A THEOREM ON FUNCTIONS OF CHARACTERISTIC FUNCTIONS AND ITS APPLICATION TO SOME RENEWAL THEORETIC RANDOM WALK PROBLEMS

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Summary

This paper is concerned with three main problems and their interrelationship. A function $M(x)$ belongs to the "moment-class" $\mathfrak{M}^*$ if it is nonnegative and nondecreasing on $[0, \infty)$, if $M(x + y) \leq M(x)M(y)$ for all $x, y \geq 0$, and if $M(2x) = O(M(x))$ for all $x \geq 0$. The class $\mathcal{B}^\ddagger(M; \nu)$, for any real $\nu \geq 0$, is the Banach algebra of functions which are Fourier-Stieltjes transforms of functions $B(x)$ of bounded total variation such that $\int_{-\infty}^{+\infty} |x|^\nu M(|x|)|dB(x)| < \infty$. Our first main result, theorem 3, demonstrates that if a characteristic function belongs to $\mathcal{B}^\ddagger(M; \nu)$, then it has a Taylor expansion whose remainder term may involve a nonintegral power of $|\theta|$ and a member of some subalgebra $\mathcal{B}^\ddagger$ whose "parameters" depend on a variety of details which we suppress in this summary. A version of the Wiener-Pitt-Lévy theorem on analytic functions of functions of $\mathcal{B}^\ddagger(M; \nu)$ is then given, and from this and our results about Taylor expansions of characteristic functions, we obtain our second main result, a "Master Theorem" (theorem 1). This Master Theorem considers a certain rational form involving several characteristic functions and shows that under appropriate conditions it will be in some algebra $\mathcal{B}^\ddagger(M; \nu)$; the form has been chosen as being liable to arise in various investigations in the theory of random walks.

The Master Theorem and the results about characteristic functions are then applied to a general problem of a renewal-theoretic nature. Suppose $\{X_n\}$ is an infinite sequence of independent and identically distributed random variables such that $0 < \mathcal{E} X_n < \infty$. Let the characteristic function of X_n belong to $\mathcal{B}^\ddagger(M; \nu)$ for some $M \in \mathfrak{M}^*$ and some $\nu \geq 0$. Then what can be said about the asymptotic nature of, for instance,

$$S_\ell(x) \equiv \sum_{n=0}^{\infty} \binom{-(\ell - 1)}{n} P\{X_1 + X_2 + \cdots + X_n \leq x\},$$

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where $\ell \geq 1$ is not necessarily an integer? A variety of results, too detailed to be accurately described here, are obtained. These include more familiar results in renewal theory, but the behavior of $S_\ell(x)$ for nonintegral ℓ seems not to have been studied before. Another novelty of the present approach comes from the use of the algebras $\mathcal{B}^\ddagger$ which enable us to show the effect of the finiteness of quite general moments of the $\{X_n\}$ upon the remainder terms which arise in our "polynomial-type" approximations to $S_\ell(x)$. As a by-product of our results we are able to make some remarks about conditions which are necessary for the finiteness of $S_\ell(x)$; these subsidiary questions tie up with a line of research initiated by Hsu and Robbins. (See also note added in proof at end.)

1. Introduction and notation

One of our concerns in this work is to allow for the existence of moments, of our random variables, of a fairly general nature. For this reason we introduce a class of functions $\mathfrak{M}$ as follows.

DEFINITION 1. *The function $M(x)$, defined for all $x \geq 0$ belongs to $\mathfrak{M}$ if*

(i) *$M(x)$ is nondecreasing in $[0, \infty)$,*

(ii) *$M(x) \geq 1$, all $x \geq 0$,*

(iii) *$M(x + y) \leq M(x)M(y)$ for all $x, y \geq 0$.*

In fact, we are essentially concerned only with the character of $M(x)$ for large x and can conveniently specify a suitable $M(x)$ by stipulating its values for all sufficiently large x. Suppose we have a function $N(x)$ such that for some large $\Delta > 0$,

(i)* $N(x)$ is nondecreasing in $(0, \infty)$,

(ii)* $N(x) > 0$ for $x \geq \Delta$,

(iii)* for some $A > 0$ and all $x \geq \Delta$, $y \geq \Delta$, $N(x + y) \leq AN(x)N(y)$.

With no loss of generality we may suppose that $AN(\Delta) \geq 1$ and define

$$(1.1) \qquad \begin{aligned} M(x) &= AN(x), & x \geq \Delta, \\ &= M(\Delta), & x \leq \Delta. \end{aligned}$$

The function $M(x)$ so defined, as can be verified, belongs to $\mathfrak{M}$ and $M(x) \asymp N(x)$ for all large x (that is, $0 < \delta_1 < M(x)/N(x) < \delta_2$, for all large x).

Our interest in the class $\mathfrak{M}$ arises from the fact that if X and Y are independent random variables, then $\mathcal{E}M(|X| + |Y|) \leq \{\mathcal{E}M(|X|)\}\{\mathcal{E}M(|Y|)\}$. Two typical $\mathfrak{M}$-functions in which we might have interest are those asymptotically equal to e^x and to $x^{1/2} \log x$. An important special $\mathfrak{M}$-function is $I(x) \equiv 1$.

It will transpire that in certain cases we shall need to be specific about the rate of growth of $M(x) \in \mathfrak{M}$, compared with the rate of growth of x^ρ, $0 < \rho < 1$. We introduce three subclasses of $\mathfrak{M}$, which, though not exhaustive, would appear to cover all situations of interest:

$\mathfrak{M}_1(\rho)$: *if $x^\rho = O(M(x))$ as $x \to \infty$, then we say $M \in \mathfrak{M}_1(\rho)$;*

$\mathfrak{M}_2(\rho)$: *if $M(x)/x^\rho$ is nonincreasing for all large x, and as $x \to \infty$,*

$$\int_x^\infty \frac{M(u)}{u^{1+\rho}}\,du = O\left(\frac{M(x)}{x^\rho}\right), \tag{1.2}$$

then we say $M \in \mathfrak{M}_2(\rho)$.

(This will be the case, in particular, if, for some $\epsilon > 0$, $M(x)/x^{(\rho-\epsilon)}$ is nonincreasing for all large x.)

$\mathfrak{M}_3(\rho)$: *If* $M(x)/x^\rho$ *is nonincreasing for all large* x *and if we can find* $M_\rho(x) \in \mathfrak{M}$ *such that, as* $x \to \infty$,

$$\int_x^\infty \frac{M_\rho(u)}{u^{1+\rho}}\,du = O\left(\frac{M(x)}{x^\rho}\right), \tag{1.3}$$

then we say $M(x) \in \mathfrak{M}_3(\rho)$, *providing* $M(x) \notin \mathfrak{M}_2(\rho)$.

The class $\mathfrak{M}_3(\rho)$ represents a "critical-borderline" class between $\mathfrak{M}_1(\rho)$ and $\mathfrak{M}_2(\rho)$. A typical member of $\mathfrak{M}_3(\rho)$ might be given by $M(x) \sim x^\rho/(\log x)$. We could then take $M_\rho(x) \sim x^\rho/(\log x)^2$. Notice that $M_\rho(x)$ will always have the meaning developed here and that, always, $M_\rho(x) = O(M(x))$, as $x \to \infty$.

In part of our investigation it will also be necessary to suppose that the following condition is satisfied:

(iv) $M(2x) = O(M(x))$ *for all* $x \geq 0$.

The class of $M \in \mathfrak{M}$ which also satisfies (iv) will be called $\mathfrak{M}^*$; similarly for $\mathfrak{M}_1^*(\rho)$, and so on. It is an easy exercise to show that if $M \in \mathfrak{M}^*$, then $M(x) = O(x^N)$, as $x \to \infty$, for some large N. An $\mathfrak{M}$-function based on e^x is not in $\mathfrak{M}^*$; an $\mathfrak{M}$-function based on $x^{1/2}\log x$ is in $\mathfrak{M}^*$. (See note added in proof.)

We shall simply write $\mathfrak{L}$ for the class of functions more usually denoted $L_1(-\infty, +\infty)$; if $f(x) \in \mathfrak{L}$, we write $f^\dagger(\theta) = \int_{-\infty}^{+\infty} e^{i\theta x} f(x)\,dx$ for its Fourier transform; thus we shall write $\mathfrak{L}^\dagger$ for the class of functions which are Fourier transforms of functions of $\mathfrak{L}$. If $g(x)$ belongs to $\mathfrak{L}$ and, for some $M(x) \in \mathfrak{M}$, and some $\nu \geq 0$, $\int_{-\infty}^{+\infty} |x|^\nu M(|x|)|g(x)|\,dx < \infty$, then we shall say $g(x) \in \mathfrak{L}(M;\nu)$. In an obvious way, $\mathfrak{L}^\dagger(M;\nu)$ denotes the class of Fourier transforms of functions of $\mathfrak{L}(M;\nu)$. If $B(x)$ is a function of bounded variation, we write

$$B^\ddagger(\theta) = \int_{-\infty}^{+\infty} e^{i\theta x}\,dB(x) \tag{1.4}$$

for its Fourier-Stieltjes transform. The symbol $\mathfrak{D}$ shall denote the class of univariate right-continuous distribution functions, and we let $\mathfrak{D}^\ddagger$ denote the class of Fourier-Stieltjes transforms of functions in $\mathfrak{D}$; thus $\mathfrak{D}^\ddagger$ is the class of characteristic functions. The class of distribution functions $F(x)$, such that $\int_{-\infty}^{+\infty} |x|^\nu M(|x|)\,dF(x) < \infty$ will be denoted $\mathfrak{D}(M;\nu)$ and the corresponding class of characteristic functions will be $\mathfrak{D}^\ddagger(M;\nu)$. We write $\mathfrak{B}$ for the class of functions which are finite linear combinations, with possibly complex coefficients, of functions from $\mathfrak{D}$ (that is, $\mathfrak{B}$ is the class of complex-valued functions of bounded variation), and $\mathfrak{B}(M;\nu)$ for the class similarly derived from $\mathfrak{D}(M;\nu)$; the classes of Fourier-Stieltjes transforms corresponding to $\mathfrak{B}$ and $\mathfrak{B}(M;\nu)$ are denoted $\mathfrak{B}^\ddagger$ and $\mathfrak{B}^\ddagger(M;\nu)$, respectively. If $F \in \mathfrak{D}(I;\nu)$, it will occasionally prove convenient to write $\mu_\nu(F) = \int_{-\infty}^{+\infty} x^\nu\,dF(x)$ for the ν-th moment of F.

If $B_1(x)$ and $B_2(x)$ are any two members of $\mathcal{B}$, then we may define the product $B_1B_2(x)$, say, as the familiar Stieltjes convolution of $B_1(x)$ and $B_2(x)$. If $M(x) \in \mathfrak{M}$ and $B(x) \in \mathcal{B}(M; \nu)$ for some $\nu \geq 0$, we can define the norm of the function $B(x)$ in terms of some $K(x)$ in $\mathfrak{M}$ such that $K(x) \asymp x^\nu M(x)$, as

$$\|B\| = \int_{-\infty}^{+\infty} K(|x|)|dB(x)| \tag{1.5}$$

and, when $B_1(x)$ and $B_2(x)$ are both members of $\mathcal{B}(M; \nu)$, we shall have $\|B_1B_2\| \leq \|B_1\|\|B_2\|$. Thus $\mathcal{B}(M; \nu)$ can be regarded as a commutative Banach algebra, and this fact explains our special interest in $\mathcal{B}(M; \nu)$. However, we shall prove all our results without appeal to the general theory of these algebras.

Many-valued functions like z^α, for α nonintegral, will occur often in this paper; we must establish a satisfactory convention to prevent ambiguities from arising. It will be supposed that the complex plane is slit by removing the negative real axis from 0 to $-\infty$. The function z^α is then defined throughout the open slit plane by analytic continuation from the positive real axis, on which z^α is taken in a natural way to be real and positive. Thus we only attempt to define z^α when $|\arg z| < \pi$. However, z^α will be a one-valued analytic function throughout the open region on which we define it. In particular, the following simple and useful algorithm is true. If the arguments of z_1, z_2, and z_1z_2 all lie in the open interval $(-\pi, +\pi)$, then $z_1^\alpha z_2^\alpha = (z_1z_2)^\alpha$.

Let $F(x) \in \mathfrak{D}$ and let $F^\ddagger(\theta)$ be the corresponding Fourier-Stieltjes transform. If $F(x)$ has a nonnull absolutely continuous component, then we shall say $F(x)$ belongs to the class $\mathfrak{J}$ and $F^\ddagger(\theta)$ belongs to the class $\mathfrak{J}^\ddagger$. If $F^\ddagger(\theta)$ does not necessarily belong to $\mathfrak{J}^\ddagger$ but we can find an integer $\kappa \geq 1$ such that $\{F^\ddagger(\theta)\}^\kappa \in \mathfrak{J}^\ddagger$, then we say that $F^\ddagger(\theta)$ belongs to the class $\mathfrak{S}^\ddagger$ and $F(x)$ belongs to the class $\mathfrak{S}$. Thus $\mathfrak{J} \subset \mathfrak{S}$ and $\mathfrak{J}^\ddagger \subset \mathfrak{S}^\ddagger$. A related class $\mathcal{U}^\ddagger$ is the class of all characteristic functions $F^\ddagger(\theta)$ such that

$$\liminf_{|\theta| \to \infty} |1 - F^\ddagger(\theta)| > 0. \tag{1.6}$$

If $F^\ddagger(\theta) \in \mathcal{U}^\ddagger$, then we say $F(x) \in \mathcal{U}$. Presumably the class $\mathcal{U} - \mathfrak{S}$ is not empty (it is easy to see $\mathfrak{S} \subset \mathcal{U}$); Stone [13] calls $\mathcal{U}$ the class of strongly nonlattice distribution functions.

One of our main objects in this paper is to establish the following generalization of Smith [12].

THEOREM 1. *Suppose that, for some* $M(x) \in \mathfrak{M}^*$, *(possibly* $M \equiv I$*),*

1(i) $\alpha_1, \alpha_2, \cdots, \alpha_n, \beta_1, \beta_2, \cdots, \beta_m$ *are strictly positive real numbers, and we define* $\gamma = (\alpha_1 + \alpha_2 + \cdots + \alpha_n) + 2(\beta_1 + \beta_2 + \cdots + \beta_m)$;

1(ii) $\varphi_1(\theta), \varphi_2(\theta), \cdots, \varphi_n(\theta)$ *are characteristic functions of random variables with finite nonzero expectations* $\mu_1^{(1)}, \mu_1^{(2)}, \cdots, \mu_1^{(n)}$, *and these characteristic functions belong to* $\mathfrak{S}^\ddagger \cap \mathfrak{D}^\ddagger(M; 1)$;

1(iii) $\psi_1(\theta), \psi_2(\theta), \cdots, \psi_m(\theta)$ *are characteristic functions of random variables with zero means and finite, strictly positive, variances, and these characteristic functions belong to* $\mathfrak{S}^\ddagger \cap \mathfrak{D}^\ddagger(M; 2)$;

1(iv) $\lambda(\theta) \in \mathcal{B}^\ddagger(M, \gamma)$ *and* $\lambda(\theta) = O(|\theta|^\gamma)$ *as* $|\theta| \to 0$;
1(v) *If* γ *is an integer, then* $\sum_{j=1}^{n} \alpha_j \operatorname{sgn} \mu_1^{(j)} = \gamma \bmod 2$;
1(vi) $\Psi^\ddagger(\theta)$ *is defined for* $\theta \neq 0$ *by*

$$(1.7) \qquad \Psi^\ddagger(\theta) = \frac{\lambda(\theta)}{\prod_{r=1}^{n} [1 - \varphi_r(\theta)]^{\alpha_r} \prod_{s=1}^{m} [1 - \psi_s(\theta)]^{\beta_s}}$$

and $\Psi^\ddagger(0)$ *is defined to make* $\Psi^\ddagger(\theta)$ *continuous.*

Then we may draw the conclusion that $\Psi^\ddagger(\theta) \in \mathcal{B}^\ddagger(M; 0)$, *if* γ *is an integer. On the other hand, if* γ *is not an integer and* $k \geq 0$ *is the greatest integer not exceeding* γ, *set* $\rho = k + 1 - \gamma$. *We can then state the following:*

(a) *if* $M \in \mathfrak{M}_2^*(\rho)$, $\Psi^\ddagger(\theta) \in \mathcal{B}^\ddagger(M; 0)$;

(b) *if* $M \in \mathfrak{M}_3^*(\rho)$, $\Psi^\ddagger(\theta) \in \mathcal{B}^\ddagger(M_\rho; 0)$, *and in this case the conditions "*$\varphi_r(\theta) \in \mathfrak{D}(M; 1)$*" can be relaxed to "*$\varphi_r(\theta) \in \mathfrak{D}(M_\rho; 1)$*", with a similar relaxation of the conditions on the characteristic functions* $\psi_s(\theta)$;

(c) *if* $M \in \mathfrak{M}_1^*(\rho)$, $\Psi^\ddagger(\theta)$ *is the Fourier-Stieltjes transform of some* $\mathcal{B}$*-function* $\Psi(x)$. *However, if every* $\varphi_r(\theta) \in \mathfrak{D}^\ddagger(I, 2)$ *and every* $\psi_s(\theta) \in \mathfrak{D}^\ddagger(I, 3)$, *then as* $x \to \infty$,

$$(1.8) \qquad x^\rho\{\Psi(x) - \Psi(\infty)\} \to \frac{\sin\left(\frac{\pi}{2}\overline{c + \gamma}\right)}{\sin(\gamma\pi)} C,$$

where $c = \sum_{j=1}^{n} \alpha_j \operatorname{sgn} \mu_1^{(j)}$ *and*

$$(1.9) \qquad C = \lim_{\theta \to 0} \left\{\frac{\Psi^\ddagger(\theta) e^{\frac{1}{2}\pi i(\gamma - c)\operatorname{sgn}\theta}}{(-i\theta)^\rho \Gamma(1 - \rho)}\right\}.$$

This limit necessarily exists.

The need for the somewhat puzzling condition 1(v) is illustrated by the following example. Consider

$$(1.10) \qquad \Psi^\ddagger(\theta) = \frac{e^{i\theta} - 1}{[1 - (1 + i\theta)^{-1}]^{1/3}[1 - (1 - i\theta)^{-1}]^{2/3}}.$$

This $\Psi^\ddagger(\theta)$ satisfies all the requirements of theorem 1 except for 1(v). However, as $\theta \to 0$, $\Psi^\ddagger(\theta) \backsim \exp\{(2\pi i/3)(\operatorname{sgn}\theta)\}$ and is therefore intrinsically discontinuous at $\theta = 0$. Thus $\Psi^\ddagger(\theta)$ cannot possibly be the Fourier-Stieltjes transform of a function of bounded variation.

In order to establish theorem 1, we find it necessary to study Taylor expansions of characteristic functions, especially when moments of nonintegral order are known to exist. It also becomes necessary to establish the following sharpening of a well-known Wiener-Pitt-Lévy theorem.

If $\varphi(\theta)$ is a characteristic function, we shall write $1 - \partial[\varphi(\theta)]$ for the total weight of probability in the absolutely continuous component of the associated distribution. We then define

$$(1.11) \qquad \rho[\varphi(\theta)] = \inf_k \{\partial[\{\varphi(\theta)\}^k]\}^{1/k}$$

and note that $\rho[\varphi(\theta)] < 1$ if $\varphi(\theta) \in \mathfrak{S}^\ddagger$.

THEOREM 2. *Suppose that for some $M(x) \in \mathfrak{M}^*$, some $\nu > 0$, some closed finite interval J, both $\varphi(\theta)$ and $\psi(\theta)$ belong to $\mathfrak{B}^{\ddagger}(M;\nu)$, and suppose that $\psi(\theta)$ vanishes identically on the complement of J. Suppose that as θ runs through J, the point $z = \varphi(\theta)$ maps out an arc $\mathfrak{C}$ in the complex plane and that $\Phi(z)$ is analytic at every point of $\mathfrak{C}$. Then $\psi(\theta)\Phi(\varphi(\theta))$ also belongs to $\mathfrak{B}^{\ddagger}(M;\nu)$.*

Moreover, if $\varphi(\theta) \in \mathfrak{D}^{\ddagger}$, then the interval J may be infinite (or semi-infinite), provided that, in addition to the above conditions, no singularity of $\Phi(z)$ is within a distance $\rho[\varphi(\theta)]$ of the origin.

Concerning the Taylor expansion of characteristic functions we shall prove the following theorem.

