$

exists and is finite on $\{x: \sum_{k=0}^{\infty} T^k p > 0\}$.

Let χ_A be the characteristic function of A. Define T_C and T_D by $T_C f = \chi_C Tf$, $T_D f = \chi_D Tf$ for $f \in L_1$. Then

$$(4) \qquad R_C f = \chi_C f + T_C(\chi_D f) + \sum_{k=1}^{\infty} T_C T_D^k(\chi_D f)$$

defines a positive contraction. Chacon [2], [3] proved the following theorem.

THEOREM D. *The invariant subsets of C form a σ-field $\mathfrak{I}$. For every $0 \leq p \in L_1$,*

(i) *the function $h(f, p) \cdot p$ is integrable;*

(ii) *the equality $h(f, p) = h(R_C f, R_C p) = E(R_C f | \mathfrak{I}) / E(R_C p | \mathfrak{I})$ holds on $C \cap \{x: \sum_{k=0}^{\infty} T^k p > 0\}$.*

3. The positive part and the null part of X

For $0 \leq f \in L_1$ with $Tf = f$, let $P(f) = \{f > 0\}$. It is easy to see that there is a maximal set P among the sets $P(f)$. We will obtain P by a different approach and characterize P in probabilistic terms.

THEOREM 1. *The space X is the disjoint union of two uniquely determined sets P and N, respectively the positive part and the null part of X, such that*

(i) *P is an invariant subset of C;*

(ii) *there exists an invariant $0 \leq \bar{f} \in L_1$ which is strictly positive in P;*

(iii) *for any $\varphi \in \Phi$ and $A \in \mathfrak{I}$, the limit*

$$(5) \qquad \lim_{n \to \infty} n^{-1} \sum_{k=0}^{n-1} \Lambda^k \varphi(A \cap P) = \lambda_\varphi(A)$$

exists ($\lambda_\varphi \in \Phi$ is invariant). Let $f = d\varphi/d\mu$; then we have

$$(6) \qquad \lambda_\varphi(A) = \int_{A \cap P} \bar{f} E(R_C f | \mathfrak{I}) E(\bar{f} | \mathfrak{I})^{-1} \, d\mu.$$

Thus, if $f \geq 0$ and $\int_{A \cap P} f \, d\mu > 0$, then $\lambda_\varphi(A) > 0$;

(iv) *$N = X - P$ is a countable union of sets X_i, $i = 1, 2, \cdots$ such that*

$$(7) \qquad \lim_{n \to \infty} n^{-1} \sum_{k=0}^{n-1} \Lambda^k \varphi(X_i \cap A) = 0$$

holds for any $A \in \mathfrak{I}$, $\varphi \in \Phi$ and $i = 1, 2, \cdots$.

PROOF. We consider $(X, \mathfrak{I}, \mu)$ as a measure algebra. A real-valued function H on $\mathfrak{I}$ is called *monotonic* if $B \subseteq A$ implies $H(B) \leq H(A)$. The construction of the decomposition is based on the following simple lemma.

LEMMA 1. *If H is a nonnegative monotonic function on $\mathfrak{I}$ with $H(0) = 0$, then X is the disjoint union of two uniquely determined sets P and N such that*

(i) *$H(B) > 0$ holds for all $0 \neq B \subseteq P$ and*

(ii) *N is the disjoint union of countably many sets X_i, with $H(X_i) = 0$.*

PROOF. Measure algebras are closed with respect to the formation of arbitrary unions. (Such unions may always be replaced by countable unions.) Let N be the union of all sets $B \in \mathfrak{I}$ with $H(B) = 0$. Passing to subsets we may assume $N = \bigcup_{i=1}^{\infty} X_i$ with disjoint X_i and $H(X_i) = 0$. Let $P = X - N$. Then (i), (ii), and the uniqueness are obvious.

For any bounded sequence $\{x_n\}$ of real numbers let

$$(8) \qquad M\{x_n\} = \lim_{n \to \infty} \left[\sup_j n^{-1} \sum_{i=0}^{n-1} x_{i+j} \right]$$

(this M is the maximal value of Banach limits, (see [6], [22])). The function $H(B) = M\{\Lambda^n\mu(B)\}$ is monotonic. Further, P and N are characterized by the conditions

(9) $M\{\Lambda^n\mu(B)\} > 0$ for every $B \subseteq P$ with $\mu(B) > 0$,

(10) N is the disjoint union of countably many sets

$$X_i, i = 1, 2, \cdots \quad \text{where} \quad M\{\Lambda^n\mu(X_i)\} = 0.$$

Before proceeding with the proof, we will collect some facts about M. It is known and not difficult to prove that

(11) $M\{x_n + y_n\} \leq M\{x_n\} + M\{y_n\}$ for every $\{x_n\},$ $\{y_n\} \in \ell^\infty$

(12) $M\{\alpha x_n\} = \alpha M\{x_n\}$ for $\alpha \geq 0,$

(13) $M\{x_n\} \leq \sup_n |x_n|.$

We will further need the equation

(14) $M\{x_n + y_n\} = M\{x_n + y_{n+1}\}$ for every $\{x_n\},$ $\{y_n\} \in \ell^\infty,$

which follows by an easy cancellation argument.