THEOREM 3. *Let $F(x) \in \mathfrak{D}(I, \ell)$ for some $\ell > 0$, and let $k \geq 0$ be the greatest integer not exceeding ℓ. Set $\rho = k + 1 - \ell$.*

When ℓ is not an integer, we can choose any real constant c and have

$$F^{\ddagger}(\theta) = 1 + \sum_{j=1}^{k} \frac{\mu_j(F)}{j!} (i\theta)^j + |\theta|^{\ell} e^{ic \operatorname{sgn} \theta} s^{\dagger}(\theta) \tag{1.12}$$

where $s^{\dagger}(\theta) \in \mathfrak{L}^{\dagger}$ is the Fourier transform of some function $s(x) \in \mathfrak{L}$ such that $s^{\dagger}(0) = 0$ and

$$\Gamma(\ell + 1) \int_{-\infty}^{+\infty} |s(x)| \, dx \leq \frac{2^{(\ell-k+1)} \left\{ \left| \sin\left(\frac{\ell\pi}{2} - c\right) \right| + \left| \sin\left(\frac{\ell\pi}{2} + c\right) \right| \right\}}{|\sin \ell\pi|} \int_{-\infty}^{+\infty} |x|^{\ell} \, dF(x). \tag{1.13}$$

If it is additionally known that $F \in \mathfrak{D}(M;\ell)$ for some $M \in \mathfrak{M}$, then: (a) *when $M \in \mathfrak{M}_2(\rho)$ we have $s(x) \in \mathfrak{L}(M;0)$;* (b) *when $M \in \mathfrak{M}_3(\rho)$ we have $s(x) \in \mathfrak{L}(M_\rho;0)$;* (c) *when $M \in \mathfrak{M}_1(\rho)$ we have*

$$\Gamma(1 - \rho) x^{\rho} \int_x^{\infty} s(y) \, dy \to \frac{(-)^k \mu_{(k+1)}(F)}{(k+1)!} \frac{\sin\left(\frac{\ell\pi}{2} - c\right)}{\sin(\ell\pi)} \tag{1.14}$$

as $x \to \infty$.

On the other hand, if $\ell \geq 1$ and r is any integer, $1 \leq r \leq \ell$, for any $F^{\ddagger}(\theta) \in \mathfrak{D}^{\ddagger}(M;\ell)$, we have

$$F^{\ddagger}(\theta) = 1 + \sum_{j=1}^{r-1} \frac{\mu_j(F)}{j!} (i\theta)^j + \frac{(i\theta)^r}{r!} t_r^{\ddagger}(\theta), \tag{1.15}$$

where $t_r^{\ddagger}(\theta) \in \mathfrak{B}^{\ddagger}(M;\ell - r)$ is the Fourier transform of some function $t_r(x) \in \mathfrak{L}$ such that $t_r^{\ddagger}(0) = \mu_r(F)$ and

$$\int_{-\infty}^{+\infty} |t_r(x)| \, dx = \int_{-\infty}^{+\infty} |x|^r \, dF(x). \tag{1.16}$$

Indeed, when r is even or, if r is odd, when $F(x)$ refers to a nonnegative random variable, $t_r^{\ddagger}(\theta) = \mu_r(F) F_{(r)}^{\ddagger}(\theta)$, where $F_{(r)}^{\ddagger}(\theta) \in \mathfrak{D}^{\ddagger}(M;(\ell - r))$. In any case, $t_r^{\ddagger}(\theta)$ is expressible as the linear combination of two characteristic functions, one of which refers to a positive random variable and the other to a negative one.

With reference to (1.12) we observe that little has been published concerning the remainder term in the Taylor expansion of a characteristic function when a moment of nonintegral order is known to exist. Doob [2] and Loève [7] give some information on this topic; another important discussion is given in a little-known paper by Hsu [4]; our remainder terms are quite different from the ones obtained by these authors. The results concerning (1.15), where the remainder term involves an integral power of θ, are foreshadowed by similar results of Smith [11] and Pitman [8].

We believe that theorem 1 will ultimately find a number of applications in the study of random walks. However, in the remainder of the present paper we shall explore just one avenue of development. Let $\{X_n\}$ be a sequence of independent and identically distributed random variables; we do not suppose that these random variables are necessarily nonnegative. Write $S_n = X_1 + X_2 + \cdots + X_n$, $n = 1, 2, \cdots$, ad infinitum. Then we shall be concerned with finding conditions under which sums like

$$\Sigma_\ell(x) = \sum_{n=1}^{\infty} n^{(\ell-2)} P\{S_n \leq x\} \tag{1.17}$$

will be convergent, and, more especially, when (1.17) does converge, we shall also be concerned with the asymptotic behavior of $\Sigma_\ell(x)$ for large positive x. The constant ℓ in (1.17) may be any real number. However, if $\ell < 1$, the series $\sum n^{(\ell-2)}$ is convergent, and our problem becomes a trivial one. We shall therefore suppose $\ell \geq 1$.

In our study of $\Sigma_\ell(x)$ our first step will be to deduce from theorems 1, 2, and 3, the following.

THEOREM 4. *If $F(x) \in \mathfrak{D}(M; \ell)$ for some $M \in \mathfrak{M}^*$ and some $\ell > 1$ (possibly $M \equiv I$), and if $\mu_1 = \mu_1(F) \neq 0$, then there exist constants $A_1(\ell), A_2(\ell), \cdots, A_k(\ell)$, where k is the greatest integer not exceeding ℓ, such that*

$$\frac{1}{\{1 - F^\ddagger(\theta)\}^{(\ell-1)}} - \sum_{j=1}^{k} \frac{A_j(\ell)}{(-\mu_1 i\theta)^{(\ell-j)}} \tag{1.18}$$

tends to zero as $|\theta| \to 0$. Moreover, if we assume $F(x) \in \mathfrak{S}$, then: (a) *when ℓ is an integer we may conclude that* (1.18) *is a member of $\mathfrak{B}^\ddagger(M; 0)$;* (b) *when ℓ is not an integer and we write $\rho = k + 1 - \ell$, we have:* (i) *if $M \in \mathfrak{M}_2^*(\rho)$, then* (1.18) *belongs to $\mathfrak{B}^\ddagger(M; 0)$;* (ii) *if $M \in \mathfrak{M}_3^*(\rho)$, then* (1.18) *belongs to $\mathfrak{B}^\ddagger(M_\rho; 0)$;* (iii) *if $M \in \mathfrak{M}_1^*(\rho)$, then* (1.18) *is the Fourier-Stieltjes transform of some $\Lambda_\ell(x) \in \mathfrak{B}$ such that, as $x \to \infty$,*

$$\begin{aligned} x^\rho\{\Lambda_\ell(x) - \Lambda_\ell(\infty)\} &\to 0 \quad \text{if} \quad \mu_1 < 0, \\ &\to C_\ell \quad \text{if} \quad \mu_1 > 0, \end{aligned} \tag{1.19}$$

where $C_\ell = \lim_{\theta\to 0} \{\Lambda_\ell^\ddagger(\theta)/(-i\theta)^\rho\Gamma(1 - \rho)\}$, and this limit must exist.

In the work that follows it is convenient to use the special function $U(x) = P\{0 \leq x\}$. From theorem 4 we then deduce the following.

THEOREM 5. *Suppose that $\{X_n\}$ is a sequence of independent and identically* *distributed random variables with distribution function* $F(x) \in \mathfrak{D}(M;\ell)$ *for some* $M \in \mathfrak{M}^*$, *and some* $\ell > 1$. *Suppose further that* $\mu_1 = \mu_1(F) > 0$ *and that* $F(x) \in \mathfrak{S}$. *Then* $\Sigma_\ell(x)$ *is finite for every* x *and, if* k *is the integer part of* ℓ,

$$(1.20)\qquad \sum_{n=0}^{\infty} \binom{n+\ell-2}{n} P\{S_n \le x\} = \sum_{j=1}^{k} \frac{A_j(\ell)}{\Gamma(\ell-j+1)} \left(\frac{x}{\mu_1}\right)^{(\ell-j)} U(x) + \Lambda(x),$$

where $\Lambda(x)$ *is a function of bounded variation such that* $\Lambda(-\infty) = 0$ *and* $\Lambda^\ddagger(\theta)$ *is given by* (1.18), *and the conclusions of theorem* 4 *apply to* $\Lambda(x)$ *appropriately.*

For ease let us, in the future, write

$$(1.21)\qquad \mathcal{P}(x) = \sum_{j=1}^{k} \frac{A_j(\ell)}{\Gamma(\ell-j+1)} \left(\frac{x}{\mu_1}\right)^{(\ell-j)} U(x),$$

$$(1.22)\qquad H_1(x) = \sum_{n=0}^{\infty} \binom{n+\ell-2}{n} P\{S_n \le x\}.$$

The requirement $F(x) \in \mathfrak{S}$ is vital to our methods of proof. One suspects that some result like (1.20) should be true even if this requirement is dropped. Certainly when $\ell = 2$ we have the so-called "Second Renewal Theorem" without this requirement. However, if we relax the requirement a trifle to $F(x) \in \mathfrak{U}$, we can at least obtain the following corollary.

COROLLARY 5.1. *Suppose* $\nu \ge \ell > 1$ *and let* $F(x) \in \mathfrak{D}(I;\nu) \cap \mathfrak{U}$. *Write* m *for the integer part of* ν. *Let* $\kappa > 0$ *be arbitrarily small. Then* $\Sigma_\ell(x)$ *is finite and:*

(a) *if* ℓ *is an integer,*

$$(1.23)\qquad H_1(x) = \mathcal{P}(x) + \Omega(x) + r(x),$$

where $\Omega(x) \in \mathcal{B}(I;\nu-\ell)$ *and* $r(x) = O(x^{-(m+2-\ell)})$, *unless* $\ell = 2$, *in which case* $r(x) = O(x^{-(m-\kappa)})$;

(b) *if* ℓ *is not an integer but* $\nu < k+1$, *then* (1.23) *still holds, with* $\Omega(x) \in \mathcal{B}(I;\nu-\ell)$ *and* $r(x) = O(x^{-(m+2-\ell)})$ *if* $\ell > 2$, *but* $r(x) = O(x^{-(m-\kappa)})$ *if* $\ell < 2$;

(c) *if* ℓ *is not an integer and* $\nu \ge k+1$, (1.23) *holds with the same conditions on* $r(x)$ *as in case* (b), *with* $\Omega(x) \in \mathcal{B}$, *and* $x^\rho\{H_1(x) - \mathcal{P}(x) - \Omega(\infty)\} \to C_\ell$ *as* $x \to \infty$ *where* C_ℓ *and* ρ *are as for theorem* 4.

Corollary 5.1 shows that "polynomial-type" approximations to $H_1(x)$ are still possible if we only require $F(x) \in \mathfrak{U}$, as Stone [13] has done; indeed, this corollary generalizes some of Stone's results. However, if no such restraint is placed on $F(x)$, we are unable to obtain such precise information about $H_1(x)$. Nevertheless, a simple trick enables us to deduce from theorem 5 the following (note that $X_n^+ = X_n$ if $X_n \ge 0$ and $X_n^+ = 0$ if $X_n < 0$, $X_n^- = |X_n|$ if $X_n < 0$, $X_n^- = 0$ if $X_n \ge 0$).

COROLLARY 5.2. *Let* $\{X_n\}$ *be a sequence of independent and identically distributed random variables such that, for some* $\ell > 1$, $\mathcal{E}\{(X_n^-)^\ell\} < \infty$. *Assume* $0 < \mu_1 = \mathcal{E}X_n \le \infty$. *Then* $\Sigma_\ell(x)$ *is finite and*

$$(1.24)\qquad \Sigma_\ell(x) \sim \frac{x^{(\ell-1)}}{(\ell-1)\mu_1^{(\ell-1)}} \qquad \text{as} \quad x \to +\infty.$$

Under the conditions of theorem 5 or corollary 5.1 it is possible to prove rather more about the "remainder terms" of bounded variation than is actually given in these particular results. By way of example, we show that the following is a straightforward consequence of our work.

(A further example is given in the addendum at the end of this paper.)

COROLLARY 5.3. *Let $\ell \geq 2$ and $\Lambda(x)$ be the remainder function of bounded variation in theorem* 5. *Then, for any $h > 0$, we have:* (a) *if ℓ is an integer, $\Lambda(x + h) - \Lambda(x) \in \mathcal{B}(M; 1)$;* (b) *if ℓ is not an integer,*

$$x^{\rho}\{\Lambda(x + h) - \Lambda(x)\} \to 0 \qquad \text{as} \quad x \to \pm\infty. \tag{1.25}$$

These remarks also apply to the remainder term $\Omega(x)$ of corollary 5.1.

A special case of this result, when $\ell = 2$, provides information about the familiar Blackwell limit theorem. If we set

$$\beta(x) = \sum_{n=0}^{\infty} P\{x < S_n \leq x + h\}, \tag{1.26}$$

we have, under the conditions of theorem 5, that $\beta(x) = (h/\mu_1) + \delta(x)$ where $\delta(x) \to 0$ as $x \to \infty$ and $\delta(x) \in \mathcal{B}(M; 1)$. Thus, for instance, if it is known that $\mathcal{E}|X_n|^{\nu} < \infty$, for some not necessarily integer-valued $\nu \geq \ell$, then we have among other things, that $\delta(x) = o(x^{-(\nu-1)})$. It is clear that under the conditions of corollary 5.1 our conclusions about $\delta(x)$ need some modification, but we can still show that $\delta(x) = o(x^{-(\nu-1)})$. Thus we see that corollary 5.3 generalizes in some directions certain other results of Stone [13] about the Blackwell theorem.

Since this work was started, a paper by A. A. Borovkov [1] has appeared. Borovkov is concerned solely with the quantity we have chosen to call $\beta(x)$ and with the case of lattice-valued random variables. We make the important remark at this place that *all the work of this paper can be repeated for the lattice case without difficulty;* in fact, the lattice case will be easier because we shall be spared the complications about absolute continuity and the class $\mathfrak{S}$. Borovkov also uses a sharpened form of the Wiener-Pitt-Lévy theorem (different from ours) and makes use of functions of slow growth (where we use the class $\mathfrak{M}^*$). Concentrating as he does on a more particular problem than we have, Borovkov obtains a variety of detailed results about $\beta(x)$. However, our results on $\beta(x)$, stemming from corollary 5.3, seem no weaker than corresponding ones of Borovkov.

Theorem 5 and its corollaries exclude the possibility $\ell = 1$; this case is not without interest, but needs special arguments. We prove the following theorem.

THEOREM 6. *Let the sequence $\{X_n\}$ have a distribution function $F(x) \in \mathfrak{S} \cap \mathfrak{D}(M; \ell)$ for some $M \in \mathfrak{M}^*$ and some $\ell \geq 1$. Let $\mu_1 = \mathcal{E}X_n > 0$; then*

$$\sum_{n=1}^{\infty} \frac{1}{n} P\{S_n \leq x\} = Q(x) + \Lambda(x), \tag{1.27}$$

where

$$\begin{aligned} Q(x) &= 0, && \text{for} \quad x \leq 0, \\ Q(x) &= \int_0^{x/\mu_1} \frac{1 - e^{-t}}{t}\, dt, && \text{for} \quad x > 0, \end{aligned} \tag{1.28}$$

and $\Lambda(x)$ belongs to $\mathcal{B}(M; \ell - 1)$ and tends to zero as $|x| \to \infty$.

COROLLARY 6.1. *Let the sequence $\{X_n\}$ merely be such that $\mathcal{E}\{X_n^-\} < \infty$ and $0 < \mathcal{E}X_n \leq \infty$. Then $\Sigma_1(x)$ is finite and when $0 < \mathcal{E}X_n < \infty$,*

$$\Sigma_1(x) \sim \log x \qquad \text{as} \quad x \to \infty. \tag{1.29}$$

When $\mathcal{E}X_n = \infty$ we may merely conclude that

$$\limsup_{x\to\infty} \frac{\Sigma_1(x)}{\log x} \leq 1. \tag{1.30}$$

The result (1.29) is actually a very special case of a general theorem proved elsewhere (Smith, [12]). However, in that latter work an appeal is made to the following corollary (an easy consequence of our results).

COROLLARY 6.2. *Suppose X_0 is a random variable which is independent of the variables $\{X_n\}$. Then, under the conditions of corollary 5.2,*

$$\sum_{n=1}^{\infty} n^{(\ell-2)} P\{X_0 + S_n \leq x\} \tag{1.31}$$

will be finite if $\mathcal{E}\{(X_0^-)^{(\ell-1)}\} < \infty$. Under the conditions of corollary 6.1, (1.31) will be finite (with $\ell = 1$), if $\mathcal{E}\{\log (1 + X_0^-)\} < \infty$.

The question naturally presents itself as to whether the conditions of corollaries 5.2 and 6.1 are necessary for the convergence of the series (1.17) or whether they are unnaturally restrictive. We do not know the full answer to this question, but the following final theorem suggests that our conditions are reasonable.

THEOREM 7. *Suppose that $P\{X_n = 0\} \neq 1$ and that, for some $\ell \geq 1$ and some finite x, the series (1.17) converges. Then this series converges for every x. If, in addition to the convergence of (1.17), it is given that $\mathcal{E}X_n$ is finite (and it then follows trivially that $\mathcal{E}X_n > 0$), then it necessarily follows that $\mathcal{E}(X_n^-)^\ell$ is finite. On the other hand, there exists a sequence $\{X_n\}$ for which both $\mathcal{E}X_n^+$ and $\mathcal{E}X_n^-$ are infinite, and yet (1.17) converges.*

It should be pointed out that questions of the finiteness of our series (1.17) have a bearing on the theory of "complete convergence" developed by Hsu and Robbins [5] and further studied by Erdös [3] and by Katz [6]. The finiteness of (1.17) could, in fact, be deduced from the direct theorems of Katz; however, here it has been a consequence of our detailed Fourier analysis of $\Sigma_\ell(x)$. Furthermore, theorem 7 shows that the "one-sided" problem considered by us can exhibit features which do not arise in the "two-sided" set-up considered by the aforementioned authors.

2. Functions of characteristic functions

By arguments very similar to those in section 3 of Smith [11], or Pitman [8], we have the following.

LEMMA 1. *Let $r \geq 1$ be an integer and $M(x) \in \mathfrak{M}$. Let $F(x) \in \mathfrak{D}(M; \ell)$, where ℓ is not necessarily an integer and $\ell \geq r$, refer to a nonnegative random variable. Then there exists an $F_{(r)}(x) \in \mathfrak{D}(M; (\ell - r))$, also associated with a nonnegative random variable, such that*

$$F^{\ddagger}(\theta) = 1 + \sum_{j=1}^{(r-1)} \frac{\mu_j(F)}{j!} (i\theta)^j + \frac{\mu_r(F)}{r!} (i\theta)^r F^{\ddagger}_{(r)}(\theta), \tag{2.1}$$

and $F_{(r)}(x)$ is absolutely continuous with a density function which is monotonically decreasing for positive x.

In conjunction with this known result we need the following new one which extends our scope to expansions of $F^{\ddagger}(\theta)$ which terminate with terms involving fractional powers of θ.

LEMMA 2. *If $F(x) \in \mathfrak{D}(I, \ell)$ for some $0 < \ell < 1$, then for any prescribed real constant γ there is a function $s^{\dagger}(\theta) \in \mathfrak{L}^{\dagger}$ such that*

$$F^{\ddagger}(\theta) = 1 + |\theta|^{\ell} e^{i\gamma \operatorname{sgn} \theta} s^{\dagger}(\theta). \tag{2.2}$$

If $s(x)$ is an $\mathfrak{L}$-function with Fourier transform $s^{\dagger}(\theta)$, then $\int_{-\infty}^{+\infty} s(x)\, dx = 0$ and

$$\Gamma(\ell + 1) \int_{-\infty}^{+\infty} |s(x)|\, dx \leq C(\ell, \gamma) \int_{-\infty}^{+\infty} |x|^{\ell}\, dF(x), \tag{2.3}$$

where $|\sin \ell\pi| C(\ell, \gamma) = 2^{\ell+1}\{|\sin((\ell\pi/2) + \gamma)| + |\sin((\ell\pi/2) - \gamma)|\}$.

If it is additionally known that, in fact, $F(x) \in \mathfrak{D}(M; \ell)$, for some $M \in \mathfrak{M}$, then: (a) *when $M \in \mathfrak{M}_2(1 - \ell)$, we have $s(x) \in \mathfrak{L}(M; 0)$;* (b) *when $M \in \mathfrak{M}_3(1 - \ell)$, we have $s(x) \in \mathfrak{L}(M_{(1-\ell)}; 0)$;* (c) *when $M \in \mathfrak{M}_1(1 - \ell)$ we have, as $x \to \infty$,*

$$x^{(1-\ell)} \int_x^{\infty} s(y)\, dy \to \frac{\sin\left(\frac{\ell\pi}{2} - \gamma\right)}{\Gamma(\rho) \sin \ell\pi} \int_{-\infty}^{+\infty} x\, dF(x). \tag{2.4}$$

PROOF. For any $\Delta \geq 0$ define the function

$$h_{\Delta}(x) = \frac{1}{x^{(1-\ell)}} - \int_{-\infty}^{x-\Delta} \frac{dF(z)}{(x - z)^{(1-\ell)}}, \qquad x > \Delta, \tag{2.5}$$

$$h_{\Delta}(x) = -\int_{-\infty}^{x-\Delta} \frac{dF(z)}{(x - z)^{(1-\ell)}}, \qquad x \leq \Delta. \tag{2.6}$$

We first show that $h_{\Delta}(x) \in \mathfrak{L}$. If $x \leq \Delta$, then $h_{\Delta}(x)$ is negative, and an application of Fubini's Theorem will show that

$$\int_{-\infty}^{\Delta} |h_{\Delta}(x)|\, dx = \frac{1}{\ell} \int_{-\infty}^{0} \{(\Delta + |z|)^{\ell} - \Delta^{\ell}\}\, dF(z) < \infty. \tag{2.7}$$

On the other hand, when $x > \Delta$ we can write

$$h_{\Delta}(x) = g_1(x) + g_2(x) + g_3(x) - g_4(x) - g_5(x), \tag{2.8}$$

where the functions $g_i(x)$, $i = 1, 2, \cdots, 5$, are all nonnegative, and are given by the following equations:

$$g_1(x) = \frac{1 - F(x - \Delta)}{x^{(1-\ell)}}, \tag{2.9}$$

$$g_2(x) = \frac{F(-x)}{x^{(1-\ell)}}, \tag{2.10}$$

$$g_3(x) = \int_{-x}^{0} \left[\frac{1}{x^{(1-\ell)}} - \frac{1}{(x-z)^{(1-\ell)}}\right] dF(z), \tag{2.11}$$

$$g_4(x) = \int_{0}^{x-\Delta} \left[\frac{1}{(x-z)^{(1-\ell)}} - \frac{1}{x^{(1-\ell)}}\right] dF(z), \tag{2.12}$$

$$g_5(x) = \int_{-\infty}^{-x} \frac{dF(z)}{(x-z)^{(1-\ell)}}. \tag{2.13}$$

If we use integration by parts it is an easy matter to verify that

$$\int_{\Delta}^{\infty} g_1(x)\, dx = \frac{1}{\ell}\int_{0}^{\infty} [(x+\Delta)^{\ell} - \Delta^{\ell}]\, dF(x), \tag{2.14}$$

$$\int_{\Delta}^{\infty} g_2(x)\, dx = \frac{1}{\ell}\int_{-\infty}^{-\Delta} [|x|^{\ell} - \Delta^{\ell}]\, dF(x). \tag{2.15}$$

A further appeal to Fubini's Theorem, followed by some slight rearrangement will show that

$$\int_{\Delta}^{x} g_4(y)\, dy = \frac{1}{\ell}\int_{0}^{x-\Delta} [(z+\Delta)^{\ell} - \Delta^{\ell}]\, dF(z) - \frac{1}{\ell}\int_{0}^{x-\Delta} [x^{\ell} - (x-z)^{\ell}]\, dF(z). \tag{2.16}$$

At this point it is convenient to state and prove the next lemma.

LEMMA 3. *If $m(n) > n$ and $m(n) - n = o(n^{(1-\ell)})$ as $n \to \infty$, then $[m(n)]^{\ell} - n^{\ell} \to 0$ as $n \to \infty$ and $[m(n)]^{\ell} - n^{\ell} \le [m(n) - n]^{\ell}$ for every n.*

PROOF OF LEMMA 3. We observe that $[m(n)]^{\ell} - n^{\ell} = \ell \int_{n}^{m(n)} (dy/y^{(1-\ell)})$ and that the integrand $y^{-(1-\ell)}$ is a strictly decreasing function. Thus, for example,

$$[m(n)]^{\ell} - n^{\ell} < \frac{\ell[m(n) - n]}{n^{(1-\ell)}} \to 0 \qquad \text{as} \quad n \to \infty. \tag{2.17}$$

On the other hand,

$$[m(n)]^{\ell} - n^{\ell} < \ell \int_{0}^{m(n)-n} \frac{dy}{y^{(1-\ell)}} = [m(n) - n]^{\ell}, \tag{2.18}$$

which proves lemma 3.

To return to the proof of lemma 2, if we now appeal to lemma 3, we can see that $x^{\ell} - (x-z)^{\ell} \le z^{\ell}$ for all $0 \le z \le x$ and $x^{\ell} - (x-z)^{\ell} \to 0$ as $x \to \infty$ for z fixed. Hence, by dominated convergence

$$\int_{0}^{x-\Delta} [x^{\ell} - (x-z)^{\ell}]\, dF(z) \to 0 \qquad \text{as} \quad x \to \infty. \tag{2.19}$$

Therefore, we may deduce from (2.16) that

$$\int_{\Delta}^{\infty} g_4(x)\, dx = \frac{1}{\ell}\int_{0}^{\infty} [(z+\Delta)^{\ell} - \Delta^{\ell}]\, dF(z). \tag{2.20}$$

Again, by using Fubini's Theorem one can show that

$$\int_{\Delta}^{x} g_3(y)\,dy = \frac{2^{\ell}-1}{\ell}\int_{-x}^{-\Delta} |z|^{\ell}\,dF(z) + \frac{1}{\ell}\int_{-\Delta}^{0} [(\Delta - z)^{\ell} - \Delta^{\ell}]\,dF(z) - \frac{1}{\ell}\int_{-x}^{0} [(x - z)^{\ell} - x^{\ell}]\,dF(z). \tag{2.21}$$

Lemma 3 shows that $(x - z)^{\ell} - x^{\ell} \le |z|^{\ell}$ whenever $z \le 0$ and that $(x - z)^{\ell} - x^{\ell} \to 0$ as $x \to \infty$ for z fixed and negative. Therefore, again by dominated convergence,

$$\int_{-x}^{0} [(x - z)^{\ell} - x^{\ell}]\,dF(z) \to 0 \qquad \text{as} \quad x \to \infty, \tag{2.22}$$

and we conclude that

$$\int_{\Delta}^{\infty} g_3(x)\,dx = \frac{2^{\ell}-1}{\ell}\int_{-\infty}^{-\Delta} |z|^{\ell}\,dF(z) + \frac{1}{\ell}\int_{-\Delta}^{0} [(\Delta - z)^{\ell} - \Delta^{\ell}]\,dF(z). \tag{2.23}$$

Finally, we notice that $g_5(x) \le g_2(x)$ so that $g_5(x) \in L_1(\Delta, \infty)$. One more appeal to Fubini's Theorem will then establish that

$$\int_{\Delta}^{\infty} g_5(x)\,dx = \frac{2^{\ell}}{\ell}\int_{-\infty}^{-\Delta} |z|^{\ell}\,dF(z) - \frac{1}{\ell}\int_{-\infty}^{-\Delta} (\Delta - z)^{\ell}\,dF(z). \tag{2.24}$$

If we now combine our findings concerning the various functions $g_1(x), \cdots, g_5(x)$, we see that $h_{\Delta}(x) \in \mathfrak{L}$ as claimed, and we also see that $\int_{-\infty}^{+\infty} h_{\Delta}(x)\,dx = 0$. Therefore,

$$\begin{aligned}\int_{-\infty}^{+\infty} |h_{\Delta}(x)|\,dx &\le 2\int_{\Delta}^{\infty} \{g_1(x) + g_2(x) + g_3(x)\}\,dx \\ &\le \frac{2}{\ell}\int_{-\Delta}^{\infty} [(\Delta + |z|)^{\ell} - \Delta^{\ell}]\,dF(z) + \frac{2^{\ell+1}}{\ell}\int_{-\infty}^{-\Delta} |z|^{\ell}\,dF(z),\end{aligned} \tag{2.25}$$

ignoring a negative term. But, again by our lemma, $(\Delta + |z|)^{\ell} - \Delta^{\ell} \le |z|^{\ell}$ and $(\Delta + |z|)^{\ell} - \Delta^{\ell} \to 0$ as $\Delta \to \infty$, $|z|$ fixed. Hence we can deduce from dominated convergence that

$$\int_{-\infty}^{+\infty} |h_{\Delta}(x)|\,dx \to 0, \qquad \text{as} \quad \Delta \to \infty. \tag{2.26}$$

We are especially concerned with the function $h_0(x)$, which is necessarily a member of $\mathfrak{L}$ by the preceding argument, and we require the Fourier transform $h_0^{\dagger}(\theta)$. In view of (2.26), we see that

$$\begin{aligned}h_0^{\dagger}(\theta) &= \lim_{\Delta\to\infty}\int_{-\infty}^{+\infty} e^{i\theta x}[h_0(x) - h_{\Delta}(x)]\,dx \\ &= \lim_{\Delta\to\infty}\lim_{T\to\infty}\int_{-T}^{+T} e^{i\theta x}[h_0(x) - h_{\Delta}(x)]\,dx.\end{aligned} \tag{2.27}$$

But, for large T,

$$(2.28)\qquad \int_{-T}^{+T} e^{i\theta x}[h_0(x) - h_\Delta(x)]\,dx = \int_0^\Delta \frac{e^{i\theta x}}{x^{(1-\ell)}}\,dx - \int_{-T}^{+T} e^{i\theta x}\left\{\int_{x-\Delta}^{x} \frac{dF(z)}{(x-z)^{(1-\ell)}}\right\}dx.$$

Let us call the double integral on the right of this last equation J. If we rearrange the order of integration in J, the resulting double integral is easily seen to be absolutely convergent. Thus we obtain

$$(2.29)\qquad \begin{aligned} J = &\int_{-T-\Delta}^{-T}\left\{\int_{-T}^{z+\Delta} \frac{e^{i\theta x}}{(x-z)^{(1-\ell)}}\,dx\right\}dF(z) \\ &+ \int_{-T}^{T-\Delta}\left\{\int_{z}^{z+\Delta} \frac{e^{i\theta x}}{(x-z)^{(1-\ell)}}\,dx\right\}dF(z) \\ &+ \int_{T-\Delta}^{T}\left\{\int_{z}^{T} \frac{e^{i\theta x}}{(x-z)^{(1-\ell)}}\,dx\right\}dF(z). \end{aligned}$$

If we substitute $x = z + u$ in the three inner integrals and rearrange the orders of integration of the resulting absolutely convergent double integrals, it transpires that

$$(2.30)\qquad J = \int_0^\Delta \frac{e^{i\theta u}}{u^{(1-\ell)}}\left\{\int_{-T-u}^{T-u} e^{i\theta z}\,dF(z)\right\}du.$$

However,

$$(2.31)\qquad \int_{-T-u}^{T-u} e^{i\theta z}\,dF(z) \to \varphi(\theta), \qquad \text{boundedly, as } T \to \infty.$$

Therefore, by bounded convergence,

$$(2.32)\qquad \lim_{T\to\infty} J = \varphi(\theta)\int_0^\Delta \frac{e^{i\theta u}}{u^{(1-\ell)}}\,du.$$

Hence

$$(2.33)\qquad \begin{aligned} h_0^\dagger(\theta) &= \lim_{\Delta\to\infty}\,[1 - \varphi(\theta)]\int_0^\Delta \frac{e^{i\theta u}}{u^{(1-\ell)}}\,du \\ &= \frac{\Gamma(\ell)}{|\theta|^\ell}\,e^{(1/2)\ell\pi i\,\mathrm{sgn}\,\theta}\,[1 - \varphi(\theta)]. \end{aligned}$$

In a similar way we can consider the function

$$(2.34)\qquad \begin{aligned} k_0(x) &= -\int_x^\infty \frac{dF(z)}{(z-x)^{(1-\ell)}}, && x \geq 0, \\ &= \frac{1}{|x|^{(1-\ell)}} - \int_x^\infty \frac{dF(z)}{(z-x)^{(1-\ell)}}, && x < 0, \end{aligned}$$

and show that $k_0(x) \in \mathfrak{L}$; that $\int_{-\infty}^{+\infty} k_0(x)\,dx = 0$; and that the Fourier transform of $k_0(x)$ is

$$k_0^\dagger(\theta) = \frac{\Gamma(\ell)}{|\theta|^\ell} e^{-\frac{1}{2}\ell\pi i \operatorname{sgn}\theta}[1 - \varphi(\theta)]. \tag{2.35}$$

Let us now define

$$s^\dagger(\theta) = \frac{(-1)}{\Gamma(\ell)}\left\{\frac{\sin\left(\frac{\ell\pi}{2} - \gamma\right)}{\sin(\ell\pi)} h_0^\dagger(\theta) + \frac{\sin\left(\frac{\ell\pi}{2} + \gamma\right)}{\sin(\ell\pi)} k_0^\dagger(\theta)\right\}. \tag{2.36}$$

Then $s^\dagger(\theta)$ belongs to $\mathfrak{L}^\dagger$ and, by our previous results,

$$s^\dagger(\theta) = \frac{\{1 - \varphi(\theta)\}}{|\theta|^\ell} T(\theta), \tag{2.37}$$

where

$$\begin{aligned} T(\theta) &= \frac{\sin\left(\frac{\ell\pi}{2} - \gamma\right)}{\sin(\ell\pi)} e^{\frac{1}{2}\ell\pi i \operatorname{sgn}\theta} + \frac{\sin\left(\frac{\ell\pi}{2} + \gamma\right)}{\sin(\ell\pi)} e^{-\frac{1}{2}\ell\pi i \operatorname{sgn}} \\ &= e^{-i\gamma \operatorname{sgn}\theta}. \end{aligned} \tag{2.38}$$

Thus (2.2) is established, and it is obvious at this stage that $\int_{-\infty}^{+\infty} s(x)\,dx = 0$. We do not need (2.3) in the sequel and have only mentioned it for completeness' sake; it can be demonstrated by routine computations based upon the fact that

$$\int_{-\infty}^{+\infty} |h_0(x)|\,dx \le \frac{2^{\ell+1}}{\ell}\int_{-\infty}^{+\infty} |x|^\ell\,dF(x), \tag{2.39}$$

which can be inferred from the previous argument.

Now suppose that $F(x) \in \mathfrak{D}(M;\ell)$ where $M \in \mathfrak{M}_2(1-\ell)$. To prove (2) of lemma 2 it will be enough to establish that $M(|x|)h_0(x) \in \mathfrak{L}$. For $x > 0$,

$$\begin{aligned} &M(x)h_0(x) \\ &\quad = \frac{M(x)}{x^{(1-\ell)}}\{1 - F(x)\} + \int_{-\infty}^{x}\left\{\frac{1}{x^{(1-\ell)}} - \frac{1}{(x-z)^{(1-\ell)}}\right\} M(x)\,dF(z) \\ &\quad = a_1(x) + a_2(x), \quad \text{say.} \end{aligned} \tag{2.40}$$

An integration by parts will easily establish that $a_1(x) \in L_1(0,\infty)$, if we use the inequality

$$\int_0^x \frac{M(u)}{u^{(1-\ell)}}\,du \le \frac{M(x)x^\ell}{\ell}. \tag{2.41}$$

Next we observe that

$$\begin{aligned} &\int_0^\infty |a_2(x)|\,dx \\ &\quad \le \int_{-\infty}^{0}\int_0^\infty + \int_0^\infty\int_z^\infty \left|\frac{1}{x^{(1-\ell)}} - \frac{1}{(x-z)^{(1-\ell)}}\right| M(x)\,dx\,dF(z) \\ &\quad = \int_{-\infty}^{0} J_1(z)\,dF(z) + \int_0^\infty J_2(z)\,dF(z), \text{ say.} \end{aligned} \tag{2.42}$$

For $x > 0$,

$$\left|\frac{1}{x^{(1-\ell)}} - \frac{1}{(x + |z|)^{(1-\ell)}}\right| < \min\left\{\frac{1}{x^{(1-\ell)}}, \frac{(1-\ell)|z|}{x^{(2-\ell)}}\right\}. \tag{2.43}$$

Thus

$$\begin{aligned} J_1(z) &\leq \int_0^{|z|} \frac{M(x)}{x^{(1-\ell)}}\, dx + \int_{|z|}^{\infty} \frac{(1-\ell)|z| M(x)}{x^{(2-\ell)}}\, dx \\ &= \frac{|z|^{\ell}}{\ell} M(|z|) + O(|z|^{\ell} M(|z|)), \end{aligned} \tag{2.44}$$

in view of the fact that $M \in \mathfrak{M}_2(1-\ell)$. Thus $\int_{-\infty}^{0} J_1(z)\, dF(z) < \infty$, because $F \in \mathfrak{D}(M;\ell)$.

To deal with $J_2(z)$, let us suppose that

$$\int_z^{\infty} \frac{M(x)}{x^{(2-\ell)}}\, dx < \frac{AM(z)}{z^{(1-\ell)}} \tag{2.45}$$

for all $z \geq \frac{1}{2}\Delta$, where A and Δ are some constants; such a hypothesis must hold because $M \in \mathfrak{M}_2(1-\ell)$. For $x \geq 2z > 0$ we have

$$\frac{1}{(x-z)^{(1-\ell)}} - \frac{1}{x^{(1-\ell)}} < \frac{2^{(2-\ell)}(1-\ell)z}{x^{(2-\ell)}} \leq \frac{2^{(1-\ell)}(1-\ell)}{x^{(1-\ell)}}. \tag{2.46}$$

Thus, if $2z \leq \Delta$, we have

$$\begin{aligned} &\int_{2z}^{\infty} \left\{\frac{1}{(x-z)^{(1-\ell)}} - \frac{1}{x^{(1-\ell)}}\right\} M(x)\, dx \\ &\qquad \leq 2^{(1-\ell)}(1-\ell) M(\Delta) \int_{2z}^{\Delta} \frac{dx}{x^{(1-\ell)}} + 2^{(2-\ell)}(1-\ell) A M(\Delta)\Delta \\ &\qquad = O(1); \end{aligned} \tag{2.47}$$

and if $2z > \Delta$, we have

$$\begin{aligned} \int_{2z}^{\infty} \left\{\frac{1}{(x-z)^{(1-\ell)}} - \frac{1}{x^{(1-\ell)}}\right\} M(x)\, dx &\leq 2^{(2-\ell)}(1-\ell) z \int_z^{\infty} \frac{M(x)}{x^{(2-\ell)}}\, dx \\ &= O(z^{\ell} M(z)), \end{aligned} \tag{2.48}$$

in view of our assumptions about $M(x)$. Furthermore, $-M(2x) \leq 2^{(1-\ell)} M(x)$, and

$$\int_z^{2z} \left\{\frac{1}{(x-z)^{(1-\ell)}} - \frac{1}{x^{(1-\ell)}}\right\} M(x)\, dx \leq M(2z)\left\{\frac{2-2^{\ell}}{\ell} z^{\ell}\right\}, \tag{2.49}$$

on performing a simple integration. Thus we may conclude that $J_2(z) = O(1 + z^{\ell} M(z))$, and hence that $\int_0^{\infty} J_2(z)\, dF(z) < \infty$. This establishes that $M(x)h_0(x) \in L_1(0,\infty)$. A similar argument will show that $M(x)h_0(x) \in L_1(-\infty, 0)$. Obviously one could similarly show $M(x)k_0(x) \in \mathfrak{L}$. Thus, part (a) of lemma 2 is proved.

It should be clear that (b) can be proved by slightly modifying the proof of (a) suitably.

Now suppose $M \in \mathfrak{M}_1(1 - \ell)$, which means that $F \in \mathfrak{D}(I, 1)$. We have, for $x > 0$,

$$\int_x^\infty h_0(y)\,dy = \int_x^\infty \frac{1 - F(y)}{y^{(1-\ell)}}\,dy + \int_x^\infty \int_{-\infty}^y \left\{\frac{1}{y^{(1-\ell)}} - \frac{1}{(y - z)^{(1-\ell)}}\right\} dF(z)\,dy = A_1(x) + A_2(x), \quad \text{say.} \tag{2.50}$$

It is easy to see that $x^{(1-\ell)}A_1(x) \to 0$ as $x \to +\infty$. The integrand of $A_2(x)$ is negative when $z \geq 0$. If we were artificially to make $F(z) = 0$ for all negative z, then the resulting $A_2(x)$ would necessarily be finite. Similarly, the argument of $A_2(x)$ is positive when $z < 0$, and if we were artificially to make $F(z) = 1$ for all $z \geq 0$, the resulting $A_2(x)$ would also necessarily be finite. Thus the double integral $A_2(x)$ is necessarily absolutely convergent and we may reverse the order of integrations. We find

$$\begin{aligned} A_2(x) &= \int_{-\infty}^x \int_x^\infty \left\{\frac{1}{y^{(1-\ell)}} - \frac{1}{(y - z)^{(1-\ell)}}\right\} dy\,dF(z) \\ &\quad + \int_x^\infty \int_z^\infty \left\{\frac{1}{y^{(1-\ell)}} - \frac{1}{(y - z)^{(1-\ell)}}\right\} dy\,dF(z) \\ &= \frac{1}{\ell}\int_{-\infty}^x \{x^\ell - (x - z)^\ell\}\,dF(z) + \frac{1}{\ell}\int_x^\infty z^\ell\,dF(z) \\ &= B_1(x) + B_2(x), \quad \text{say.} \end{aligned} \tag{2.51}$$

Since $F(x) \in \mathfrak{D}(I; 1)$, it is easy to see that $x^{(1-\ell)}B_2(x) \to 0$ as $x \to +\infty$. Also, since $|x\{1 - (1 - z/x)^\ell\}| \leq |z|$, it follows by dominated convergence that

$$-x^{(1-\ell)}B_1(x) \to \int_{-\infty}^{+\infty} z\,dF(z), \qquad \text{as} \quad x \to \infty. \tag{2.52}$$

Thus, after combining our various results, we find

$$-x^{(1-\ell)} \int_x^\infty h_0(y)\,dy \to \int_{-\infty}^{+\infty} z\,dF(z), \qquad \text{as} \quad x \to \infty. \tag{2.53}$$

A much easier argument will also show that

$$x^{(1-\ell)} \int_x^\infty k_0(y)\,dy \to 0, \qquad \text{as} \quad x \to \infty. \tag{2.54}$$

The limit (2.4) now follows and lemma 2 is proved.