With the above, the proof of (iv) is immediate: we may assume $\varphi \geq 0$ and $A = X$. Let $f = d\varphi/d\mu$, $f_n = \min\{f, n\}$, $g_n = f - f_n$, and let φ_n and ψ_n be respectively the measures with $f_n = d\varphi_n/d\mu$ and $g_n = d\psi_n/d\mu$. For every $\epsilon > 0$, $\|\psi_{n_0}\| = \|g_{n_0}\| < \epsilon$ for sufficiently large n_0. For every n,

(15) $|\Lambda^n\varphi(X_i)| \leq |\Lambda^n\varphi_{n_0}(X_i)| + |\Lambda^n\psi_{n_0}(X_i)| \leq n_0|\Lambda^n\mu(X_i)| + \epsilon.$

Therefore,

(16) $M\{\Lambda^n\varphi(X_i)\} \leq n_0 M\{\Lambda^n\mu(X_i)\} + \epsilon = \epsilon.$

Since $\epsilon > 0$ was arbitrary, this proves $M\{\Lambda^n\varphi(X_i)\} = 0$, a statement which is slightly stronger than (iv), since

(17) $M\{x_n\} \geq \limsup_{n \to \infty} n^{-1} \sum_{k=0}^{n-1} x_k.$

Now for every $F \in \mathfrak{F}$ let Γ_F be the operator in Φ defined by

(18) $\Gamma_F\psi(A) = \psi(F \cap A),$ $A \in \mathfrak{F}.$

The next lemma will be used in the proof of the invariance of P.

LEMMA 2. *Let $0 \leq \varphi \in \Phi$, $\varphi(A^c) = 0$, and $\Lambda\varphi(E) > 0$ for some set E. Then there exists an $\epsilon > 0$ and a set $B \subseteq A$ with $\mu(B) > 0$ such that*

(19) $\Lambda\psi(E) \geq \epsilon\psi(B)$

for all $0 \leq \psi \in \Phi$.

PROOF. Let $g = d\varphi/d\mu$. Then $A' = \{x: g(x) > 0\}$ is the smallest carrier of φ. We may assume that $A = A'$. Define $\eta(F) = \Lambda\Gamma_F\varphi(E)$. It is easy to show that $\eta \in \Phi$ and $\eta \ll \varphi$. If $f = d\eta/d\varphi$, it follows from $\eta(X) = \Lambda\varphi(E) > 0$ and $\eta \ll \varphi \ll \mu$ that $\mu\{x: f(x) > 0\} > 0$; and, hence, that for some $\epsilon > 0$, $B =$

$\{s: f(s) > \epsilon\}$ has positive measure. Changing f on a φ-null set, we may assume that $B \subseteq A$.

If $0 \leq \psi \in \Phi$ is carried by A, then $A = A'$ implies $\psi \ll \varphi$. To prove (19), we first assume $d\psi/d\varphi = \chi_H$ for some $H \subseteq A$.
Then

$$(20) \qquad \Lambda\psi(E) = \Lambda\Gamma_H\varphi(E) = \eta(H) = \int_H f \, d\varphi \geq \epsilon \int_{H \cap B} d\varphi = \epsilon\psi(B).$$

The usual extension procedures yield (19) for arbitrary $0 \leq \psi \ll \varphi$, equivalently for every $\Gamma_A\psi$, $(0 \leq \psi \in \Phi)$. If $0 \leq \psi \in \Phi$ is arbitrary, then the inequalities

$$(21) \qquad \Lambda\psi(E) \geq \Lambda\Gamma_A\psi(E) \geq \epsilon\Gamma_A\psi(B) = \epsilon\psi(B)$$

complete the proof.

We proceed to prove the invariance of P. Assume that P is not invariant. Then there exists some nonnegative $\varphi \in \Phi$ with $\varphi(N) = 0$ and $\Lambda\varphi(N) > 0$. Hence, there is an index i_0 with $\Lambda\varphi(X_{i_0}) > 0$. Apply lemma 2 with $A = P$ and $X_{i_0} = E$. From (9) we obtain

$$(22) \qquad M\{\Lambda^n\mu(X_{i_0})\} = M\{\Lambda^{n+1}\mu(X_{i_0})\} \geq \epsilon M\{\Lambda^n\mu(B)\} > 0,$$

which contradicts (10).

Let us now consider the influence of the null part. We define inductively for every $\varphi \in \Phi$,

$$(23) \qquad \begin{aligned} \varphi_0 &= \Gamma_P\varphi, & \varphi_0^* &= \Gamma_N\varphi, \\ \varphi_{k+1} &= \Gamma_P\Lambda\varphi_k^*, & \varphi_{k+1}^* &= \Gamma_N\Lambda\varphi_k^*. \end{aligned}$$

Then

$$(24) \qquad \Lambda^n\varphi = \Lambda^n\varphi_0 + \cdots + \Lambda^1\varphi_{n-1} + \varphi_n + \varphi_n^*$$

follows by induction and $\|\varphi_{k+1}\| + \|\varphi_{k+1}^*\| \leq \|\varphi_k^*\|$ implies $\sum_{k=0}^\infty \|\varphi_k\| \leq \|\varphi\|$. Therefore, $\Psi_P\varphi = \sum_{k=0}^\infty \Gamma_P(\Lambda\Gamma_N)^k\varphi$ defines a contraction.

LEMMA 3. *For every* $B \subseteq P$ *we have* $M\{\Lambda^n\mu(B)\} = M\{\Lambda^n\Psi_P\mu(B)\}$.

PROOF. Define μ_k, μ_k^* by (23). First note that $B \subseteq P$ implies $\mu_n^*(B) = 0$ for every n since P is invariant. For every $\epsilon > 0$ we may choose n_0 so large that $\sum_{k=n_0}^\infty \|\mu_k\| < \epsilon$. Then from (13) and (24) we derive the inequality

$$(25) \qquad |M\{\Lambda^{n_0+n}\mu(B)\} - M\{(\Lambda^{n_0+n}\mu_0 + \cdots + \Lambda^n\mu_{n_0})(B)\}| < \epsilon.$$

Equation (14) implies both $M\{\Lambda^{n_0+n}\mu(B)\} = M\{\Lambda^n\mu(B)\}$ and

$$(26) \qquad \begin{aligned} M\{(\Lambda^{n_0+n}\mu_0 + \cdots + \Lambda^n\mu_{n_0})(B)\} &= M\{\Lambda^{n_0+n}(\mu_0 + \cdots + \mu_{n_0})(B)\} \\ &= M\{\Lambda^n(\mu_0 + \cdots + \mu_{n_0})(B)\}. \end{aligned}$$

Therefore, $M\{\Lambda^n(\mu_0 + \cdots + \mu_{n_0})(B)\}$ tends (for $n_0 \to \infty$) to $M\{\Lambda^n\mu(B)\}$ and to

$$(27) \qquad M\left\{\Lambda^n\left(\sum_{k=0}^\infty \mu_k\right)(B)\right\} = M\{\Lambda^n\Psi_P\mu(B)\}.$$

The essential step in the proof of theorem 1 is an application of theorem A. Observe that P is invariant and that $\Psi_P\mu$ is equivalent to $\Gamma_P\mu$, that is, $\Gamma_P\mu \ll \Psi_P\mu \ll \Gamma_P\mu$. If $\mu(P) = 0$, theorem 1 is now trivial. If $\mu(P) > 0$, we may assume that $\Psi_P\mu(P) = 1$ by normalizing the measure. Lemma 3 says that Λ

satisfies condition (ii) of theorem A applied to $(P, \mathfrak{F} \cap P, \Psi_P\mu, \Lambda)$. Hence, there exists an invariant measure $\bar{\varphi}$ on P which is equivalent to $\Psi_P\mu$, and therefore equivalent to $\Gamma_P\mu$. Set $\bar{\varphi}(N) = 0$. Then $\bar{f} = d\bar{\varphi}/d\mu$ is invariant, strictly positive in P and 0 in N, which proves (ii). That $P \subseteq C$ follows from (ii) and theorem B, since $\sum_{k=0}^{\infty} T^k\bar{f} = \sum_{k=0}^{\infty} \bar{f}$ diverges in P. The proof of (iii) rests on (ii), theorem C, theorem D, and the following lemma.

LEMMA 4.   *For every $f \in L_1$ the sequence $\{\chi_P T^k f, k = 0, 1, \cdots\}$ is uniformly integrable.*

PROOF.   For $g \in L_1$ put $T_P g = \chi_P T g$ and $T_N g = \chi_N T g$. We define for $f \in L_1$ the sequences $\{f_j\}$ and $\{f_j^*\}$ by

$$(28) \qquad \begin{aligned} f_0 &= \chi_P f, & f_0^* &= \chi_N f, \\ f_{k+1} &= T_P f_k^*, & f_{k+1}^* &= T_N f_k^*. \end{aligned}$$

Then $\|f_{k+1}\| + \|f_{k+1}^*\| \le \|f_k^*\|$ implies $\sum_{j=0}^{\infty} \|f_j\| \le \|f\|$, and for $j \ge 1$ we have $f_j = T_P T_N^{j-1}(\chi_N f)$. From the invariance of P we conclude that $\chi_N T^n g = T_N^n g$ and $T^n(\chi_P g) = T_P^n(\chi_P g)$ for all $g \in L_1$. It is now intuitively clear and also follows by induction that

$$(29) \qquad \chi_P T^k f = \chi_P \sum_{j=0}^{k} T^j f_{k-j}.$$

To prove uniform integrability of $\{\chi_P T^k f\}$, we may and do assume $f \ge 0$. For a given $\epsilon > 0$, choose $\ell = \ell_\epsilon$ so large that

$$(30) \qquad \sum_{j=\ell+1}^{\infty} \|f_j\| < \epsilon/6.$$

Let $h = \chi_P T^\ell f$, and let $0 \le \bar{f} \in L_1$ be invariant and strictly positive on P. We choose m so large that

$$(31) \qquad \int (h - m\bar{f})^+ \, d\mu < \epsilon/6,$$

where $g^+ = \max\{g, 0\}$. Finally, $a_\epsilon > 0$ may be chosen so large that

$$(32) \qquad \int_{\{m\bar{f} > a_\epsilon\}} m\bar{f} \, d\mu < \epsilon/6.$$

It follows from (30) and (31) that for any $k \ge \ell$,

$$(33) \qquad A_k = \{x: T^{k-\ell}(h - m\bar{f})^+ + T^{k-\ell-1}f_{\ell+1} + \cdots + T^0 f_k > a_\epsilon\}$$

has measure $\mu(A_k) < a_\epsilon^{-1} \cdot \epsilon/3$. The inequality

$$(34) \qquad \chi_P T^k f \le m\bar{f} + T^{k-\ell}(h - m\bar{f})^+ + T^{k-\ell-1}f_{\ell+1} + \cdots + T^0 f_k$$

implies

$$(35) \qquad P \cap \{T^k f > 2a_\epsilon\} \subseteq \{m\bar{f} > a_\epsilon\} + \{m\bar{f} \le a_\epsilon\} \cap A_k.$$

Therefore,

$$(36) \qquad \int_{\{T^k f > 2a_\epsilon\}} \chi_P T^k f \, d\mu \le \int_{\{m\bar{f} > a_\epsilon\}} m\bar{f} \, d\mu + \int (h - m\bar{f})^+ \, d\mu$$
$$+ \sum_{j=\ell+1}^{\infty} \|f_j\| + \int_{A_k} a_\epsilon \, d\mu < \epsilon.$$

This establishes the inequality

(37) $$\int_{\{T^k f > c_\epsilon\}} \chi_P T^k f \, d\mu < \epsilon$$

for $c_\epsilon \geq 2a_\epsilon$ and all $k \geq \ell$ so that, for c_ϵ large enough, (37) holds for all $k \geq 0$.