PROOF OF THEOREM 3. Suppose that X is an arbitrary random variable (not necessarily nonnegative) with a distribution function $F(x) \in \mathfrak{D}(M; \ell)$. Then X^+ and X^- are nonnegative random variables with finite moments of order ℓ. Therefore, by lemma 1, we have that $F_+^\dagger$, the characteristic function of X^+, can be expressed as follows:

$$F_+^\dagger(\theta) = 1 + \sum_{j=1}^{(k-1)} \frac{\mu_j(F_+)}{j!}(i\theta)^j + \frac{\mu_k(F_+)}{k!}(i\theta)^k F_{+(k)}^\dagger(\theta) \tag{2.55}$$

where k is the greatest integer not exceeding ℓ, $F^{\dagger}_{+(k)}(\theta)$ is the characteristic function of a nonnegative random variable with an absolutely continuous distribution function, and this distribution function belongs to $\mathfrak{D}(M, \ell - k)$. An expansion similar to (2.55) also holds for $F^{\dagger}_{-}(\theta)$, the characteristic function of X^{-}. Evidently, if $F^{\ddagger}(\theta)$ is the characteristic function of X, we have

$$F^{\ddagger}(\theta) = F^{\dagger}_{+}(\theta) + F^{\dagger}_{-}(-\theta) - 1. \tag{2.56}$$

Moreover, by repeatedly differentiating (2.56) and putting $\theta = 0$, or by more direct arguments, we find that

$$\mu_j(F) = \mu_j(F_+) + (-1)^j \mu_j(F_-), \qquad j = 1, 2, \cdots, k. \tag{2.57}$$

Thus by combining (2.55) and the corresponding expansion of $F^{\dagger}_{-}(\theta)$ together, in accordance with (2.56), we deduce that (1.15) holds with k in place of r and

$$t^{\ddagger}_k(\theta) = \mu_k(F_+)F^{\dagger}_{+(k)}(\theta) + (-1)^k \mu_k(F_-)F^{\dagger}_{-(k)}(-\theta). \tag{2.58}$$

Clearly, a similar result can be obtained for any integer $r < k$; the properties claimed for $t_r(x)$ in theorem 3 flow easily from this representation of $t^{\ddagger}_r(\theta)$.

Parts (1.12), (1.13), and (1.14) are all immediate from lemma 2 if $k = 0$. Suppose that $k > 0$. By applying lemma 2 to $F^{\dagger}_{+(k)}(\theta)$ and $F^{\dagger}_{-(k)}(\theta)$ separately we find

$$F^{\dagger}_{+(k)}(\theta) = 1 + |\theta|^{(\ell-k)} e^{i\gamma_1 \operatorname{sgn} \theta} s^{\dagger}_1(\theta), \tag{2.59}$$

$$F^{\dagger}_{-(k)}(\theta) = 1 + |\theta|^{(\ell-k)} e^{i\gamma_2 \operatorname{sgn} \theta} s^{\dagger}_2(\theta), \tag{2.60}$$

where γ_1 and γ_2 are arbitrary constants and $s_1(x)$, $s_2(x)$ are the appropriate $\mathfrak{L}$-functions, as described in lemma 2. If we now choose $\gamma_1 = c - \frac{1}{2}\pi k$ and $\gamma_2 = -c - \frac{1}{2}\pi k$, then we find

$$\frac{(i\theta)^k}{k!} t^{\ddagger}_k(\theta) = \frac{\mu_k(F)}{k!} (i\theta)^k + |\theta|^{\ell} e^{ic \operatorname{sgn} \theta} s^{\dagger}(\theta), \tag{2.61}$$

where

$$k! s^{\dagger}(\theta) = \mu_k(F_+) s^{\dagger}_1(\theta) + \mu_k(F_-) s^{\dagger}_2(-\theta). \tag{2.62}$$

If we put $r = k$ in (1.15) and substitute for $t^{\ddagger}_k(\theta)$, using (2.61), then (1.12) follows. The function $s^{\dagger}(\theta)$ clearly belongs to $\mathfrak{L}^{\dagger}$.

Routine computation based on the fact that, for $x > 0$,

$$dF_{+(k)}(x) = \frac{1 - F_{+(k-1)}(x)}{\mu_1(F_{+(k-1)})} dx \tag{2.63}$$

will lead to the result

$$\int_0^{\infty} x^{(\ell-k)} \, dF_{+(k)}(x) = \frac{k! \Gamma(\ell - k + 1)}{\Gamma(\ell + 1)\mu_k(F_+)} \int_0^{\infty} x^{\ell} \, dF(x), \tag{2.64}$$

and, of course, a similar result is true for $F_{-(k)}(x)$. From these results, (2.62), and (2.3) of lemma 2, it then follows that if $s(x) \in \mathfrak{L}$ is the original of the transform $s^{\dagger}(\theta)$, we must have

$$\int_{-\infty}^{+\infty} |s(x)|\,dx \leq \frac{2^{\ell-k+1}\left\{\left|\sin\left(\frac{\ell\pi}{2}-c\right)\right| + \left|\sin\left(\frac{\ell\pi}{2}+c\right)\right|\right\}}{|\sin(\ell\pi)|} A, \tag{2.65}$$

where

$$A = \frac{1}{\Gamma(\ell+1)} \int_{-\infty}^{\infty} |x|^{\ell}\,dF(x). \tag{2.66}$$

Thus we have proved (1.13).

What we have so far proved only needs $F \in \mathfrak{D}(I;\ell)$. Evidently, if $F \in \mathfrak{D}(M;\ell)$, then, by lemma 1, both $F_{+(k)}(x)$ and $F_{-(k)}(x)$ belong to $\mathfrak{D}(M;\ell - k)$. Set $\rho = 1 - \ell + k$. Then (a) if $M \in \mathfrak{M}_2(\rho)$, we can infer from lemma 2 that $s_1(x)$ and $s_2(x)$ belong to $\mathfrak{L}(M;0)$; thus $s(x) \in \mathfrak{L}(M;0)$ as claimed; (b) if $M \in \mathfrak{M}_3(\rho)$, then, by similar reasoning, $s(x) \in \mathfrak{L}(M_\rho;0)$; (c) if $M \in \mathfrak{M}_1(\rho)$, we infer from lemma 2 that as $x \to \infty$,

$$\begin{aligned}\Gamma(\ell-k)x^{(1-\ell-k)} \int_x^{\infty} s_1(y)\,dy &\to \frac{\sin\left(\frac{\ell\pi}{2}-c\right)}{\sin(\overline{\ell-k}\pi)} \int_0^{+\infty} x\,dF_{+(k)}(x) \\ &= \frac{\sin\left(\frac{\ell\pi}{2}-c\right)}{\sin(\overline{\ell-k}\pi)} \frac{\mu_{(k+1)}(F_+)}{(k+1)\mu_k(F_+)},\end{aligned} \tag{2.67}$$

and similarly, but allowing carefully for signs, we find

$$\begin{aligned}\Gamma(\ell-k)x^{(1-\ell+k)} &\int_x^{\infty} s_2(-y)\,dy \\ &\to \frac{\sin\left(\frac{\ell\pi}{2}-c-k\pi\right)}{\sin(\overline{\ell-k}\pi)} \frac{(-)^{(k-1)}\mu_{(k+1)}(F_-)}{(k+1)(-)^k\mu_k(F_-)} \\ &= -\frac{\sin\left(\frac{\ell\pi}{2}-c\right)}{\sin(\ell\pi)} \frac{\mu_{(k+1)}(F_-)}{(k+1)\mu_k(F_-)}.\end{aligned} \tag{2.68}$$

Therefore,

$$\Gamma(\ell-k)x^{(1-\ell+k)} \int_x^{\infty} s(y)\,dy \to \frac{(-1)^k\mu_{(k+1)}(F)}{(k+1)!} \frac{\sin\left(\frac{\ell\pi}{2}-c\right)}{\sin(\ell\pi)} \tag{2.69}$$

as $x \to \infty$. This completes the proof of theorem 3.

PROOF OF THEOREM 2. To begin with we must define what we mean by a smooth mutilator function (S.M.F.). Let us define

$$\begin{aligned} p^{\dagger}(x) &= \exp\left\{\frac{1}{(x^2-\frac{1}{4})}\right\}, & |x| &< \tfrac{1}{2}, \\ &= 0, & |x| &\geq \tfrac{1}{2}. \end{aligned} \tag{2.70}$$

Then $p^\dagger(x)$ is differentiable for all x an arbitrary number of times and each derivative is a bounded $\mathfrak{L}$-function. Moreover, $p^\dagger(x)$ and all its derivatives vanish when $|x| \geq \frac{1}{2}$. Suppose we are given $\alpha < \beta < \gamma < \delta$ as four points on the real axis. We define $q^\dagger(x; \alpha, \beta, \gamma, \delta)$ as the S.M.F. based on these points by the equation

$$q^\dagger(x; \alpha, \beta, \gamma, \delta) = \frac{\int_{-\infty}^{x} \left\{\frac{1}{(\beta - \alpha)} p^\dagger\left(\frac{y - \frac{1}{2}(\alpha + \beta)}{(\beta - \alpha)}\right) - \frac{1}{(\delta - \gamma)} p^\dagger\left(\frac{y - \frac{1}{2}(\gamma + \delta)}{(\delta - \gamma)}\right)\right\} dy}{\int_{-\infty}^{+\infty} p^\dagger(y)\, dy}. \tag{2.71}$$

The S.M.F. has the following properties. It vanishes when $x \leq \alpha$ or when $x \geq \delta$. It has the constant value unity on the interval $\beta \leq x \leq \gamma$. It is monotonically increasing on $\alpha < x < \beta$ and decreasing on $\gamma < x < \delta$. Furthermore, $q^\dagger(x; \alpha, \beta, \gamma, \delta)$ is differentiable for all x an arbitrary number of times and each derivative is a bounded $\mathfrak{L}$-function. The derivatives all vanish identically, except when $\alpha < x < \beta$ or $\gamma < x < \delta$.

If we put

$$q(x; \alpha, \beta, \gamma, \delta) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\theta x} q^\dagger(\theta; \alpha, \beta, \gamma, \delta)\, d\theta, \tag{2.72}$$

then we can, by a familiar argument involving repeatedly integrating by parts, show that $q(x; \alpha, \beta, \gamma, \delta) = O(1/1 + |x|^N)$ for N arbitrarily large. Thus $q(x; \alpha, \beta, \gamma, \delta) \in \mathfrak{L}(M; \nu)$ for any $M \in \mathfrak{M}^*$ and any $\nu \geq 0$.

We shall write $q^\dagger(\theta)$ for the special S.M.F. $q^\dagger(\theta; -2, -1, +1, +2)$. Let θ_0 be any fixed point in the closed interval J (possibly an end-point). Then $\Phi(z)$ is analytic at $z = \varphi(\theta_0)$ and so admits of an expansion

$$\Phi(z) = \Phi(\varphi(\theta_0)) + \sum_{n=1}^{\infty} c_n(z - \varphi(\theta_0))^n \tag{2.73}$$

about the point $z = \varphi(\theta_0)$ with a strictly positive radius of convergence. Since $\varphi(\theta)$ must be continuous, it follows that for all sufficiently small $|\theta - \theta_0|$

$$\Phi(\varphi(\theta)) = \Phi(\varphi(\theta_0)) + \sum_{n=1}^{\infty} c_n(\varphi(\theta) - \varphi(\theta_0))^n. \tag{2.74}$$

For some $\lambda > 0$ let us define

$$\vartheta_\lambda^\dagger(\theta) = q^\dagger\left(\frac{\theta - \theta_0}{\frac{1}{2}\lambda}\right) \Phi(\varphi(\theta)). \tag{2.75}$$

Then the functions $\vartheta_\lambda^\dagger(\theta)$ and $\Phi(\varphi(\theta))$ are identical for $|\theta - \theta_0|$ sufficiently small. For any $n > 0$,

$$q^\dagger\left(\frac{\theta - \theta_0}{\frac{1}{2}\lambda}\right) = q^\dagger\left(\frac{\theta - \theta_0}{\frac{1}{2}\lambda}\right)\left[q^\dagger\left(\frac{\theta - \theta_0}{\lambda}\right)\right]^n, \tag{2.76}$$

and so

$$(2.77)\qquad \vartheta_\lambda^\dagger(\theta) = \Phi(\varphi(\theta_0))q^\dagger\left(\frac{\theta-\theta_0}{\frac{1}{2}\lambda}\right) + \sum_{n=1}^{\infty} c_n q^\dagger\left(\frac{\theta-\theta_0}{\frac{1}{2}\lambda}\right)\left[q^\dagger\left(\frac{\theta-\theta_0}{\lambda}\right)\{\varphi(\theta)-\varphi(\theta_0)\}\right]^n.$$

Now $q^\dagger(\theta-\theta_0)/\lambda)\{\varphi(\theta)-\varphi(\theta_0)\}$ is the Fourier transform of $r_\lambda(x)$, say, where

$$(2.78)\qquad r_\lambda(x) = \lambda\int_{-\infty}^{+\infty} e^{-i\theta_0(x-z)}\{q(\lambda(x-z)) - q(\lambda x)\}\,dB(z),$$

and $B(x)$ is the function from $\mathfrak{B}(M;\nu)$ of which $\varphi(\theta)$ is the Fourier-Stieltjes transform. Thus, after an application of Fubini's theorem, we have

$$(2.79)\qquad \int_{-\infty}^{+\infty}|r_\lambda(x)|\,dx < \lambda\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty}|(q(\lambda(x-z)) - q(\lambda x)|\,dx\right\}|dB(z)| = \int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty}|q(u-\lambda z) - q(u)|\,du\right\}|dB(z)|.$$

The inner integral tends to zero boundedly as $\lambda\to 0$. Thus, by choosing λ sufficiently small, we can make $\int_{-\infty}^{+\infty}|r_\lambda(x)|\,dx = \rho_\lambda$, say, as small as we please.

Since $r_\lambda^\dagger(\theta)$ is the product of a function in $\mathfrak{B}^\dagger(M;\nu)$ and an S.M.F., it is apparent that $r_\lambda(x)$ belongs to $\mathfrak{L}(M;\nu)$. Define convolutions of $r_\lambda(x)$ as follows:

$$(2.80)\qquad r_\lambda^{*1}(x) = r_\lambda(x),$$

$$(2.81)\qquad r_\lambda^{*n}(x) = \int_{-\infty}^{+\infty} r_\lambda(x-z)r_\lambda^{*(n-1)}(z)\,dz, \qquad \text{for } n = 2, 3, \cdots, \text{ad inf.}$$

Then $r_\lambda^{*n}(x)$ also belongs to $\mathfrak{L}(M;\nu)$, and we can define

$$(2.82)\qquad \alpha_\nu(n) = \int_{-\infty}^{+\infty} M(|x|)|x|^\nu|r_\lambda^{*n}(x)|\,dx.$$

If $M(x)\in\mathfrak{M}^*$ and we set $K(x) = M(x)(1+x)^\nu$, then it is easy to verify that $K(x)\in\mathfrak{M}^*$ also. Thus there is some A such that $K(2x)\le AK(x)$ for all $x\ge 0$. From this it is easy to show the existence of $d>0$ such that $K(nx)\le n^dK(x)$ for all positive integer values of n and all $x\ge 0$. Thus

$$(2.83)\qquad \begin{aligned} K(|z_1+\cdots+z_n|) &\le K(|z_1|+\cdots+|z_n|)\\ &\le \sum_{j=1}^n K(n|z_j|)\\ &\le n^d\sum_{j=1}^n K(|z_j|).\end{aligned}$$

Thus

$$(2.84)\qquad \begin{aligned}\alpha_\nu(n) &\le \int\cdots\int K(|z_1+\cdots+z_n|)\prod_{j=1}^n|r_\lambda(z_j)|\,dz_j\\ &\le n^{d+1}\rho_\lambda^{(n-1)}\int_{-\infty}^{+\infty}K(|z|)|r_\lambda(z)|\,dz.\end{aligned}$$

If we choose λ small enough to make $\sum_{n=1}^{\infty} n^d c_n(\rho_\lambda)^n$ absolutely convergent, then it follows that $\sum_{n=1}^{\infty} c_n r_\lambda^{*n}(x) = r(x)$, say, is a function of the class $\mathcal{L}(M;\nu)$ and its Fourier transform is

$$r^\dagger(\theta) = \sum_{n=1}^{\infty} c_n \left[q^\dagger\left(\frac{\theta-\theta_0}{\lambda}\right)\{\varphi(\theta)-\varphi(\theta_0)\}\right]^n. \tag{2.85}$$

From this it is an easy step to see that $\vartheta_\lambda^\dagger(\theta) \in \mathcal{L}^\dagger(M;\nu)$, since the product of two functions in $\mathcal{L}^\dagger(M;\nu)$ is a further function in $\mathcal{L}^\dagger(M;\nu)$. Thus we have shown that if θ_0 is any point of the closed interval J (including the end-points), then we can find a function $\vartheta_\lambda^\dagger(\theta)$, say, which belongs to $\mathcal{L}^\dagger(M;\nu)$ and is such that $\Phi(\varphi(\theta)) = \vartheta_\lambda^\dagger(\theta)$ for all θ in a closed subinterval centered upon θ_0. By the Heine-Borel theorem we can cover the finite closed interval J with a finite number of these subintervals. Suppose (β_1, γ_1), (β_2, γ_2) are two overlapping such subintervals, $\beta_1 < \beta_2 < \gamma_1 < \gamma_2$, and suppose $\vartheta_{\lambda_1}^\dagger(\theta)$ and $\vartheta_{\lambda_2}^\dagger(\theta)$ are the appropriate $\mathcal{L}^\dagger(M;\nu)$-functions associated with these intervals. Then the function

$$q^\dagger(\theta;\beta_1-1,\beta_1,\beta_2,\gamma_1)\vartheta_{\lambda_1}^\dagger(\theta) + q^\dagger(\theta;\beta_2,\gamma_1,\gamma_2,\gamma_2+1)\vartheta_{\lambda_2}^\dagger(\theta) \tag{2.86}$$

is a new $\mathcal{L}^\dagger(M;\nu)$-function which is identically equal to $\Phi(\varphi(\theta))$ throughout the larger interval (β_1, γ_2). It is clear how this argument may be continued, with appropriate modifications at the end-points of J, so that we end up with a single function $\vartheta^\dagger(\theta)$, say, which belongs to $\mathcal{L}^\dagger(M;\nu)$ and is identically equal to $\Phi(\varphi(\theta))$ throughout J. But $\psi(\theta) = 0$ for all $\theta \notin J$. Thus $\psi(\theta)\vartheta^\dagger(\theta) = \psi(\theta)\Phi(\varphi(\theta))$ for all θ; and $\psi(\theta)\vartheta^\dagger(\theta)$ belongs to $\mathcal{L}^\dagger(M;\nu)$. This actually proves a little more than is claimed in theorem 2, namely $\psi(\theta)\Phi(\varphi(\theta))$ belongs to $\mathcal{L}^\dagger(M;\nu)$ rather than to $\mathcal{B}^\ddagger(M;\nu)$.