For the proof of (iii) let $\varphi \in \Phi$ and $f = d\varphi/d\mu$. Theorem C applied to f and to $p = \bar{f}$ states

(38) $$\lim_{n \to \infty} n^{-1} \sum_{k=0}^{n-1} T^k f = h(f, \bar{f}) \cdot \bar{f}$$

in P. Since $\bar{f} = \chi_P \bar{f}$ we have $R_C \bar{f} = \bar{f}$. By theorem D

(39) $$\chi_P h(f, \bar{f}) \cdot \bar{f} = \chi_P \bar{f} E(R_C f \,|\, \mathfrak{I}) / E(\bar{f} \,|\, \mathfrak{I}) \in L_1.$$

Lemma 4 implies that $\{\chi_P n^{-1} \sum_{k=0}^{n-1} T^k f\}$ is uniformly integrable; hence, $\chi_P n^{-1} \sum_{k=0}^{n-1} T^k f$ tends to $\chi_P h(f, \bar{f}) \cdot \bar{f}$ in norm. This limit is shown to be invariant by a cancellation argument. Norm convergence implies convergence of $\lim_{n \to \infty} n^{-1} \sum_{k=0}^{n-1} \Lambda^k \varphi(A \cap P)$ for any $A \in \mathfrak{F}$ since $\Lambda^k \varphi(A \cap P) = \int_{A \cap P} T^k f \, d\mu$. The theorem is thereby completely proved.

REMARK. (1) Dean and Sucheston ([6], theorem 3) have shown the following: for $\varphi = \mu$, and $P = X$ $n^{-1} \sum_{k=0}^{n-1} \Lambda^{k+i} \varphi(A \cap P)$ converges *uniformly* in i. We mention that this remains true for general $\varphi \in \Phi$ and P as may be shown by extending their method to the present case. The application of their proposition 3 must then be replaced by an application of lemma 4 of this paper.

(2) Let $f_0 \in L_1$ be strictly positive. Neveu [20] mentions that P is the intersection of all sets $\{\sum_{k=0}^{\infty} T^{n_i} f_0 = +\infty\}$ where $\{n_i\}$ runs through all subsequences of the nonnegative integers. Another characterization of P in terms of T is given by proposition 1.

PROPOSITION 1. *Define P_f for $0 \leq f \in L_1$ by*

(40) $$P_f = \left\{ x : \liminf_{n \to \infty} n^{-1} \sum_{k=0}^{n-1} T^k f(x) > 0 \right\}.$$

Then $P = P_f$ holds for all $f \in L_1$, which are strictly positive in X.

PROOF. Theorem C implies that P_f does not depend on the choice of a strictly positive $f \in L_1$. Taking $f = \bar{f} + \chi_N$ with $\bar{f} = T\bar{f}$ strictly positive in P, we obtain $P_f \supseteq P$. Next take $f \equiv 1$. For every X_i,

(41) $$0 \leq \int_{X_i} \left(\liminf_{n \to \infty} n^{-1} \sum_{k=0}^{n-1} T^k f \right) d\mu \leq \liminf_{n \to \infty} n^{-1} \sum_{k=0}^{n-1} \int_{X_i} T^k f \, d\mu$$

$$= \liminf_{n \to \infty} n^{-1} \sum_{k=0}^{n-1} \Lambda^k \mu(X_i) \leq M\{\Lambda^n \mu(X_i)\} = 0;$$

hence, all X_i belong to P_f^c.

Property (iv) of theorem 1 might lead one to expect the convergence to 0 of (1) even with X_i replaced by N. However, in that case the Césaro averages of $\Lambda^n \mu(N)$ will usually decrease to a positive lower bound, and for appropriate $A \subseteq N$ the numbers $n^{-1} \sum_{k=0}^{n-1} \Lambda^k \mu(A)$ may oscillate. This observation is due to

Mrs. Dowker ([7], theorem 3), who considered ergodic point mappings. We mention the following generalization of her result.

PROPOSITION 2.   *If μ is nonatomic (that is, every A with $\mu(A) > 0$ contains a B with $0 < \mu(B) < \mu(A)$) and $P = 0$, and $\|\Lambda^k \mu\| = 1$ for all k, then for any given α, β with $0 \leq \alpha \leq \beta \leq 1$, there exists a set $A_{\alpha,\beta} \in \mathfrak{F}$ with*

(42)
$$\limsup_{n \to \infty} n^{-1} \sum_{k=0}^{n-1} \Lambda^k \mu(A_{\alpha,\beta}) = \beta,$$

$$\liminf_{n \to \infty} n^{-1} \sum_{k=0}^{n-1} \Lambda^k \mu(A_{\alpha,\beta}) = \alpha.$$

PROOF.   Clearly, the measures $\mu_n = n^{-1} \sum_{k=0}^{n-1} \Lambda^k \mu$ are equivalent to μ. By theorem 1 (iv) the following lemma of Mrs. Dowker is applicable.

LEMMA A.   *Let X be the disjoint union of countably many sets X_i, and let μ be nonatomic and $\{\mu_n\}$ a sequence of normalized measures which are equivalent to μ and such that $\lim_{n \to \infty} \mu_n(X_i) = 0$ for every X_i. Then for any given α, β with $0 \leq \alpha \leq \beta \leq 1$ there exists a set $A_{\alpha,\beta}$ with (42).*

4. Weighted conservative parts

We now introduce "weighted conservative parts C_w" by considering expressions $\sum_{k=0}^{\infty} w_k T^k f$ with monotonically decreasing weights:

(43) $w = \{w_k\}, \qquad k = 0, 1, \cdots; \; w_k \geq w_{k+1} > 0.$

The result will be a finer splitting of the conservative null part $C \cap N$ (see example 1). In the elementary special case of Markov chains we may consider this splitting as a classification of the null-recurrent states according to the speed of convergence to 0 of the "probabilities of return at time n." This classification does not yield nearly as precise statements about $C \cap N$ as some results of Vere-Jones [24] and Kingman [19] do about D. It seems, however, to be the first suggestion of any method for further classification of $C \cap N$ and might be of interest as a common generalization of both the decompositions $X = C + D$ and $X = P + N$.

Ergodic theorems with weighted averages were first introduced by Baxter [1], who used recurrence probabilities as weights. Jamison, Orey, and Pruitt [18] showed that far more general weights may be used for the summation of independent identically distributed random variables.

We first state two elementary lemmas which make it possible to replace Césaro means by weighted averages with monotonically decreasing weights in practically all theorems of pointwise ergodic theory.

LEMMA 5.   *Let $\{f_k\}, \{p_k\}, \{w_k\}$ be three sequences of real numbers $(k = 0, 1, \cdots)$ with $p_k \geq 0, \sum_{k=0}^{\infty} p_k > 0$, and (43). If*

(44)
$$\sum_{k=0}^{n-1} f_k \Big/ \sum_{k=0}^{n-1} p_k$$

converges to a finite limit, then so does

$$(45) \qquad \sum_{k=0}^{n-1} w_k f_k \Big/ \sum_{k=0}^{n-1} w_k p_k.$$

In the case $\sum_{k=0}^{\infty} w_k p_k = \infty$, the limit is the same.

(The special case where all $p_k > 0$ and $\sum w_k p_k = \infty$ is equivalent to a known theorem of the theory of summability; see, for example, Hardy ([11], p. 309).)

PROOF. Put $s_n = \sum_0^n f_k$, $r_n = \sum_0^n p_k$, $s_{w,n} = \sum_0^n w_k f_k$, and $r_{w,n} = \sum_0^n w_k p_k$. By a translation $f_k \to f_k - \lambda p_k$ we may and do assume the limit of (44) to be 0. Furthermore, it is sufficient to assume $p_0 > 0$. We investigate the ratios

$$(46) \qquad \frac{s_{w,n}}{r_{w,n}} = \frac{\displaystyle\sum_{k=0}^{n-1} s_k(w_k - w_{k+1}) + s_n w_n}{\displaystyle\sum_{k=0}^{n-1} r_k(w_k - w_{k+1}) + r_n w_n},$$

which are obtained by means of the Abel transformation. The ratios $b_k = s_k r_k^{-1}$ tend to 0. Put $s_k = b_k r_k$ in (46). If the denominator remains bounded, then $r_n w_n$ is bounded and $\sum_{k=0}^{\infty} r_k(w_k - w_{k+1}) < \infty$. In that case, $\sum_{k=0}^{\infty} b_k r_k(w_k - w_{k+1})$ converges and $s_n w_n$ tends to 0. If the denominator tends to infinity and $\epsilon > 0$ is given, choose K such that $|b_n| < \epsilon/2$ for $n \geq K$, and then choose $L \geq K$ such that $|r_{w,L}^{-1} \sum_{k=0}^{K} s_k(w_k - w_{k+1})| < \epsilon/2$. Then $|s_{w,n} r_{w,n}^{-1}| < \epsilon$ for all $n \geq L$.

While lemma 5 carries over the convergence theorems and the theorems on identification of the limit to the case of monotone averages, we need a second lemma for the proof of a maximal ergodic theorem and a dominated ergodic theorem.

LEMMA 6. *If $x_0, \cdots, x_n$ are any real numbers $y_0 > 0$, $y_1, \cdots, y_n \geq 0$, and $0 < w_0 \geq w_1 \geq \cdots \geq w_n \geq 0$, then*

$$(47) \qquad \max_{0 \leq k \leq n} \frac{x_0 + \cdots + x_k}{y_0 + \cdots + y_k} \geq \max_{0 \leq k \leq n} \frac{w_0 x_0 + \cdots + w_k x_k}{w_0 y_0 + \cdots + w_k y_k}.$$

PROOF. We may assume $w_k > 0$, $(k = 0, \cdots, n)$. Let λ be the expression on the right-hand side of (47), and let k be the first index for which the quotient equals λ. Then

$$(48) \qquad w_j x_j + \cdots + w_k x_k \geq \lambda(w_j y_j + \cdots + w_k y_k)$$

for all j with $0 \leq j \leq k$. Find $\alpha_0, \cdots, \alpha_k$ successively by solving the equations $w_j \sum_{i=0}^{j} \alpha_i = 1$ for $j = 0, \cdots, k$. The monotonicity of the w_j implies $\alpha_i \geq 0$, and then (48) implies

$$\alpha_0(w_0 x_0 + w_1 x_1 + \cdots + w_k x_k) \geq \lambda \alpha_0(w_0 y_0 + w_1 y_1 + \cdots + w_k y_k),$$
$$(49) \qquad \alpha_1(w_1 x_1 + \cdots + w_k x_k) \geq \lambda \alpha_1(w_1 y_1 + \cdots + w_k y_k),$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\alpha_k w_k x_k \geq \lambda \alpha_k w_k y_k.$$

Adding these inequalities we obtain $(x_0 + \cdots + x_k) \geq \lambda(y_0 + \cdots + y_k)$.