Next, for the sake of argument, suppose that J is semi-infinite; say $J = (\eta, +\infty)$. Write ρ for $\rho[\varphi(\theta)]$. Since no singularity of $\Phi(z)$ is within a distance ρ of the origin, we can find an $\epsilon > 0$ such that $\Phi(z) = \sum_0^\infty c_n z^n$ for all $|z| < \rho + 2\epsilon$, the series converging absolutely in this region. We can then find a k such that $\{\varphi(\theta)\}^k = \alpha a^\dagger(\theta) + \beta B^\ddagger(\theta)$, where $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta = 1$, and where $a^\dagger(\theta) \in \mathcal{L}^\dagger \cap \mathfrak{D}^\ddagger(M;\nu)$ and $B^\ddagger(\theta) \in \mathfrak{D}^\ddagger(M;\nu)$ and $\beta^k < (\rho + \frac{1}{2}\epsilon)^k$. For all $|\theta| > 2\lambda$, say, $|\varphi(\theta)| < \rho + \epsilon$ and so

$$\begin{aligned} \Phi(\varphi(\theta)) &= \sum_0^\infty c_n(\varphi(\theta))^n \\ &= \sum_{j=0}^{(k-1)} \sum_{n=0}^{\infty} c_{nk+j}(\varphi(\theta))^{nk+j} \\ &= \sum_{j=0}^{(k-1)} (\varphi(\theta))^j \sum_{n=0}^{\infty} c_{nk+j}(\alpha a^\dagger(\theta) + \beta B^\ddagger(\theta))^n \\ &= \sum_{j=0}^{(k-1)} (\varphi(\theta))^j \Xi_j(\theta), \quad \text{say.} \end{aligned} \tag{2.87}$$

Now for $\theta \geq 2\lambda$,

$$\Xi_j(\theta) = \sum_{n=0}^{\infty} c_{nk+j}\left(\alpha a^\dagger(\theta) + \beta B^\ddagger(\theta) - \alpha a^\dagger(\theta) q^\dagger\left(\frac{\theta}{\lambda}\right)\right)^n, \tag{2.88}$$

and this series converges absolutely. As $a^\dagger(\theta)$ is the Fourier transform of the $\mathfrak{L}(M;\nu)$-function $a(x)$, then $a^\dagger(\theta) - a^\dagger(\theta)q^\dagger(\theta/\lambda)$ is the Fourier transform of the $\mathfrak{L}(M;\nu)$-function $g_\lambda(x)$, where

$$g_\lambda(x) = \int_{-\infty}^{+\infty} \left[a(x) - a\left(x - \frac{u}{\lambda}\right)\right] q(u)\,du, \tag{2.89}$$

if we use the fact that $\int_{-\infty}^{+\infty} q(u)\,du = q^\dagger(0) = 1$. By arguments very similar to those we have employed in the first part of the present proof, we can next show that $\int_{-\infty}^{+\infty} |g_\lambda(x)|\,dx \to 0$ as $\lambda \to \infty$. Thus, by choosing λ large enough we can ensure that $\beta B^\ddagger(\theta) + \alpha g_\lambda^\ddagger(\theta)$ is a function of $\mathfrak{B}^\ddagger(M;\nu)$ whose total variation is less than $(\rho + \epsilon)^k$. Much as before, it will follow that $\Xi_j(\theta) \in \mathfrak{B}^\ddagger(M;\nu)$ if we recall that $\Phi(z)$ is analytic in $|z| < \rho + 2\epsilon$. Since $\{\varphi(\theta)\}^j \in \mathfrak{B}^\ddagger(M;\nu)$ for every integer, it follows that there is some function $\vartheta_\infty^\ddagger(\theta)$, say, belonging to $\mathfrak{B}^\ddagger(M;\nu)$ and such that $[1 - q^\dagger(\theta/2\lambda)]\psi(\theta)\Phi(\varphi(\theta)) \equiv \vartheta_\infty^\ddagger(\theta)$ for all θ. By the first part of this proof we can say there is also a function $\vartheta_0^\ddagger(\theta)$, belonging to $\mathfrak{B}^\ddagger(M;\nu)$ and such that $q^\dagger(\theta/2\lambda)\psi(\theta)\Phi(\varphi(\theta)) \equiv \vartheta_0^\ddagger(\theta)$, all θ. We combine $\vartheta_0^\ddagger$ and $\vartheta_\infty^\ddagger(\theta)$, and conclude the proof of this theorem, as before.

LEMMA 4. *Suppose that for some $\gamma > 0$ and some $M \in \mathfrak{M}$ (possibly $M = I$) we have $\lambda(\theta) \in \mathfrak{B}^\ddagger(M;\gamma)$, and suppose further that $\lambda(\theta) = O(|\theta|^\gamma)$ as $|\theta| \to 0$. Then*

(a) *when γ is an integer, $\lambda(\theta)/(-i\theta)^\gamma \in \mathfrak{B}^\ddagger(M;0)$;*

(b) *when γ is not an integer and k is the greatest integer not exceeding γ, set $\rho = k + 1 - \gamma$ and let w be any prescribed constant and write $\Lambda^\ddagger(\theta) = e^{iw\,\mathrm{sgn}\,\theta}\lambda(\theta)/(-i\theta)^\gamma$. Then $\Lambda^\ddagger(\theta) = o(1)$ as $|\theta| \to 0$ and*

(i) *if $M \in \mathfrak{M}_2(\rho)$, $\Lambda^\ddagger(\theta) \in \mathfrak{B}^\ddagger(M;0)$;*

(ii) *if $M \in \mathfrak{M}_3(\rho)$, $\Lambda^\ddagger(\theta) \in \mathfrak{B}^\ddagger(M_\rho;0)$;*

(iii) *if $M \in \mathfrak{M}_1(\rho)$, $\Lambda^\ddagger(\theta)$ is the Fourier-Stieltjes transform of some function $\Lambda(x) \in \mathfrak{B}$ such that, as $x \to \infty$, $x^\rho\{\Lambda(x) - \Lambda(\infty)\} \to (\sin(\gamma\pi + w)/\sin(\gamma\pi))C$, where*

$$C = \lim_{\theta\to 0} \frac{\lambda(\theta)}{(-i\theta)^{(k+1)}\Gamma(1-\rho)}, \tag{2.90}$$

this limit, C, necessarily existing.

PROOF. Since $\lambda(\theta) \in \mathfrak{B}^\ddagger(M;\gamma)$, we can infer that there must be four members of $\mathfrak{D}^\ddagger(M;\gamma)$, $A_1^\ddagger(\theta)$, $A_2^\ddagger(\theta)$, $A_3^\ddagger(\theta)$, $A_4^\ddagger(\theta)$ and four constants a_1, a_2, a_3, a_4, such that $\lambda(\theta) = \sum_{m=1}^4 a_m A_m^\ddagger(\theta)$.

Let us suppose first that γ is nonintegral and recall that k is the greatest integer not exceeding γ. By (1.12) of theorem 3 we have, for any prescribed constants c_1, c_2, c_3, c_4,

$$A_m^\ddagger(\theta) = 1 + \sum_{j=1}^{k} \frac{\mu_j(A_m)}{j!}(i\theta)^j + |\theta|^\gamma e^{ic_m\,\mathrm{sgn}\,\theta} s_m^\dagger(\theta) \tag{2.91}$$

for $m = 1, 2, 3, 4$, where $s_1(\theta), \cdots, s_4(\theta)$ are the members of some $\mathfrak{L}^\dagger$-class, depending on M. Since $\gamma > k$ and $\lambda(\theta) = O(|\theta|^\gamma)$, we have that

$$\sum_{m=1}^{4} a_m\mu_j(A_m) = 0, \qquad j = 0, 1, \cdots, k. \tag{2.92}$$

Thus, if we choose $c_1 = c_2 = c_3 = c_4 = -w - \frac{1}{2}(\pi\gamma)$, we have

$$\frac{\lambda(\theta)}{(-i\theta)^\gamma} = e^{-iw \operatorname{sgn} \theta} \sum_{m=1}^{4} a_m s_m^\dagger(\theta), \tag{2.93}$$

and it follows that $e^{iw \operatorname{sgn} \theta}\lambda(\theta)/(-i\theta)^\gamma$ belongs to the $\mathfrak{B}^\ddagger$-class as claimed. The remainder of lemma 4, for γ nonintegral, follows from the properties proved for the $s_m^\dagger(\theta)$ in theorem 3. The only point that, perhaps, calls for any remark concerns the final limiting result. We can infer from theorem 3 that, as $x \to \infty$,

$$\Gamma(\gamma - k)x^{k-\gamma+1} \int_x^\infty s_m(y)\, dy \to \frac{(-)^k}{(k+1)!} \left\{\frac{\sin(\gamma\pi + w)}{\sin(\gamma\pi)}\right\} \mu_{(k+1)}(A_m) \tag{2.94}$$

for $m = 1, 2, 3, 4$. Thus, as $x \to +\infty$,

$$\Gamma(\gamma - k)x^{k-\gamma+1}\{\Lambda(\infty) - \Lambda(x)\} \to \frac{(-)^k}{(k+1)!} \left\{\frac{\sin(\gamma\pi + w)}{\sin(\gamma\pi)}\right\} \sum_{m=1}^{4} a_m \mu_{(k+1)}(A_m). \tag{2.95}$$

However, since in this part of the argument we may assume $A_m(x) \in \mathfrak{D}(I; k+1)$ for $m = 1, 2, 3, 4$, we can make use of expansions like (1.15) of theorem 3, ending with terms involving $(i\theta)^{(k+1)}$, and find that

$$\frac{1}{(k+1)!} \sum_{m=1}^{4} a_m \mu_{(k+1)}(A_m) = \lim_{\theta \to 0} \frac{\lambda(\theta)}{(i\theta)^{(k+1)}}, \tag{2.96}$$

and the final limiting result is verified.

The part of lemma 4 that concerns integer values for γ is easier, and it will be obvious at this stage how it follows from the relevant parts of theorem 3.

Proof of theorem 1. We begin the proof of theorem 1 by noting that, for any $\lambda > 0$,

$$\begin{aligned} \Psi^\ddagger(\theta) &= q^\dagger\left(\frac{\theta}{\lambda}\right)\Psi^\ddagger(\theta) + \left[1 - q^\dagger\left(\frac{\theta}{\lambda}\right)\right]\Psi^\ddagger(\theta) \\ &= \Psi_1^\ddagger(\theta) + \Psi_2^\ddagger(\theta), \quad \text{say.} \end{aligned} \tag{2.97}$$

We shall prove that both $\Psi_1^\ddagger(\theta)$ and $\Psi_2^\ddagger(\theta)$ belong to appropriate $\mathfrak{B}^\ddagger$-classes.

We put $\sup_{|\theta| > 1/2\lambda} |\varphi_1(\theta)| = \alpha$, say. Then because $\varphi_1(\theta) \in \mathfrak{S}^\ddagger$, it follows that $\alpha < 1$. Hence, as θ runs from $\frac{1}{2}\lambda$ to $+\infty$, the point $z = \varphi_1(\theta)$ maps out a continuous curve which lies everywhere in the circle $|z| \leq \alpha$. Thus it follows from theorem 2 that

$$\frac{\left[1 - q^\dagger\left(\frac{2\theta}{\lambda}\right)\right]}{[1 - \varphi_1(\theta)]^{\alpha_1}} \tag{2.98}$$

belongs to $\mathfrak{B}^\ddagger(M; 1)$ if $\varphi_1(\theta) \in \mathfrak{D}^\ddagger(M; 1)$. Hence

$$\frac{\left[1 - q^\dagger\left(\frac{2\theta}{\lambda}\right)\right]^{(n+m)}}{\prod_{r=1}^{n} [1 - \varphi_r(\theta)]^{\alpha_r} \prod_{s=1}^{m} [1 - \psi_s(\theta)]^{\beta_s}} \tag{2.99}$$

is a member of $\mathfrak{B}^\ddagger(M; 1)$ if every $\varphi_r(\theta)$ and $\psi_s(\theta)$ belongs to $\mathfrak{D}^\ddagger(M; 1)$. But

$$(2.100)\qquad \left[1 - q^\dagger\left(\frac{\theta}{\lambda}\right)\right] = \left[1 - q^\dagger\left(\frac{\theta}{\lambda}\right)\right]\left[1 - q^\dagger\left(\frac{2\theta}{\lambda}\right)\right]^{n+m},$$

so that it follows easily that $\Psi_2^\ddagger(\theta) \in \mathfrak{B}(M; 0)$, if $\lambda(\theta) \in \mathfrak{B}^\ddagger(M; \gamma)$, $\gamma \geq 0$.

To deal with $\Psi_1^\ddagger(\theta)$ we first note that, as a consequence of theorem 3,

$$(2.101)\qquad 1 - \varphi_r(\theta) = (-i\theta\mu_1^{(r)})u_r(\theta),$$

where we have written $\mu_1^{(r)}$ for the mean of the distribution associated with $\varphi_r(\theta)$, $\mu_1^{(r)} \neq 0$, and $u_r(\theta)$ is a member of $\mathfrak{L}^\dagger(M; 0)$ which assumes the value unity at the origin. For any small $\delta > 0$, because of the absolute continuity present, we have $\sup_{\delta \leq |\theta| \leq 4\lambda} |\varphi_r(\theta)| < 1$. Therefore, there is an angle v, say, $v < \pi/2$ such that $|\arg (1 - \varphi_r(\theta)| < v$ for all $\delta \leq |\theta| \leq 4\lambda$. From this it follows that $|\arg u_r(\theta)| < v + \frac{1}{2}\pi < \pi$ for all θ in the same range. For $|\theta| \leq \delta$ the function $u_r(\theta)$ takes on values near unity if we choose δ small enough. We can therefore draw the following conclusion. As θ runs from -4λ to $+4\lambda$, the function $u_r(\theta)$ maps out a continuous closed curve C, say, in the complex plane. The curve C is a strictly positive distance from the negative real axis. Thus C lies in some open subset of the slit complex plane, throughout which our definition of $z^{-\alpha_r}$ is a one-valued analytic function. From theorem 2, therefore, $q^\dagger(\theta/2\lambda)/\{u_r(\theta)\}^{\alpha_r}$ belongs to $\mathfrak{B}^\ddagger(M; 0)$ if $u_r(\theta) \in \mathfrak{B}^\ddagger(M; 0)$, and theorem 3 assures us this will be so if $\varphi_r(\theta) \in \mathfrak{B}^\ddagger(M; 1)$.

We can similarly show that $1 - \psi_s(\theta) = \frac{1}{2}\theta^2\nu_s w_s(\theta)$ where $\nu_s > 0$ is the variance of the distribution associated with $\psi_s(\theta)$ and $w_s(\theta) \in \mathfrak{L}^\dagger(M; 0)$ if $\psi_s(\theta) \in \mathfrak{B}^\ddagger(M; 2)$, and $w_s(0) = 1$. Indeed, $w_s(\theta)$ is a characteristic function (theorem 3). As before, we can show that $q^\dagger(\theta/2\lambda)/\{w_s(\theta)\}^{\beta_s}$ is a member of $\mathfrak{B}^\ddagger(M; 0)$. But

$$(2.102)\qquad q^\dagger\left(\frac{\theta}{\lambda}\right) = q^\dagger\left(\frac{\theta}{\lambda}\right)\left[q^\dagger\left(\frac{\theta}{2\lambda}\right)\right]^{n+m}.$$

Hence we may conclude that

$$(2.103)\qquad \frac{q^\dagger\left(\dfrac{\theta}{\lambda}\right)}{\prod_{r=1}^{n} [u_r(\theta)]^{\alpha_r} \prod_{s=1}^{m} [w_s(\theta)]^{\beta_s}}$$

belongs to $\mathfrak{B}^\ddagger(M; 0)$.

For $|\theta| \leq 4\lambda$ we have seen that the three numbers $u_r(\theta)$, $1 - \varphi_r(\theta)$, $-i\theta\mu_1^{(r)}$ all have arguments between $-\pi$ and $+\pi$. Thus we can write

$$(2.104)\qquad [u_r(\theta)]^{\alpha_r} = \left[\frac{1 - \varphi_r(\theta)}{-i\theta\mu_1^{(r)}}\right]^{\alpha_r} = \frac{[1 - \varphi_r(\theta)]^{\alpha_r}}{[-i\theta\mu_1^{(r)}]^{\alpha_r}},$$

and therefore,

$$(2.105)\qquad [u_r(\theta)]^{\alpha_r} = \frac{[1 - \varphi_r(\theta)]^{\alpha_r}}{|\theta|^{\alpha_r}|\mu_1^{(r)}|^{\alpha_r} \exp\left\{\dfrac{i\pi\alpha_r}{2}\operatorname{sgn}(\theta\mu_1^{(r)})\right\}}.$$

Similarly,

$$[w_s(\theta)]^{\beta_s} = \frac{[1 - \psi_s(\theta)]^{\beta_s}}{|\frac{1}{2}\nu_s|^{\beta_s}|\theta|^{2\beta_s}}, \tag{2.106}$$

and we therefore have that

$$\frac{|\theta|^{\gamma} q^{\dagger}\left(\frac{\theta}{\lambda}\right) \prod_{r=1}^{n} |\mu_1^{(r)}|^{\alpha_r} \prod_{s=1}^{m} |\frac{1}{2}\nu_s|^{\beta_s}}{\prod_{r=1}^{n} [1 - \varphi_r(\theta)]^{\alpha_r} \prod_{s=1}^{m} [1 - \psi_s(\theta)]^{\beta_s}} e^{-\frac{1}{2}c\pi i \operatorname{sgn}\theta} \tag{2.107}$$

belongs to $\mathcal{B}^{\ddagger}(M; 0)$, where $c = \sum_{r=1}^{n} \alpha_r \operatorname{sgn} \mu_1^{(r)}$.

Let us now suppose, for the time being, that γ is an integer. By $1(v)$ we can suppose that $c = \gamma + 2p$, where p is a positive or negative integer, or zero. By lemma 4 we see that

$$\frac{\lambda(\theta)}{(-i\theta)^{\gamma}} = \frac{\lambda(\theta)}{|\theta|^{\gamma}} e^{\frac{1}{2}\gamma\pi i \operatorname{sgn}\theta} \tag{2.108}$$

belongs to $\mathcal{B}^{\ddagger}(M; 0)$ if $\lambda(\theta) \in \mathcal{B}^{\ddagger}(M, \lambda)$. On multiplying (2.107) and (2.108) together, and ignoring constant terms, and noting especially that

$$e^{\frac{1}{2}(\gamma - c)\pi i \operatorname{sgn}\theta} = (-1)^{p}, \quad \text{a constant,} \tag{2.109}$$

we find that $\Psi_1^{\ddagger}(\theta) \in \mathcal{B}^{\ddagger}(M; 0)$ as desired. The proof of theorem 1 is now complete for the case of γ having an integer value.

If γ is not an integer, recall that ρ is the difference between γ and the least integer exceeding γ. Suppose, to begin with, that $M \in \mathfrak{M}_2^{*}(\rho)$. By lemma 4 we can state that

$$\frac{e^{\frac{1}{2}\pi(c-\gamma)i \operatorname{sgn}\theta}\lambda(\theta)}{(-i\theta)^{\gamma}} = \frac{e^{\frac{1}{2}c\pi i \operatorname{sgn}\theta}\lambda(\theta)}{|\theta|^{\gamma}} \tag{2.110}$$

belongs to $\mathcal{B}^{\ddagger}(M; 0)$, if $\lambda(\theta) \in \mathcal{B}^{\ddagger}(M; \gamma)$. On multiplying (2.110) and (2.107) as before we reach the desired conclusion that $\Psi_1^{\ddagger}(\theta) \in \mathcal{B}^{\ddagger}(M; 0)$. It should be clear that a similar argument will cover the case $M \in \mathfrak{M}_3^{*}(\rho)$.

Finally we come to the case when $M \in \mathfrak{M}_1^{*}(\rho)$. Since $\lambda(\theta) \in \mathcal{B}^{\ddagger}(M; \gamma)$ implies $\lambda(\theta) \in \mathcal{B}^{\ddagger}(I; \gamma)$, and so on, it is clear that $\Psi_1^{\ddagger}(\theta) \in \mathcal{B}^{\ddagger}$. However, let us now assume every $\varphi_r(\theta) \in \mathfrak{D}^{\ddagger}(M; 2)$ and every $\psi_s(\theta) \in \mathfrak{D}^{\ddagger}(M; 3)$. Then, from (2.107) we see that $Q(\theta)$, say, defined as

$$\frac{|\theta|^{\gamma} q^{\dagger}(\theta/\lambda) e^{-\frac{1}{2}c\pi i \operatorname{sgn}\theta}}{\prod_{r=1}^{n} [1 - \varphi_r(\theta)]^{\alpha_r} \prod_{s=1}^{m} [1 - \psi_s(\theta)]^{\beta_s}} \tag{2.111}$$

belongs to $\mathcal{B}^{\ddagger}(I, 1)$, and, from lemma 4, $\lambda(\theta)/(-i\theta)^{k} \in \mathcal{B}^{\ddagger}(I, 1)$. Therefore,

$$\lambda_1(\theta) \equiv \left\{\frac{\lambda(\theta)}{(-i\theta)^{k}}\right\} Q(\theta) \tag{2.112}$$

must belong to $\mathcal{B}^{\ddagger}(I, 1)$ and must be $O(|\theta|^{\gamma-k})$ as $|\theta| \to 0$. Hence,

$$\frac{e^{iw \operatorname{sgn} \theta}\lambda_1(\theta)}{(-i\theta)^{(\gamma-k)}}, \tag{2.113}$$

for any constant w, is the Fourier-Stieltjes transform of some function $\Lambda_1(x) \in \mathcal{B}$, say, such that, as $x \to \infty$,

$$\Gamma(1-\rho)x^\rho\{\Lambda_1(x) - \Lambda_1(\infty)\} \to \frac{\sin(\gamma\pi + w)}{\sin(\gamma\pi)}\left\{\lim_{\theta\to 0}\frac{\lambda_1(\theta)}{(-i\theta)}\right\}. \tag{2.114}$$

If we choose $w = \frac{1}{2}\pi(c-\gamma)$, then we find that $\Psi_1^\ddagger(\theta)$ is the Fourier-Stieltjes transform of some $\Psi_1(x) \in \mathcal{B}$ such that, as $x \to \infty$,

$$x^\rho\{\Psi_1(x) - \Psi_1(\infty)\} \to \frac{\sin(\frac{1}{2}\pi(c+\gamma))}{\sin(\gamma\pi)}C, \tag{2.115}$$

where the constant C is defined in the enunciation of theorem 1.

However, under present conditions, $\Psi_2 \in \mathcal{B}(I; 2)$, and it is therefore trivial that $x^\rho\{\Psi_2(x) - \Psi_2(\infty)\} \to 0$, as $x \to \infty$. Thus theorem 1 is proved.

3. On certain sums arising from random walks

Proof of theorem 4. We shall deal with the case of nonintegral ℓ, when ℓ is an integer the proof is similar. Once again, recall that k is the greatest integer not exceeding ℓ. Since $\ell > 1$, we have (by theorem 3) that

$$F^\ddagger(\theta) = 1 + \mu_1(i\theta) + \cdots + \frac{\mu_k(i\theta)^k}{k!} + o(|\theta|^\ell). \tag{3.1}$$

For θ small, the arguments of $1 - F^\ddagger(\theta)$ and $\{1 - F^\ddagger(\theta)\}/(-i\mu_1\theta)$ both lie in $(-\pi, +\pi)$ so we may claim

$$\begin{aligned}\frac{1}{\{1-F^\ddagger(\theta)\}^{(\ell-1)}} &= \frac{1}{(-\mu_1 i\theta)^{(\ell-1)}}\left\{1 + \frac{\mu_2}{2!\mu_1}(i\theta) + \cdots + \frac{\mu_k}{k!\mu_1}(i\theta)^{(k-1)} + o(|\theta|^{(\ell-1)})\right\}^{-(\ell-1)} \\ &= \sum_{j=1}^{k}\frac{A_j(\ell)}{(-\mu_1 i\theta)^{(\ell-j)}} + o(1),\end{aligned} \tag{3.2}$$

where the constants $A_j(\ell)$ are obtained by expansion of the previous line. This proves (1.18). Now suppose that $M \in \mathfrak{M}_2^*(\rho)$. Let us note that, by theorem 3,

$$1 - F^\ddagger(\theta) = -i\theta\mu_1 t_1^\ddagger(\theta), \tag{3.3}$$

where $t_1^\ddagger(\theta) \in \mathcal{B}^\ddagger(M; \ell - 1)$, and let us write

$$\Lambda^\ddagger(\theta) = \frac{1}{\{1-F^\ddagger(\theta)\}^{(\ell-1)}} - \sum_{j=1}^{k}\frac{A_j(\ell)}{(-\mu_1 i\theta)^{(\ell-j)}}. \tag{3.4}$$

Then

$$\Lambda^\ddagger(\theta) = \frac{1}{\{1-F^\ddagger(\theta)\}^{(\ell-1)}} - \sum_{j=1}^{k}\frac{A_j(\ell)\{t_1^\ddagger(\theta)\}^{(\ell-j)}}{\{1-F^\ddagger(\theta)\}^{(\ell-j)}}. \tag{3.5}$$

If ℓ is an integer, the argument which follows could be simplified; however, let us suppose ℓ is not an integer. By theorem 2 we can state that $q^{\dagger}(\theta/\lambda)\{t_1^{\ddagger}(\theta)\}^{(\ell-j)}$ belongs to $\mathcal{B}^{\ddagger}(M;\ell-1)$ for any $\lambda>0$ (we use $F\in\mathfrak{S}$ to ensure that $t_1^{\ddagger}(\theta)$ stays away from zero). Thus

$$A_1^{\ddagger}(\theta)\equiv q^{\dagger}(\theta/\lambda)\Lambda^{\ddagger}(\theta)=\frac{V^{\ddagger}(\theta)}{\{1-F^{\ddagger}(\theta)\}^{(\ell-1)}},\tag{3.6}$$

where $V^{\ddagger}(\theta)\in\mathcal{B}^{\ddagger}(M;\ell-1)$. Furthermore, since $\Lambda^{\ddagger}(\theta)=O(1)$ as $\theta\to 0$, it follows that $V(\theta)=O(|\theta|^{(\ell-1)})$. From theorem 1, with $\gamma=(\ell-1)$ we can now infer that $A_1^{\ddagger}(\theta)\in\mathcal{B}^{\ddagger}(M;0)$. Similarly, if $M\in\mathfrak{M}_3^*(\rho)$, we can prove that $A_1^{\ddagger}(\theta)\in\mathcal{B}^{\ddagger}(M_\rho;0)$.

Theorem 2 shows that

$$A_2^{\ddagger}(\theta)\equiv\{1-q^{\dagger}(\theta/\lambda)\}/\{1-F^{\ddagger}(\theta)\}^{(\ell-1)}\tag{3.7}$$

belongs to $\mathcal{B}^{\ddagger}(M;\ell)$, and it is easy to see from lemma 4 that, for every j, $1\leq j\leq k$,

$$A_{3_j}^{\ddagger}(\theta)\equiv\{1-q^{\dagger}(\theta/\lambda)\}/(-\mu_1 i\theta)^{(\ell-j)}\tag{3.8}$$

belongs either to $\mathcal{B}^{\ddagger}(M;0)$ if $M\in\mathfrak{M}_2^*(\rho)$ or to $\mathcal{B}^{\ddagger}(M_\rho;0)$ if $M\in\mathfrak{M}_3^*(\rho)$. Since

$$\Lambda^{\ddagger}(\theta)=A_1^{\ddagger}(\theta)+A_2^{\ddagger}(\theta)+\sum_{j=1}^{k}A_{3_j}^{\ddagger}(\theta),\tag{3.9}$$

it is clear that the theorem is proved except if $M\in\mathfrak{M}_1^*(\rho)$.

Suppose, therefore, that $M\in\mathfrak{M}_1^*(\rho)$. It is easy to see that $A_1(x)$, in this case, is of bounded variation and simple computations will show that, as $x\to\infty$, $x^{\rho}\{A_1(x)-A_1(\infty)\}$ tends to 0 or C_ℓ, according as $\mu_1<0$ or $\mu_1>0$. The same limits for $A_2(x)$ must be zero because $A_2(x)\in\mathcal{B}(M;\ell)$. The same limits for $A_{3j}(x)$ must be zero because $A_{3j}^{\ddagger}(\theta)$ vanishes identically in a neighborhood of the origin. Thus the theorem is proved.

It is convenient at this point to give the following proof.

Proof of corollary 5.3. Suppose that $\ell\geq 2$ is an integer. We must prove that $(e^{-i\theta h}-1)\Lambda^{\ddagger}(\theta)$ belongs to $\mathcal{B}^{\ddagger}(M;1)$. This plainly amounts to showing that $(e^{-i\theta h}-1)A_1^{\ddagger}(\theta)\in\mathcal{B}^{\ddagger}(M;1)$. But

$$(e^{-i\theta h}-1)A_1^{\ddagger}(\theta)=A_{11}^{\ddagger}(\theta)A_{12}^{\ddagger}(\theta),\quad\text{say,}\tag{3.10}$$

where

$$A_{11}^{\ddagger}(\theta)=\frac{(e^{-i\theta h}-1)q^{\dagger}(\theta/2\lambda)}{\{1-F^{\ddagger}(\theta)\}}\tag{3.11}$$

and

$$A_{12}^{\ddagger}(\theta)=\frac{V^{\ddagger}(\theta)}{\{1-F^{\ddagger}(\theta)\}^{(\ell-2)}}.\tag{3.12}$$

Theorem 1 shows that both $A_{11}^{\ddagger}(\theta)$ and $A_{12}^{\ddagger}(\theta)$ belong to $\mathcal{B}^{\ddagger}(M;1)$. Thus this proves the corollary for ℓ integral.

If ℓ is not integral, we remark that both $A_{11}^{\ddagger}(\theta)$ and $V^{\ddagger}(\theta)$ belong to $\mathcal{B}(M;\ell-1)$.

The final part follows on writing $\lambda(\theta) = A^{\ddagger}_{11}(\theta)V^{\ddagger}(\theta)$,

$$(e^{-i\theta h} - 1)A^{\ddagger}_{1}(\theta) = \frac{\lambda(\theta)}{\{1 - F^{\ddagger}(\theta)\}^{(\ell-2)}} \tag{3.13}$$

and appealing to theorem 1.

Note that in proving theorem 4 for an integer value of ℓ, the awkward condition 1(v) must be checked; it is easily found to be satisfied in the present circumstances.

Before we prove theorem 5 we shall establish the following.

LEMMA 5. *Suppose that*

(i) $\{A_n(x)\}$ *is a sequence of bounded and nondecreasing functions,*

(ii) $\{B_n(x)\}$ *is a sequence of functions of bounded variation which are uniformly bounded in any finite interval, and* $B_n(x) \to B(x)$ *for almost all* x *as* $n \to \infty$, *where* $B(x)$ *is of bounded variation in any finite interval,*

(iii) $A^{\ddagger}_n(\theta) - B^{\ddagger}_n(\theta) \to 0$ *boundedly as* $n \to \infty$.

Then we can conclude that $A_n(a) - A_n(b) \to B(a) - B(b)$, *as* $n \to \infty$, *whenever* a *and* b *are continuity points of* $B(x)$.

PROOF. Let us introduce the special function (recognizable as the so-called triangular probability density function):

$$\begin{aligned} \Delta_a(x) &= a^{-1} - |xa^{-2}|, \ |x| \leq a, \\ &= 0, \text{ otherwise.} \end{aligned} \tag{3.14}$$

Its Fourier transform is $\Delta^{\dagger}_a(\theta) = (\sin^2(a\theta/2)/(a\theta/2)^2)$, which conveniently happens to be a nonnegative member of $\mathfrak{L}$ also, like $\Delta_a(x)$. Let us also write

$$f_n(x) = \int_{-\infty}^{+\infty} \Delta_a(x - z)\, dA_n(z), \tag{3.15}$$

$$g_n(x) = \int_{-\infty}^{+\infty} \Delta_a(x - z)\, dB_n(z). \tag{3.16}$$

Then the functions $f_n(x)$ and $g_n(x)$ are continuous functions in $\mathfrak{L}$, and the two transforms

$$f^{\dagger}_n(\theta) = \Delta^{\dagger}_a(\theta)A^{\ddagger}_n(\theta), \qquad g^{\dagger}_n(\theta) = \Delta^{\dagger}_a(\theta)B^{\ddagger}_n(\theta) \tag{3.17}$$

are both also functions in $\mathfrak{L}$. Therefore,

$$f_n(x) - g_n(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\theta x}\Delta^{\dagger}_a(\theta)\{A^{\ddagger}_n(\theta) - B^{\ddagger}_n(\theta)\}\, d\theta. \tag{3.18}$$

In view of assumption (iii) of the present lemma, $\Delta^{\dagger}_a(\theta)\{A^{\ddagger}_n(\theta) - B^{\ddagger}_n(\theta)\} \to 0$ as $n \to \infty$, and this convergence is dominated by some multiple of the $\mathfrak{L}$-function $\Delta^{\dagger}_a(\theta)$. Thus we can infer from (3.18) that

$$f_n(x) - g_n(x) \to 0, \qquad \text{as} \quad n \to \infty. \tag{3.19}$$

Integration by parts shows that

$$g_n(x) = -\int_{x-a}^{x+a} B_n(z)\, d_z\Delta_a(x - z). \tag{3.20}$$

However, by (ii), $B_n(z) \to B(z)$ boundedly in any finite interval. Therefore,

$$g_n(x) \to -\int_{x-a}^{x+a} B(z)\, d_z\Delta_a(x - z), \qquad \text{as} \quad n \to \infty. \tag{3.21}$$

If we now recall that $B(z)$ is of bounded variation in every finite interval, we shall see that a further integration by parts is legitimate, and yields

$$g_n(x) \to \int_{x-a}^{x+a} \Delta_a(x - z)\, dB(z), \qquad \text{as} \quad n \to \infty. \tag{3.22}$$

Thus we can infer from (3.19) that, as $n \to \infty$,

$$\int_{x-a}^{x+a} \Delta_a(x - z)\, dA_n(z) \to \int_{x-a}^{x+a} \Delta_a(x - z)\, dB(z), \tag{3.23}$$

and this convergence holds for every x and every $a > 0$. The deduction of lemma 5 from (3.23) now proceeds on fairly routine lines, by sandwiching rectangular step-functions between suitable linear combinations of triangular functions, and so on. An example of such a procedure will be found in section 8 of Smith [12]. It is in this stage of the proof that the nondecreasing property of $A_n(x)$ is needed.

PROOF OF THEOREM 5. We shall use the special $\mathcal{L}$-functions defined for $\lambda > 0$ and $n > 0$,

$$\begin{aligned} e_n(\lambda; x) &= \frac{e^{-\lambda x} x^{n-1}}{\Gamma(n)}, \; x \geq 0, \\ &= 0, \text{ otherwise.} \end{aligned} \tag{3.24}$$

The corresponding Fourier transforms are $e_n^{\dagger}(\lambda; \theta) = (1/(\lambda - i\theta)^n)$. We shall also write $F_n(x) = P\{S_n \leq x\}$ and, for $0 < \zeta < 1$, define

$$H_\zeta(x) = \sum_{n=0}^{\infty} \zeta^r \binom{-(\ell - 1)}{n} F_n(x). \tag{3.25}$$

Then $H_\zeta(x)$ is a nondecreasing and bounded function. Moreover, if we write $\varphi(\theta) = F_1^{\dagger}(\theta)$ for the characteristic function of X_1, it is easily seen that

$$H_\zeta^{\dagger}(\theta) = \frac{1}{(1 - \zeta\varphi(\theta))^{(\ell-1)}}. \tag{3.26}$$

Let J be some small open interval centered at the origin and let us suppose for the time being that θ is confined to J. Write $\beta = (1 - \zeta) - \zeta\mu_1 i\theta$. Then

$$\left|\frac{\theta}{\beta}\right|^2 = \frac{\theta^2}{(1 - \zeta)^2 + \zeta^2\mu_1^2\theta^2}, \tag{3.27}$$

and by elementary calculus we can show that $|\theta/\beta|^2 \leq (1/\mu_1^2) + \theta^2$ uniformly for $0 < \zeta < 1$. Thus, for all θ in J we have that $|\theta| = O(|\beta|)$, uniformly with respect to ζ. In addition, we might notice that β will be uniformly bounded for θ in J and $0 < \zeta < 1$.

Since $\mathcal{E}|X_1|^\ell < \infty$, we have from theorem 3, after a little algebraic manipulation, that

$$\{1 - \zeta\varphi(\theta)\}^{-(\ell-1)} = \beta^{-(\ell-1)}\left\{1 - \frac{\zeta}{\beta}\sum_{j=2}^{k}\frac{\mu_j}{j!}(i\theta)^j + O(|\theta|^{(\ell-1)})\right\}^{-(\ell-1)}, \tag{3.28}$$

where k is the greatest integer $\leq \ell$, and the summation on the right vanishes for $k < 2$. We shall suppose $k \geq 2$ in what follows; the case $k = 1$ is simpler. Note that we have already started to use the fact that $|\theta| = O(|\beta|)$ and that the O-term on the right is uniform with respect to ζ. By expansion from (3.28) one obtains

$$H_\zeta^\ddagger(\theta) = \beta^{-(\ell-1)}S, \tag{3.29}$$

where

$$\begin{aligned} S = 1 &+ \frac{(\ell-1)\zeta}{\beta}\sum_{j=2}^{k}\frac{\mu_j}{j!}(i\theta)^j \\ &+ \frac{(\ell-1)\ell\zeta^2}{2!\beta^2}\left[\sum_{j=2}^{k-1}\frac{\mu_j}{j!}(i\theta)^j\right]^2 \\ &+ \cdots \\ &+ \frac{(\ell-1)\ell\cdots(\ell+k-3)\zeta^{(k-1)}}{(k-1)!\beta^{(k-1)}}\left[\frac{\mu_2}{2!}(i\theta)^2\right]^{(k-1)} \\ &+ O(|\theta|^{(\ell-1)}). \end{aligned} \tag{3.30}$$

On substituting $i\theta = \{(1-\zeta) - \beta\}/(\zeta\mu_1)$, performing several expansions, and collecting terms, we evidently obtain a relation as follows:

$$H_\zeta^\ddagger(\theta) = \sum_{j=-k+2}^{j=k}{}^{*}\frac{A_j(\ell,\zeta)}{\beta^{(\ell-j)}} + O(1). \tag{3.31}$$

where Σ^* means "omit the term $j = k$, if $\ell = k$." Note that we continue to make use of the uniform $|\theta| = O(|\beta|)$ result. The coefficients $A_j(\ell, \zeta)$ which appear in (3.31) are rational functions of ζ and of the moments $\mu_1, \mu_2, \cdots, \mu_k$. If we refer to theorem 4 and note that $-\beta \to \mu_1 i\theta$ as $\zeta \to 1 - 0$, then it becomes apparent that, as $\zeta \to 1 - 0$,

$$\begin{aligned} A_j(\ell,\zeta) &\to A_j(\ell), && \text{for } j = 1, 2, \cdots, k. \\ &\to 0, && \text{for } j = 0, -1, -2, \cdots, -k+2. \end{aligned} \tag{3.32}$$

where the constants $A_j(\ell)$ are defined in theorem 4.

At this point we recognize $\zeta^j\mu_1^j/\beta^j$ as the transform $e_j^\dagger(\lambda, \theta)$ with $\lambda = (1-\zeta)/(\zeta\mu_1)$. Let us define

$$\Psi_\zeta(x) = H_\zeta(x) - \sum_{j=-k+2}^{j=k}{}^{*}\frac{A_j(\ell,\zeta)}{(\zeta\mu_1)^{(\ell-j)}}\int_{-\infty}^{x} e_{(\ell-j)}(\lambda, y)\,dy. \tag{3.33}$$

Then $\Psi_\zeta(x) \in \mathcal{B}$ and

$$\Psi_\zeta^\ddagger(\theta) = H_\zeta^\ddagger(\theta) - \sum_{j=-k+2}^{j=k}{}^{*}\frac{A_j(\ell,\zeta)}{\beta^{(\ell-j)}}. \tag{3.34}$$

By (3.31) the transform $\Psi_\zeta^\ddagger(\theta)$ is bounded for θ in the interval J, uniformly with respect to ζ, $0 < \zeta < 1$.

Since $\varphi(\theta)$ is assumed to belong to $\mathfrak{S}^{\ddagger}$, it will be bounded away from unity on the complement of J; using this fact it is not a difficult matter to prove that $\Psi_{\zeta}^{\ddagger}(\theta)$ is bounded on the complement of J, uniformly with respect to ζ. Therefore,

$$\Psi_{\zeta}^{\ddagger}(\theta) \to \Psi_{1}^{\ddagger}(\theta) = \frac{1}{(1 - \varphi(\theta))^{(\ell-1)}} - \sum_{j=1}^{k}{}^{*} \frac{A_j(\ell)}{(-\mu_1 i\theta)^{(\ell-j)}} \tag{3.35}$$

as $\zeta \to 1 - 0$, and this convergence takes place boundedly.

By theorem 4, the function $\Psi_1^{\ddagger}(\theta)$ is the Fourier-Stieltjes transform of some function $\Psi_1(x)$, say, of $\mathfrak{B}(M; 0)$, or some related class, depending on M and ℓ. We are now in a position to apply lemma 5. The functions $A_n(x)$ of the lemma here become the functions $H_{\zeta}(x)$. We see that these functions $H_{\zeta}(x)$ are bounded and nondecreasing and, from (3.25) and the principle of monotone convergence, we see that as $\zeta \to 1 - 0$,

$$H_{\zeta}(x) \to H_1(x) = \sum_{n=0}^{\infty} \binom{-(\ell - 1)}{n} F_n(x); \tag{3.36}$$

the limit $H_1(x)$, so far as we know at this stage in the argument, may be infinite.

The role of the functions $B_n(x)$ of the lemma is to be played by the functions $\Xi_{\zeta}(x)$, say, where

$$\Xi_{\zeta}(x) = \Psi_1(x) + \sum_{j=-k+2}^{j=k}{}^{*} \frac{A_j(\ell, \zeta)}{(\zeta\mu_1)^{(\ell-j)}} \int_{-\infty}^{x} e_{(\ell-j)}(\lambda, y)\, dy \tag{3.37}$$

(recall that $\lambda = (1 - \zeta)/(\zeta\mu_1)$). It is obvious that for $0 < \zeta_1 \leq \zeta \leq 1$ the functions $\Xi_{\zeta}(x)$ are uniformly bounded in any finite interval, and as $\zeta \to 1 - 0$,

$$\Xi_{\zeta}(x) \to \Xi_1(x) = \Psi_1(x) + \sum_{j=1}^{k}{}^{*} \frac{A_j(\ell) x^{(\ell-j)}}{(\ell - j)!\mu_1^{(\ell-j)}} U(x). \tag{3.38}$$

The limit $\Xi_1(x)$ is of bounded variation in every finite interval.