REMARK. We mention another inequality which may easily be derived from

(47). Let f_k be real numbers ($k = 0, 1, 2, \cdots$) such that $\sum_{k=0}^{\infty} \lambda^k f_k$ converges for every λ with $0 \leq \lambda < 1$, and let p_k be nonnegative and $p_0 > 0$. Then

$$(50) \qquad \sup_{0 \leq \lambda < 1} \frac{\sum_0^{\infty} \lambda^k f_k}{\sum_0^{\infty} \lambda^k p_k} \leq \sup_{0 \leq n < \infty} \frac{\sum_0^n f_k}{\sum_0^n p_k}.$$

Apply lemma 6 to the classical dominated ergodic theorem (see, for instance, Jacobs [17]) and observe that it is sufficient to prove this theorem for $f \geq 0$ and positive operators T. Then a dominated ergodic theorem with monotone weights follows.

Next let us apply lemma 6 to Hopf's maximal ergodic theorem [12]. Put for $f \in L_1$,

$$(51) \qquad E_n = \left\{ x \in X \colon \max_{0 < k \leq n} \sum_{i=0}^{k-1} T^i f(x) > 0 \right\},$$

$$(52) \qquad E_{w,n} = \left\{ x \in X \colon \max_{0 < k \leq n} \sum_{i=0}^{k-1} w_i T^i f(x) > 0 \right\}.$$

Hopf's theorem states that

$$(53) \qquad \int_{E_n} f \, d\mu \geq 0$$

for any $f \in L_1$ and $n \geq 1$. Applying lemma 6 we obtain $E_{w,n} \subseteq E_n$. Since $\{x \colon f(x) > 0\} \subseteq E_{w,n}$, (53) now implies $\int_{E_{w,n}} f \, d\mu \geq 0$. This is the desired maximal ergodic theorem with monotone weights. We mention that Garcia [8] has presented a very short proof of (53).

Rota's [21] basic lemma as well as his dominated ergodic theorem of the Abel type, follow in a similar way from (50). A generalization of the Riesz lemma may be obtained by applying the idea of the proof of lemma 6. An ergodic theorem for continuous flows and monotone weights w_t ($t \geq 0$) may be proved by using lemma 5 and an extension of the usual method applied to $w_t \equiv 1$ (see, for example, Jacobs [17]).

The main result of this section needs for its proof only the maximal ergodic theorem with monotone weights. However, as we assume knowledge of theorem C in this paper, the easiest derivation uses lemma 5.

Take $w = \{w_k\}$ as in (43) and define the operators $S_{w,n}$ in L_1 by

$$(54) \qquad S_{w,n} g = \sum_{k=0}^{n-1} w_k T^k g.$$

From theorem C and lemma 5 we infer that for $f, p \in L_1$, $p \geq 0$ the ratios

$$(55) \qquad Q_{w,n}(f, p) = S_{w,n} f / S_{w,n} p$$

converge to a finite limit on $\{x \colon \sum_{k=0}^{\infty} T^k p(x) > 0\}$. If $p_1, p_2 \in L_1$ are strictly positive in X, we derive from (55) with $f = p_1$, $p = p_2$ and with $f = p_2$, $p = p_1$ that $C_{w,i} = \{x \colon \sum_{k=0}^{\infty} w_k T^k p_i = \infty\}$ is independent of i. We define the *w-con-servative part C_w of X* (with respect to T) to be $C_{w,i}$. The set $D_w = X - C_w$

is called the *w-dissipative part of* X (with respect to T). We now generalize theorem B.

THEOREM 2. *If T is a positive contraction in L_1, then X is the disjoint union of two uniquely determined sets C_w and D_w such that*

(i) *for every $f \geq 0$, $\sum_{k=0}^{\infty} w_k T^k f$ converges in D_w;*

(ii) *for every $f \geq 0$, $\sum_{k=0}^{\infty} w_k T^k f$ diverges in $C_w \cap \{x: \sum_{k=0}^{\infty} T^i f > 0\}$;*

(iii) *C_w is invariant.*

PROOF. Properties (i) and (ii) are immediate by the same kind of argument which proved that $C_{w,1}$ equals $C_{w,2}$.

To prove (iii) we first remark that for any integrable g the inequality $Tg^+ \geq (Tg)^+$ holds since T preserves order. If $p = p_1$, $p^{w,k} = \sum_{i=0}^{k} w_i T^i p$, and $\bar{p} = \sum_{i=0}^{\infty} (w_i - w_{i+1}) T^{i+1} p$, then $\bar{p}$ is integrable; hence $Tp^{w,k} \leq p^{w,\infty} + \bar{p}$ is bounded in D_w.

If C_w is not invariant, then there exists an integrable $f \geq 0$ such that $f = 0$ in D_w and Tf is positive in a subset of D_w of positive measure. Then there is some $n \geq 1$ with

(56) $$\|(nTf - p^{w,\infty} - \bar{p})^+\| = a > 0.$$

Since $p^{w,k} \uparrow \infty$ in C_w,

(57) $$\|(nf - p^{w,k})^+\| \leq a/2$$

for sufficiently large k. The estimate

(58) $$\|(nTf - p^{w,\infty} - \bar{p})^+\| \leq \|(nTf - Tp^{w,k})^+\| \leq \|T(nf - p^{w,k})^+\|$$
$$\leq \|(nf - p^{w,k})^+\|$$

makes it evident that (57) contradicts (56). (This proof uses the same idea as known proofs of the special case $w_k \equiv 1$, but avoids unnecessary complications.)

We now describe C_w and D_w in terms of Λ and Φ, since Φ is the space of most interest if Λ is given by a stochastic kernel. Such a description will also be useful in the proof of theorem 3 and it seems to make the relation to the concept of recurrent states in the theory of Markov chains a bit more transparent. It is, however, equivalent to theorem 2.

THEOREM 2*. *If Λ is a positive contraction in Φ, then X is the disjoint union of two uniquely determined sets C_w and D_w such that*

(i) *for every $\varphi \in \Phi$ with $\varphi \geq 0$, D_w is a countable union of sets X_ν with the property $\sum_{k=0}^{\infty} w_k \Lambda^k \varphi(X_\nu) < \infty$;*

(ii) *for every $0 \leq \varphi \in \Phi$ and any $A \subseteq C_w$, the series $\sum_{k=0}^{\infty} w_k \Lambda^k \varphi(A)$ diverges or is 0;*

(iii) *C_w is invariant.*

PROOF. (i) Let $f = d\varphi/d\mu$ and $X_\nu = \{x: \nu \leq \sum_{k=0}^{\infty} w_k T^k f(x) < \nu + 1\}$, $\nu = 0, 1, \cdots$. Then (i) follows from $\Lambda^k \varphi(B) = \int_B T^k f \, d\mu$, and (ii) follows from the same relation, since $\sum_{k=0}^{\infty} w_k T^k f(x)$ diverges in $C_w \cap \{\sum_{k=0}^{\infty} T^k f(x) > 0\}$.

It would be of interest to know whether the decomposition $\{X_\nu\}$ of D_w (like that of N) may be given for all $\varphi \in \Phi$ simultaneously.

Let us now show that for any Λ the positive part P equals C_w for some $w = \{w_k\}$.

LEMMA 7. *Let $\{a_{m,k}\}$ $(m = 0, 1, \cdots \; ; \; k = 0, 1, \cdots)$ be an infinite matrix of nonnegative numbers such that*

$$
\text{(59)} \qquad \lim_{n \to \infty} n^{-1} \sum_{k=0}^{n-1} a_{m,k} = 0
$$

for any m. Then there exists a decreasing sequence $w = \{w_k\}$, $(k = 0, 1, \cdots)$ of positive numbers such that $\sum_{k=0}^{\infty} w_k$ diverges and $\sum_{k=0}^{\infty} w_k a_{m,k}$ converges for every m.

PROOF. Put $n_0 = 0$, and choose $n_1 \geq 1$ such that $n \geq n_1$ implies $n^{-1} \sum_0^{n-1} a_{0,k} < \frac{1}{2}$. If $n_1 < n_2 < \cdots < n_i$ are chosen, we next choose $n_{i+1} > n_i$ so large that $n_{i+1} - n_i > n_i - n_{i-1}$ and such that for $n \geq n_{i+1}$ and $m = 0, \cdots, i$,

$$
\text{(60)} \qquad (n - n_i)^{-1} \sum_{k=n_i}^{n-1} a_{m,k} < 2^{-(i+1)}.
$$

We put

$$
\text{(61)} \qquad w_0 = \cdots = w_{n_1-1} = n_1^{-1}, \cdots, w_{n_i} = \cdots = w_{n_{i+1}-1} = (n_{i+1} - n_i)^{-1}.
$$

Then $\{w_k\}$ decreases, $\sum_{k=0}^{\infty} w_k = \infty$ since any weight $(n_{i+1} - n_i)^{-1}$ occurs $(n_{i+1} - n_i)$ times, and we have

$$
\text{(62)} \qquad \sum_{k=n_i}^{n_{i+1}-1} w_k a_{m,k} = (n_{i+1} - n_i)^{-1} \sum_{k=n_i}^{n_{i+1}-1} a_{m,k} < 2^{-(i+1)}
$$

for $m = 0, \cdots, i$. Therefore $\sum_{k=0}^{\infty} w_k a_{m,k}$ converges for every m.

THEOREM 3. *The positive part P of X with respect to Λ is the intersection of all parts C_w for which $\sum_{k=0}^{\infty} w_k = \infty$. More precisely, for every Λ there exists a $w = \{w_k\}$ with (43) and $\sum_{k=0}^{\infty} w_k = \infty$ and $P = C_w$.*

PROOF. Let $\sum_{k=0}^{\infty} w_k = \infty$. If $0 \leq \varphi \in \Phi$ is invariant and equivalent to $\Gamma_P \mu$, then $\sum_{k=0}^{\infty} w_k \Lambda^k \varphi(A) = \sum_{k=0}^{\infty} w_k \varphi(A) = \infty$ for any $A \subseteq P$ with $\mu(A) > 0$. Therefore $P \subseteq C_w$.