From (3.35) we have that $H_{\zeta}^{\ddagger}(\theta) - \Xi_{\zeta}^{\ddagger}(\theta) \to 0$ boundedly as $\zeta \to 1 - 0$. Hence we can infer from lemma 5 that whenever a and b are continuity points of $\Xi_1(x)$,

$$H_{\zeta}(a) - H_{\zeta}(b) \to \Xi_1(a) - \Xi_1(b) \tag{3.39}$$

as $\zeta \to 1 - 0$. If we assume $a > b$ and appeal to monotone convergence, this last limiting result implies

$$\sum_{n=0}^{\infty} \binom{-(\ell - 1)}{n} \{F_n(a) - F_n(b)\} = \Xi_1(a) - \Xi_1(b). \tag{3.40}$$

We now let $b \to \infty$ and deduce that

$$\sum_{n=0}^{\infty} \binom{-(\ell - 1)}{n} F_n(a) = \Xi_1(a) - \Psi_1(-\infty), \tag{3.41}$$

from which the theorem is proved. The various properties of the "remainder term" $\Omega(x)$ all come by applying theorem 4 to $\Psi_1(x)$.

Proof of corollary 5.1. For this part we have $F(x) \in \mathfrak{B}(I; \nu)$ for some $\nu \geq \ell > 1$, but $F^{\ddagger}(\theta) = \varphi(\theta) \in \mathfrak{U}^{\ddagger}$ rather than $\mathfrak{S}^{\ddagger}$. Thus a certain part of the argument leading up to theorem 5 does not apply. In particular, the difficulty occurs in the proof of theorem 1 when dealing with $\Psi_2^\ddagger(\theta)$; we cannot state that

$$\{1 - q^\dagger(\theta/\lambda)\}/\{1 - \varphi(\theta)\}^{(\ell-1)} \tag{3.42}$$

belongs to a class $\mathcal{B}^\ddagger$. It is, however, still true that $\Psi_\zeta^\ddagger(\theta) \to \Psi_1^\ddagger(\theta)$ boundedly as $\zeta \to 1 - 0$, and that $q^\dagger(\theta/\lambda)\Psi_1^\ddagger(\theta)$ belongs to an appropriate $\mathcal{B}^\ddagger$-class. Recall that

$$\mathcal{P}(x) = \sum_{j=1}^{j=k} \frac{A_j(\ell)x^{(\ell-j)}}{(\ell - j)!\mu_1^{(\ell-j)}} U(x). \tag{3.43}$$

Then, as $\zeta \to 1 - 0$, we have that

$$\Psi_\zeta(x) \to \Psi_1(x) = H_1(x) - \mathcal{P}(x). \tag{3.44}$$

The fact that $H_1(x) < \infty$ is a consequence of corollary 5.2, whose proof is independent of the present argument. By a familiar inversion theorem we can show, for any $a > 0$, any u, v,

$$\begin{aligned}\int_u^{u+a} \Psi_\zeta(y)\, dy - \int_v^{v+a} \Psi_\zeta(y)\, dy \\ = \int_{-\infty}^{+\infty} \left(\frac{e^{-iu\theta} - e^{-iv\theta}}{-i\theta}\right)\left(\frac{e^{ia\theta} - 1}{i\theta}\right) \Psi_\zeta^\ddagger(\theta)\, d\theta.\end{aligned} \tag{3.45}$$

We note that, since $\Psi_\zeta^\ddagger(\theta)$ is uniformly bounded as $\zeta \to 1 - 0$, we can appeal to dominated convergence to deduce that the above equation holds with $\zeta = 1$.

Let us write $\Omega(x)$ for the function of bounded variation such that $\Omega(-\infty) = 0$ and $\Omega^\ddagger(\theta) = q^\dagger(\theta/\lambda)\Psi_1^\ddagger(\theta)$. Then

$$\begin{aligned}\int_u^{u+a} \Omega(y)\, dy - \int_v^{v+a} \Omega(y)\, dy \\ = \int_{-\infty}^{+\infty} \left(\frac{e^{-iu\theta} - e^{-iv\theta}}{-i\theta}\right)\left(\frac{e^{ia\theta} - 1}{i\theta}\right) \Omega^\ddagger(\theta)\, d\theta.\end{aligned} \tag{3.46}$$

Thus, if we set $\Theta(\theta) = (1 - q^\dagger(\theta/\lambda)/\theta^2)\Psi_1^\ddagger(\theta)$, then it follows that

$$\begin{aligned}\int_u^{u+a} \{\Psi_1(y) - \Omega(y)\}\, dy - \int^{v+a} \{\Psi_1(y) - \Omega(y)\}\, dy \\ = \int_{-\infty}^{+\infty} (e^{-iu\theta} - e^{-iv\theta})(e^{ia\theta} - 1)\Theta(\theta)\, d\theta.\end{aligned} \tag{3.47}$$

Since $\varphi(\theta) \in \mathcal{U}^\ddagger$, $\Theta(\theta)$ is in $\mathcal{L}$ and so we can appeal to the Riemann-Lebesgue lemma to infer (letting $v \to -\infty$),

$$\int_u^{u+a} \{\Psi_1(y) - \Omega(y)\}\, dy = \int_{-\infty}^{+\infty} (e^{i(a-u)\theta} - e^{-iu\theta})\Theta(\theta)\, d\theta. \tag{3.48}$$

We are supposing that $F(x) \in \mathcal{B}(I; \nu)$. Let m be the greatest integer not exceeding ν. Then $\Theta(\theta)$ can be differentiated m times and it is not hard to see that $\Theta^{(m)}(\theta)$ also belongs to $\mathcal{L}$. Then, by repeated integrations by parts, we have

$$\int_{-\infty}^{+\infty} (e^{i(a-x)\theta} - e^{-ix\theta})\Theta(\theta)\, d\theta = \int_{-\infty}^{+\infty} T(\theta)\Theta^{(m)}(\theta)\, d\theta, \text{ say,} \tag{3.49}$$

where

$$T(\theta) = \frac{e^{-i\theta x}}{(-ix)^m} - \frac{e^{-i(x-a)\theta}}{(-i(x-a))^m}. \tag{3.50}$$

Let κ be small and $\kappa > 0$; suppose we substitute $a = x^{-m}$. We consider a number of cases: suppose $x > 0$ in their arguments.

Case (i) $0 < |\theta| < x^{\kappa}$:

$$|T(\theta)| < \frac{1}{x^m}\left|1 - \frac{e^{ix^{-(m-\kappa)}}}{(1 - x^{-(m+1)})^m}\right| = O(x^{-(2m-\kappa)}). \tag{3.51}$$

Now we can write $|\Theta^{(m)}(\theta)| = \Delta(\theta)/\theta^2$, say, where $\Delta(\theta)$ is a bounded function which vanishes identically in $[-\lambda, +\lambda]$. Thus

$$\int_{-x^{\kappa}}^{+x^{\kappa}} |T(\theta)||\Theta^{(m)}(\theta)|\, d\theta = O(x^{-(2m-\kappa)}). \tag{3.52}$$

Case (ii) $x^{j\kappa} \leq |\theta| \leq x^{(j+1)\kappa}$, *for integer* j *such that* $(j+1)\kappa \leq m$:

$$|T(\theta)| = O(x^{-(2m-(j+1)\kappa)}). \tag{3.53}$$

Thus if A_j is the θ-set: $x^{j\kappa} \leq |\theta| \leq x^{(j+1)\kappa}$,

$$\begin{aligned}\int_{A_j} |T(\theta)|\,\frac{\Delta(\theta)}{\theta^2}\, d\theta &= O\left(\frac{1}{x^{(2m-(j+1)\kappa)}}\int_{A_j}\frac{\Delta(\theta)}{\theta^2}\, d\theta\right),\\ &= O(x^{-(2m-\kappa)}),\end{aligned} \tag{3.54}$$

since it is easily shown that $\int_{A_j} (\Delta(\theta)/\theta^2)\, d\theta = O(x^{-j\kappa})$.

Case (iii) $x^{j\kappa} \leq |\theta|$ *for integer* j *such that* $j\kappa \leq m < (j+1)\kappa$: here we can only state that $|T(\theta)| = O(x^{-m})$. However,

$$\int_{x^{j\kappa}}^{\infty} \frac{\Delta(\theta)}{\theta^2}\, d\theta = O(x^{-j\kappa}), \tag{3.55}$$

and since $j\kappa > m - \kappa$, we have that

$$\int_{x^{j\kappa} \leq |\theta|} |T(\theta)|\,\frac{\Delta(\theta)}{\theta^2}\, d\theta = O(x^{-(2m-\kappa)}). \tag{3.56}$$

Thus, on combining the results of the finite number of cases together, and using the result in (3.48), we have that, as $x \to \infty$,

$$\int_x^{x+x^{-m}} \{H_1(y) - \mathcal{P}(y) - \Omega(y)\}\, dy = O(x^{-(2m-\kappa)}), \tag{3.57}$$

where $\kappa > 0$ can be arbitrarily small.

We note that

$$x^m \int_x^{x+x^{-m}} y^{(\ell-j)}\, dy = x^{(\ell-j)} + O(x^{(\ell-j-1-m)}) \tag{3.58}$$

so that

$$x^m \int_x^{x+x^{-m}} \mathcal{P}(y)\, dy = \mathcal{P}(x) + O(x^{(\ell-2-m)}). \tag{3.59}$$

Now $\Omega^{\ddagger}(\theta)$ is bounded and is identically zero outside the interval $(-2\lambda, +2\lambda)$. Thus there is a bounded function $w(x)$, say, such that $w(x) \to 0$ as $|x| \to \infty$ and $\Omega(x) = \int_{-\infty}^{x} w(y)\,dy$. Thus

$$x^m \int_x^{x+x^{-m}} \Omega(y)\,dy = \Omega(x) + x^m \int_x^{x+x^{-m}} \left\{\int_x^y w(u)\,du\right\} dy. \tag{3.60}$$

However,

$$\int_x^{x+x^{-m}} \left\{\int_x^y w(u)\,du\right\} dy = o\left(\int_x^{x+x^{-m}} (y - x)\,dy\right) = o(x^{-2m}). \tag{3.61}$$

Thus,

$$x^m \int_x^{x+x^{-m}} \Omega(y)\,dy = \Omega(x) + o(x^{-m}). \tag{3.62}$$

(a) *If ℓ is an integer.* We must have $\ell \geq 2$ in this case and, from all we know about $\Psi_1^{\ddagger}(\theta)$, we can also state that $\Omega(x) \in \mathcal{B}(I; \nu - \ell)$. From (3.57), (3.59), and (3.62) and the monotone character of $H_1(x)$ we have

$$H_1(x) \leq \mathcal{P}(x) + \Omega(x) + o(x^{-m}) + O(x^{-(m+2-\ell)}) + O(x^{-(m-\kappa)}), \tag{3.63}$$

where $\kappa > 0$ can be arbitrarily small. Evidently if $\ell = 2$, we can simplify the last inequality to

$$H_1(x) \leq \mathcal{P}(x) + \Omega(x) + O(x^{-(m-\kappa)}) \tag{3.64}$$

whereas, if $\ell > 2$, we must have

$$H_1(x) \leq \mathcal{P}(x) + \Omega(x) + O(x^{-(m+2-\ell)}). \tag{3.65}$$

It is clear that similar arguments will produce the needed reverse inequalities on $H_1(x)$ to complete the proof (when ℓ is an integer).

(b) *If ℓ is not an integer.* Recall that the integer k is such that $k \leq \ell < k + 1$. If $\nu < k + 1$, then we can state that $\Omega(x) \in \mathcal{B}(I; \nu - \ell)$ and the argument proceeds much as before. If $\ell < 2$, we have the result (3.64) and if $\ell > 2$, we have (3.65). If, however, $\nu \geq k + 1$, then we set $\rho = k - \ell + 1$ as usual and have

$$\limsup_{x\to\infty} x^{\rho}\{H_1(x) - \mathcal{P}(x) - \Omega(\infty)\} \leq C_\ell, \tag{3.66}$$

where C_ℓ is defined in the enunciation of theorem 4. Similarly we can obtain a result concerning the lim inf and complete the proof of the corollary.

Proof of corollary 5.2. Let Δ be a large positive number, and define a sequence of truncated variables $\{X_n(\Delta)\}$ by

$$\begin{aligned} X_n(\Delta) &= X_n \quad \text{if} \quad X_n \leq \Delta, \\ &= \Delta \quad \text{if} \quad X_n > \Delta. \end{aligned} \tag{3.67}$$

Then $\{X_n(\Delta)\}$ are independent and identically distributed and have their first ℓ absolute moments finite. We shall assume Δ chosen sufficiently large to make $\mathcal{E}X_n(\Delta) > 0$; clearly the assumptions of corollary 5.2 make this possible. Let $\{Y_n\}$ be a sequence of independent random variables, each uniformly distributed over the interval (0, 1); assume the $\{Y_n\}$ and the $\{X_n(\Delta)\}$ to be independent. We shall write $S_n(\Delta) = X_1(\Delta) + X_2(\Delta) + \cdots + X_n(\Delta)$, $T_n = Y_1 + Y_2 + \cdots + Y_n$, and we choose a small positive $\epsilon < \mathcal{E}X_n(\Delta)$. Then $\{X_n(\Delta) - \epsilon Y_n\}$ is a sequence of independent and identically distributed random variables with strictly positive first moments, finite absolute moments of order ℓ, and an absolutely continuous distribution function. We can therefore appeal to theorem 5 to deduce that

$$\sum_{n=0}^{\infty} \binom{-(\ell-1)}{n} P\{S_n(\Delta) - \epsilon T_n \leq x\} = \sum_{j=1}^{k} \frac{A_j(\ell)}{\Gamma(\ell - j + 1)} \left(\frac{x}{\mathcal{E}X_n(\Delta) - \epsilon}\right)^{(\ell-j)} U(x) + \Lambda(x), \tag{3.68}$$

where $\Lambda(x)$ is a function of bounded variation and the coefficients $A_j(\ell)$ now depend on the moments of $X_n - \epsilon Y_n$ rather than X_n. Since $S_n \geq S_n(\Delta) - \epsilon T_n$, it follows from (3.68) that

$$\sum_{n=0}^{\infty} \binom{-(\ell-1)}{n} P\{S_n \leq x\} \leq \sum_{j=1}^{k} \frac{A_j(\ell)}{\Gamma(\ell - j + 1)} \left(\frac{x}{\mathcal{E}X_n(\Delta) - \epsilon}\right)^{(\ell-j)} U(x) + \Lambda(x), \tag{3.69}$$

and this implies the finiteness claimed in the theorem for the sum on the left-hand side.

It will be seen by referring to the proof of theorem 3 that $A_1(\ell) = 1$. Thus, from (3.69), we have

$$\limsup_{x\to\infty} \frac{\Gamma(\ell)}{x^{(\ell-1)}} \sum_{n=0}^{\infty} \binom{-(\ell-1)}{n} P\{S_n \leq x\} \leq \frac{1}{[\mathcal{E}(X_n(\Delta) - \epsilon)]^{(\ell-1)}}. \tag{3.70}$$

Let $\epsilon \to 0$ and then $\Delta \to \infty$ on the right of this inequality; we then find

$$\limsup_{x\to\infty} \frac{\Gamma(\ell)}{x^{(\ell-1)}} \sum_{n=0}^{\infty} \binom{-(\ell-1)}{n} P\{S_n \leq x\} \leq \frac{1}{\mu_1^{(\ell-1)}}. \tag{3.71}$$

From the fact that, as $n \to \infty$, $\Gamma(\ell)\binom{-(\ell-1)}{n} \backsim (\ell-1)n^{(\ell-2)}$, it is now an easy matter to infer that

$$\limsup_{x\to\infty} \frac{(\ell-1)}{x^{(\ell-1)}} \sum_{n=1}^{\infty} n^{(\ell-2)} P\{S_n \leq x\} \leq \frac{1}{\mu_1^{(\ell-1)}}. \tag{3.72}$$

If $\mu_1 = \infty$, then (3.72) proves the corollary. If $\mu_1 < \infty$ we must now obtain an inequality reverse to (3.72) for the inferior limit. This can be obtained fairly easily by invoking the weak law of large numbers. Since μ_1 is finite, it follows from this law that $S_n/n \to \mu_1$ in probability as $n \to \infty$. Therefore, given an $\epsilon > 0$ we can find $n_0(\epsilon)$ such that $P\{S_n \leq n\ (\mu_1 + \epsilon)\} > (1 - \epsilon)$ for all $n > n_0(\epsilon)$. If we employ the notation i.p. $[\xi]$ for the integer part of ξ, then it follows that for all large x

$$\sum_{n=1}^{\infty} n^{(\ell-2)} P\{S_n \geq x\} \geq (1 - \epsilon) \sum_{n=n_0(\epsilon)}^{\text{i.p.}[x/\mu_1+\epsilon]} n^{(\ell-2)}. \tag{3.73}$$

From this inequality we can infer that

$$\liminf_{x\to\infty} \frac{(\ell-1)}{x^{(\ell-1)}} \sum_{n=1}^{\infty} n^{(\ell-2)} P\{S_n \geq x\} \geq \frac{(1-\epsilon)}{(\mu_1+\epsilon)^{(\ell-1)}}. \tag{3.74}$$

The required inequality, and hence the corollary, follows by letting $\epsilon \to 0$ on the right-hand side. Incidentally it is interesting to note how much easier it has been to prove this last result, compared with (3.72).

Proof of theorem 6. In this proof we shall use the many-valued function $\log z$. We shall suppose as usual the complex plane to be slit along the negative real axis from 0 to $-\infty$ and $\log z$, defined on the slit plane, is to be that branch which is real on the positive real axis.

For ease, write $\varphi(\theta) = F^{\ddagger}(\theta)$ and write $\psi(\theta) = (1 - \mu_1 i\theta)^{-1}$ for the characteristic function of the negative-exponential distribution with the same mean as $\varphi(\theta)$. Choose a small $\epsilon > 0$ and define

$$\sigma_\epsilon(\varphi) = \sup_{|\theta| \geq \epsilon} |\varphi(\theta)|. \tag{3.75}$$

Because of the assumption $\varphi(\theta) \in \mathfrak{S}^{\ddagger}$, it follows that $\sigma_\epsilon(\varphi) < 1$. Similarly we have $\sigma_\epsilon(\psi) < 1$. Let $0 \leq \zeta \leq 1$, and $|\theta| \geq \epsilon$. Then the complex number $1 - \zeta\varphi(\theta)$ necessarily lies in the circle $|z - 1| \leq \sigma_\epsilon(\varphi)$ in the complex plane. From this it follows that there is an angle $\alpha_\epsilon(\varphi)$, say, such that

$$|\arg [1 - \zeta\varphi(\theta)]| \leq \alpha_\epsilon(\varphi) < \pi/2 \tag{3.76}$$

for all $0 \leq \zeta \leq 1$ and all $|\theta| \geq \epsilon$. In a similar way there must be an angle $\alpha_\epsilon(\psi)$ such that

$$|\arg [1 - \zeta\psi(\theta)]| \leq \alpha_\epsilon(\psi) < \pi/2, \tag{3.77}$$

also for $0 \leq \zeta \leq 1$ and $|\theta| \geq \epsilon$. From all this it follows that if we define the complex quantity

$$z_\zeta(\theta) = \frac{1 - \zeta\varphi(\theta)}{1 - \zeta\psi(\theta)}, \tag{3.78}$$

then

$$|\arg z_\rho(\theta)| \leq \alpha_\epsilon(\varphi) + \alpha_\epsilon(\psi) < \pi \tag{3.79}$$

for all $0 \leq \zeta \leq 1$, $|\theta| \geq \epsilon$.

In addition, for the same range of ζ and θ, we see that

$$0 < \frac{1 - \sigma_\epsilon(\varphi)}{2} \leq |z_\zeta(\theta)| \leq \frac{2}{1 - \sigma_\epsilon(\psi)} < \infty. \tag{3.80}$$

Thus $z_\zeta(\theta)$ always lies in a certain bounded closed region A, say, of the complex plane. This region A is defined by the inequalities (3.79) and (3.80). It is a circular annulus centered on the origin minus that part which also belongs to a narrow sector containing, and symmetrical with respect to, the negative real axis.