Next let $\{X_i, i = 1, 2, \cdots$ be a decomposition of N such that

$$
\text{(63)} \qquad \lim_{n \to \infty} n^{-1} \sum_{k=0}^{n-1} \Lambda^k \mu(X_i) = 0.
$$

By lemma 7 we may find a sequence $w = \{w_k\}$ for which $\sum_{k=0}^{\infty} w_k = \infty$ and $\sum_{k=0}^{\infty} w_k \Lambda^k \mu(X_i) < \infty$ for all i. Theorem 2* (ii) now implies $\mu(X_i \cap C_w) = 0$. Hence, $N \cap C_w = 0$, or equivalently, $P \supseteq C_w$.

COROLLARY 1. *The following condition is necessary and sufficient for the existence of a finite invariant measure φ with $\mu \ll \varphi \ll \mu$: $\sum_{k=0}^{\infty} w_k T^k \chi_X$ diverges in X for every $\{w_k\}$ with (43) and $\sum_{k=0}^{\infty} w_k = \infty$.*

The next corollary may replace the ergodic theorem in those cases when the ergodic theorem does not hold. That occurs quite frequently according to the following result of A. Ionescu Tulcea [14]: among all positive, linear, invertible isometries T of the space $L_1\langle 0, 1 \rangle$ of Lebesgue-integrable functions on $\langle 0, 1 \rangle$, the operators T which do not satisfy the pointwise ergodic theorem form a set of the second category in the strong operator topology.

COROLLARY 2. *For any positive contraction T in L_1 there exists a decreasing sequence $w = \{w_k\}$, with $\sum_{k=0}^{\infty} w_k = \infty$, such that*

$$(64) \qquad \lim_{n \to \infty} \left(\sum_{k=0}^{n-1} w_k T^k f \right) \Big/ \left(\sum_{k=0}^{n-1} w_k \right)$$

exists a.e. for any $f \in L_1$. The limit is invariant, vanishes in N, and is given by (39) in P. In particular it is strictly positive in $P \cap \{\sum_{k=0}^{\infty} T^k f > 0\}$ for $f \geq 0$.

PROOF. Choose w with $P = C_w$. Then $\sum_{k=0}^{\infty} w_k T^k f$ converges in N. The other statements follow from lemma 5 and the considerations at the end of the proof of theorem 1.

REMARK. The assumption $\mu(X) = 1$ is not essential in this paper; μ may be σ-finite, since Φ depends only on the null sets of μ. Without this assumption, however, the formulation of theorem D and its applications is somewhat more complicated.

5. The elementary special case of Markov chains

We now adopt the terminology of Chung [5]. Let I be a countable state space, $(p_{i,j}^{(n)})$ $i, j \in I$ the matrix of the n-step transition probabilities of a stationary Markov chain, and $f_{i,j}^{(n)}$ the probability that starting at i the first visit to j takes place at time n. Let $w = \{w_k, k \geq 0\}$ with $w_k \geq w_{k+1} > 0$. We call the state $i \in I$

(i) *w-recurrent* in the case $\sum_{k=0}^{\infty} w_k p_{i,i}^{(k)} = \infty$, and

(ii) *w-nonrecurrent* otherwise.

In the special case $w_k \equiv 1$ we say recurrent and nonrecurrent respectively. The relation to the results of sections 3 and 4 becomes obvious if we put $X = I$ and let μ be a measure which assigns positive measure to every point. Then Φ consists of all signed finite measures on X. If Λ is generated by $(p_{i,j}^{(1)})$, then C_w consists of the w-recurrent states and P consists of the positive states. In particular, in this case, theorem 2* states the following.

The property of being w-recurrent or w-nonrecurrent is a class property. The set C_w of w-recurrent states is closed. Of course this also follows easily from estimates of the type

$$(65) \qquad p_{j,j}^{(m+k+n)} \leq p_{j,i}^{(m)} p_{i,i}^{(k)} p_{i,j}^{(n)}$$

and from $\sum_{k=0}^{\infty} (w_k - w_{k+1}) \leq w_0$.

Let us convince ourselves that the classification by the sets C_w may split the set of recurrent null states into proper subclasses.

EXAMPLE 1. There are Markov chains such that for one state i the first return probabilities $f_{i,i}^{(n)}$, $(n \geq 0)$ are an arbitrary probability distribution with $f_{i,i}^{(0)} = 0$. Simply take $i = 1$ and pass from 1 to state k with probability $f_{i,i}^{(k)}$; then from state $k \geq 2$ pass through $(k - 2)$ auxiliary states associated with k deterministically, finally going back to 1. The generating functions $F(s)$ and $P(s)$ of

$\{f_{i,i}^{(n)}\}$ and $\{p_{i,i}^{(n)}\}$ satisfy $P(s) = (1 - f(s))^{-1}$. Take $F(s) = 1 - (1 - s)^p$ for some p with $0 < p < 1$. Then

(66) $P(s) = (1 - s)^{-p} = \Gamma(p)^{-1} \sum\limits_{n=0}^{\infty} \Gamma(n + p)\Gamma(n + 1)^{-1}s^n$

has coefficients $p_{i,i}^{(n)}$ with $p_{i,i}^{(n)}\Gamma(p)n^{1-p} \to 1$ (see, for example, Titchmarsh [23], p. 57–58). We may construct a decomposable Markov chain which has only recurrent null states using two such chains, say with $p = \frac{1}{2}$ and $p = \frac{1}{3}$. However, only the states of the part with $p = \frac{1}{2}$ are w-recurrent with $w = \{(1 + k)^{-1/2}\}$. Of course one would not need monotone weights to separate the classes if the probabilities $p_{i,i}^{(n)}$ behave so regularly. The advantage of the classification C_w is its *general* applicability.

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