Since in proving theorem 6 we may assume $\mu_1 = \mu_1(F) > 0$, it follows from theorem 3 that $\varphi(\theta) = 1 + \mu_1 i\theta\varphi_{(1)}(\theta)$, where $\varphi_{(1)}(\theta) \in \mathfrak{B}^{\ddagger}(M; \ell - 1)$ and $\varphi_{(1)}(0) = 1$. Computation based upon (3.78) then shows that

$$z_\zeta(\theta) = 1 - \frac{\zeta\mu_1 i\theta}{1 - \mu_1 i\theta - \zeta}\,[\varphi_{(1)}(\theta) - \varphi(\theta)], \tag{3.81}$$

if we recall the special form of $\psi(\theta)$.

Suppose now that $0 \leq \zeta \leq 1$ and that $|\theta| < \epsilon$. Given a small $\delta > 0$ we can make ϵ small enough to ensure $|\varphi_{(1)}(\theta) - \varphi(\theta)| < \delta$ because $\varphi_{(1)}(\theta) - \varphi(\theta) \to 0$ as $|\theta| \to 0$. Moreover, it is easy to show that when $0 \leq \zeta \leq 1$, $|\zeta\mu_1 i\theta| \leq |1 - \mu_1 i\theta - \zeta|$. Therefore, from (3.81), it follows that $z_\zeta(\theta)$ will lie within the small circle $|z - 1| \leq \delta$ for all $0 \leq \zeta \leq 1$ and all $|\theta| < \epsilon$. This circle is clearly within the region A. We can now state the following.

LEMMA 6. *Let the complex plane be slit along the negative real axis from* 0 *to* $-\infty$ *and define* $\log z$ *on the slit plane to be the branch which takes real values for real positive* z. *Then there is a bounded open set* A^* *in the slit plane, this set* A^* *being a strictly positive distance from the slit, and such that*

$$\frac{1 - \zeta\varphi(\theta)}{1 - \zeta(1 - \mu_1 i\theta)^{-1}} \tag{3.82}$$

lies within A^* *for all* θ *and all* ζ, $0 \leq \zeta \leq 1$. *As a consequence of these facts and of* (3.81), *it follows that*

$$\log\left(\frac{1 - \zeta\varphi(\theta)}{1 - \zeta(1 - \mu_1 i\theta)^{-1}}\right) \to \log\,(1 + \varphi_{(1)}(\theta) - \varphi(\theta)) \tag{3.83}$$

boundedly, as ζ *increases to unity.*

In addition to lemma 6, we shall need the following.

LEMMA 7. *Suppose that both* $F(x)$ *and* $G(x)$ *belong to* $\mathfrak{S} \cap \mathfrak{D}(M; \ell)$ *for some* $M \in \mathfrak{M}^*$ *and some* $\ell \geq 1$. *Suppose further that* $\mu_1(F)\cdot\mu_1(G) > 0$. *Then*

$$\log\left(\frac{1 - F^\ddagger(\theta)}{1 - G^\ddagger(\theta)}\right) \tag{3.84}$$

belongs to $\mathfrak{B}^\ddagger(M, \ell - 1)$; *the logarithm is intended to be the branch taking real values for real positive arguments.*

PROOF. Clearly it will be enough if we prove the lemma when $G(x)$ refers to a negative exponential distribution. Suppose first that $\mu_1 = \mu_1(F) = \mu_1(G) > 0$. Thus $G^\ddagger(\theta) = (1 - \mu_1 i\theta)^{-1}$. Then, by lemma 6, and the arguments leading thereto, we see that

$$z_1(\theta) = \frac{1 - F^\ddagger(\theta)}{1 - G^\ddagger(\theta)} = 1 + F^\ddagger_{(1)}(\theta) - F^\ddagger(\theta), \tag{3.85}$$

and, evidently, $z_1(\theta) \in \mathfrak{B}^\ddagger(M; \ell - 1)$. From theorem 2 and lemma 6 it appears that, for any $\lambda > 0$,

$$q^\dagger(\theta/\lambda)\log z_1(\theta) \tag{3.86}$$

belongs to $\mathfrak{B}^\ddagger(M; \ell - 1)$, where $q^\dagger(\cdot)$ is the usual S.M.F. Furthermore if, in theorem 2, we take the interval J as (λ, ∞) then we may infer that both

$$[1 - q^\dagger(\theta/\lambda)]\log\,[1 - F^\ddagger(\theta)] \quad \text{and} \quad [1 - q^\dagger(\theta/\lambda)]\log\,[1 - G^\ddagger(\theta)] \tag{3.87}$$

are in $\mathfrak{B}^{\ddagger}(M;\ell)$. Consequently,

$$[1 - q^{\dagger}(\theta/\lambda)] \log z_1(\theta) \tag{3.88}$$

belongs to $\mathfrak{B}^{\ddagger}(M;\ell)$. Combining (3.94) and (3.96) proves the lemma under our extra assumptions.

We must now remove the assumption that $\mu_1(\varphi) = \mu_1(\psi)$. Reflection will show that it will be sufficient to prove that if $0 < a < b < \infty$, then

$$\log \frac{1 - (1 - ai\theta)^{-1}}{1 - (1 - bi\theta)^{-1}} \tag{3.89}$$

belongs to $\mathfrak{B}^{\ddagger}(M;\ell - 1)$. To achieve this conclusion, consider the function

$$\begin{aligned} g(x) &= \frac{e^{-(x/a)} - e^{-(x/b)}}{x}, & x > 0 \\ &= 0, & x \leq 0. \end{aligned} \tag{3.90}$$

It is easy to see that $g(x)$ belongs to $\mathfrak{L}(M, \ell - 1)$. By evaluating an integral of the familiar Frullani type we can show that the Fourier transform of $g(x)$ is

$$g^{\dagger}(\theta) = \log \left(\frac{b^{-1} - i\theta}{a^{-1} - i\theta}\right), \tag{3.91}$$

and simple computation shows (3.89) to be the same function as $g^{\dagger}(\theta)$.

Finally, and to complete the proof of lemma 7, we must cover the situation in which both $\mu_1(\varphi) < 0$ and $\mu_1(\psi) < 0$. We content ourselves with the observation that this case can be dealt with by arguments similar to the ones already employed or, alternatively, by using the fact that the complex conjugate of a function of $\mathfrak{B}^{\ddagger}$ is also a function of $\mathfrak{B}^{\ddagger}$.

Now that we have established lemmas 6 and 7, the proof of theorem 6 presents little difficulty. Suppose $0 < \zeta < 1$ and consider the functions

$$H_{\zeta}(x) = \sum_{n=1}^{\infty} \frac{\zeta^n}{n} F_n(x); \tag{3.92}$$

$$\begin{aligned} B_{\zeta}(x) &= U(x) \sum_{n=1}^{\infty} \frac{\zeta^n}{\mu_1 n} \int_0^x \frac{e^{-(y/\mu_1)}(y/\mu_1)^{n-1}\, dy}{(n-1)!} \\ &= U(x) \int_0^x \frac{e^{-(1-\zeta)(y/\mu_1)} - e^{-(y/\mu_1)}}{y}\, dy. \end{aligned} \tag{3.93}$$

It is apparent that $H_{\zeta}(x)$ and $B_{\zeta}(x)$ are nondecreasing functions of $\mathfrak{B}$, and their Fourier-Stieltjes transforms are such that

$$B_{\zeta}^{\ddagger}(\theta) - H_{\zeta}^{\ddagger}(\theta) = \log \left(\frac{1 - \zeta\varphi(\theta)}{1 - \zeta(1 - \mu_1 i\theta)^{-1}}\right). \tag{3.94}$$

Moreover, if we now appeal to lemmas 6 and 7, we see that as ζ increases to unity

$$B_{\zeta}^{\ddagger}(\theta) - H_{\zeta}^{\ddagger}(\theta) \to \Psi^{\ddagger}(\theta), \tag{3.95}$$

where $\Psi^{\ddagger}(\theta)$ is the Fourier-Stieltjes transform of some function $\Psi(x)$ of $\mathfrak{B}(M;\ell - 1)$, and this convergence takes place boundedly.

We can now appeal to lemma 5, in much the same way as was done in proving theorem 4, and discover that

$$(3.96) \qquad \sum_{n=1}^{\infty} \frac{1}{n} F_n(x) = U(x) \int_0^x \frac{1 - e^{-(y/\mu_1)}}{y} \, dy - \Psi(x) + \Psi(-\infty).$$

However, since $\Psi^{\ddagger}(\theta) = \log(1 + \varphi_{(1)}(\theta) - \varphi(\theta))$, it is clear that $\Psi(+\infty) - \Psi(-\infty) = \Psi^{\ddagger}(0) = 0$. From this fact and (3.96) the theorem follows.

We shall not bother to give the detailed arguments for corollary 6.1 because it is deducible from theorem 6 in much the same way as we have deduced corollary 5.2 from theorem 5.

The proof of corollary 6.2 needs only the slightest sketch. If, taking $\ell \geq 2$, we define $K(x) = \sum_{n=1}^{\infty} n^{(\ell-2)} P\{S_n \leq x\}$, then by corollary 5.2 $K(x)$ is a nondecreasing function which is $O(x^{(\ell-1)})$ for large positive x; also $K(x) \to 0$ as $x \to -\infty$. If we write $G(x) = P\{X_0 \leq x\}$, then

$$(3.97) \qquad \sum_{n=1}^{\infty} n^{(\ell-2)} P\{S_n + X_0 \leq x\} = \int_{-\infty}^{+\infty} K(x - z) \, dG(z).$$

It is straightforward to verify that the assumption $\int_{-\infty}^{0} |z|^{(\ell-1)} \, dG(z) < \infty$ will ensure the finiteness of the convolution on the right of (3.97). Similarly, one can discuss the case $\ell = 1$.

Proof of theorem 7. We begin by proving that if $\Sigma_\ell(x)$ is finite for one value of x, then it is finite for every value of x. If $P\{X_n \geq 0\} = 1$, then it follows from familiar renewal theory for positive random variables (see, for example, Smith [10]) that $P\{S_n \leq x\}$ tends to zero geometrically and from this the finiteness of $\Sigma_\ell(x)$ follows; let us suppose therefore that $P\{X_n \geq 0\} < 1$. On the other hand, if $P\{X_n \leq 0\} = 1$, then it follows from the same familiar theory that $\Sigma_\ell(x)$ is infinite for every x; we may therefore suppose that $P\{X_n \leq 0\} < 1$. In other words, we may suppose $0 < P\{X_n < 0\} < 1$, $P\{X_n = 0\} \neq 1$, and $0 < P\{X_n > 0\} < 1$.

Let us now be given that $\Sigma_\ell(x_0)$ is finite, and for argument's sake suppose $x_0 < 0$. It is trivial that $\Sigma_\ell(x)$ will be finite for $x < x_0$; we must therefore demonstrate such finiteness for $x > x_0$. Since $P\{X_n < 0\} > 0$, there must be an integer r such that $P\{S_r \leq x_0 - x\} = \zeta$, say, >0. Then

$$(3.98) \qquad \begin{aligned} \Sigma_\ell(x_0) &> \sum_{n=r+1}^{\infty} n^{(\ell-2)} P\{S_n \leq x_0 \,\&\, S_r \leq x_0 - x\} \\ &= \sum_{n=r+1}^{\infty} n^{(\ell-2)} P\{S_n - S_r \leq x \,\&\, S_r \leq x_0 - x\} \\ &= \zeta \sum_{n=r+1}^{\infty} n^{(\ell-2)} P\{S_{n-r} \leq x\} \\ &> A\zeta\Sigma_\ell(x), \end{aligned}$$

where A is the greatest lower bound to the numbers $\{n^{(\ell-2)}/(n-r)^{(\ell-2)}\}$ for $n = r + 1, r + 2, \cdots$ ad inf.; obviously $A > 0$. Thus the finiteness of $\Sigma_\ell(x)$ is demonstrated; similar arguments can be employed if $x_0 = 0$ or if $x_0 > 0$.

Next let us assume $\Sigma_\ell(x)$ to be finite for every x and suppose it is given that $\mathcal{E}X_n$ is finite. If $\mathcal{E}X_n < 0$, then it follows from the weak law of large numbers that $P\{S_n \le x\} \to 1$ as $n \to \infty$, for every $x > 0$. This would imply that $\Sigma_\ell(x) = \infty$, a contradiction of our hypothesis. The case when $\mathcal{E}X_n = 0$ is covered by a result of Spitzer [9]. Therefore, we must have $\mathcal{E}X_n > 0$, as claimed in the statement of theorem 8. Let us write $\mu_1 = \mathcal{E}X_n > 0$. By the weak law of large numbers we know that $S_n/n \to \mu_1$ in probability as $n \to \infty$. Therefore, we can find $\rho > 0$ and $\eta > \mu_1$ such that $P\{0 < S_n \le n\eta\} > \rho$ for all large n. Moreover, since $\mathcal{E}|X_1| < \infty$, we will also have that $nP\{X_1 \le -n\eta\} < \frac{1}{2}\rho$ for all sufficiently large n.

Define A_r, for $r = 1, 2, \cdots, n$, to be the event

$$\{S_n - X_r \le n\eta; X_r \le -n\eta; X_s > -n\eta \quad \text{for} \quad s = 1, 2, \cdots, (r-1)\}. \tag{3.99}$$

Clearly the events $A_1, A_2, \cdots, A_n$ are disjoint, and any one of them implies the event $\{S_n \le 0\}$. Therefore,

$$P\{S_n \le 0\} \ge \sum_{r=1}^{n} P\{A_r\}. \tag{3.100}$$

However,

$$\begin{aligned} P\{A_r\} &= P\{S_n - X_r \le n\eta; X_r \le -n\eta\} \\ &\quad - P\{S_n - X_r \le n\eta; X_r \le -n\eta; X_s \le -n\eta \quad \text{for some} \quad s \le (r-1)\} \\ &\ge P\{S_{n-1} \le n\eta\}P\{X_1 \le -n\eta\} - (r-1)[P\{X_1 \le -n\eta]^2 \\ &\ge \tfrac{1}{2}\rho P\{X_1 \le -n\eta\}. \end{aligned} \tag{3.101}$$

From (3.100) it then follows that $P\{S_n \le 0\} \ge \frac{1}{2}\rho n P\{X_1 \le -n\eta\}$. Hence, if $\Sigma_\ell(0)$ is given to be finite,

$$\sum_{n=1}^{\infty} n^{(\ell-1)} P\{X_1 \le -n\eta\} < \infty. \tag{3.102}$$

This inequality implies that $\mathcal{E}(X_1^-)^\ell < \infty$, as was to be proved.

The following example completes the proof of theorem 7. Suppose that Y_1, $Y_2, \cdots$, is an infinite sequence of independent, identically distributed, nonnegative random variables with distribution function $F_1(x)$, where $1 - F_1(x) = 1/x$ for all large x. Let $Z_1, Z_2, \cdots$ be a similar sequence of independent, identically distributed, nonnegative random variables with distribution function $F_2(x)$, where

$$F_2(x) = \exp\left\{-\left(\frac{\ell}{\ell+2}\right)\left(\frac{\log x}{x^{1/(2+\ell)}}\right)\right\} \tag{3.103}$$

for all large x. Assume, moreover, that the $\{Y_n\}$ and the $\{Z_n\}$ are independent of each other.

If A_r is the event $\{Y_r > n^{(\ell+1)}\}$, then it is clear that

$$\begin{aligned} P\{Y_1 + Y_2 + \cdots + Y_n > n^{(\ell+2)}\} &\le P\{A_1 \cup A_2 \cup \cdots \cup A_n\} \\ &\le nP\{A_1\} \\ &= n^{-\ell}, \quad \text{for all large } n. \end{aligned} \tag{3.104}$$

Also, if B_r is the event $\{Z_r \leq n^{(\ell+2)}\}$, we see that

$$\begin{aligned}(3.105)\qquad P\{Z_1 + Z_2 + \cdots + Z_n \leq n^{(\ell+2)}\} &\leq P\{B_1 \cap B_2 \cap \cdots \cap B_n\} \\ &= [F_2(n^{(\ell+2)})]^n \\ &= \left[\exp\left\{-\frac{\ell \log n}{n}\right\}\right]^n \\ &= n^{-\ell}, \quad \text{for all large } n.\end{aligned}$$

Define the sequence $\{X_n\}$ by putting $X_n = Z_n - Y_n$. Then

$$\begin{aligned}(3.106)\qquad P\{S_n \leq 0\} &= P\{Z_1 + Z_2 + \cdots + Z_n \leq Y_1 + Y_2 + \cdots + Y_n\} \\ &\leq P\{Z_1 + Z_2 + \cdots + Z_n \leq n^{(\ell+2)}\} \\ &\quad + P\{Y_1 + Y_2 + \cdots + Y_n > n^{(\ell+2)}\} \\ &\leq 2n^{-\ell}, \quad \text{for all large } n.\end{aligned}$$

Hence,

$$(3.107)\qquad \sum_{n=1}^{\infty} n^{(\ell-2)} P\{S_n \leq 0\} < \infty,$$

and we have constructed a sequence $\{X_n\}$ which makes $\Sigma_\ell(x)$ finite for $x = 0$, and therefore for all x. To see that $\mathcal{E}X_n^+$ is infinite, we can argue as follows. Choose any constant $c > 0$, and let x be arbitrary and positive. Then

$$(3.108)\qquad P\{X_n > x\} \geq P\{X_n > x;\, Y_n < c\} \geq P\{Z_n > x + c\}P\{Y_n < c\}.$$

Thus, if any moment of Z_n is infinite then so is the corresponding moment of X_n^+. It is a simple matter to see that $\mathcal{E}Z_n = \infty$.

Similarly, one can show that

$$(3.109)\qquad P\{X_n < -x\} \geq P\{Z_n < c\}P\{Y_n > c + x\}$$

and prove that $\mathcal{E}X_n^- = \infty$.

◇ ◇ ◇ ◇ ◇

ADDENDUM

The function $A_1^\ddagger(\theta)$ which arises in the proof of theorem 4 can be treated in a slightly different fashion to yield modified forms for the remainder terms in theorem 5 and corollary 5.1. For simplicity we deal here with the case of integer $\ell \geq 2$. Evidently we can write

$$(\text{A.1})\qquad A_1^\ddagger(\theta) = q^\dagger(\theta/\lambda)\lambda(\theta)/\{1 - F^\ddagger(\theta)\}^{(\ell-1)}$$

where $\lambda(\theta) \in \mathcal{B}^\ddagger(M; \ell - 1)$ and $\lambda(\theta) = O(|\theta|^{(\ell-1)})$. Thus $\lambda(\theta)/(-i\theta)^{(\ell-1)} = B^\ddagger(\theta)$ say, where $B(-\infty) = 0$ and $B(x) \in \mathcal{B}(M; 0)$. Now we differ here from the proof of theorem 4 in writing

$$(\text{A.2})\qquad \mu_1^{(\ell-1)} A_1^\ddagger(\theta) = \left\{\frac{q^\dagger(\theta/\lambda)}{[F_{(1)}^\ddagger(\theta)]^{(\ell-1)}}\right\} B^\ddagger(\theta).$$

If we write $V^{\ddagger}(\theta)$ for the expression in braces, then we can say $V^{\ddagger}(\theta) \in \mathcal{B}^{\ddagger}(M; \ell - 1)$. Thus

$$\mu_1^{(\ell-1)} A_1(x) = \int_{-\infty}^{+\infty} B(x - z)\, dV(z) \tag{A.3}$$

and so

$$\mu_1^{(\ell-1)} A_1(x) - B(x) = -\int_{-\infty}^{+\infty} [B(x) - B(x - z)]\, dV(z). \tag{A.4}$$

Now,

$$\int_{|z| > 1/2x} [B(x) - B(x - z)]\, dV(z) = O\left(\frac{1}{M(x/2)} \int_{|z| > x/2} dV(z)\right) \tag{A.5}$$
$$= o(1/x^{(\ell-1)} M(x)).$$

But, for large positive x, $B(x)$ must be representable as the finite linear combination of a constant and terms like $\int_x^\infty [1 - D(u)]\, du$ where $D \in \mathfrak{D}(M; 1)$. Thus $[1 - D(x)] = o(1/xM(x))$ and hence,

$$\int_{-x/2}^{+x/2} \left[\int_{x-z}^{x} [1 - D(u)]\, du\right] dV(z) \tag{A.6}$$
$$= o\left(\int_{-x/2}^{+x/2} \frac{|z|}{(x - z) M(x - z)}\, dV(z)\right)$$
$$= o\left(\frac{1}{xM(x)} \int_{-\infty}^{+\infty} |z| M(|z|)\, dV(z)\right).$$

Thus

$$\int_{-1/2x}^{+1/2x} [B(x) - B(x - z)]\, dV(z) = o(1/xM(x)), \tag{A.7}$$

since $\int_{-\infty}^{+\infty} |z\, dV(z)| < \infty$. It follows that

$$A_1(x) = \frac{B(x)}{\mu_1^{(\ell-1)}} + o\left(\frac{1}{xM(x)}\right). \tag{A.8}$$

Moreover, it is easy to show that $A_2(x)$ and $A_3(x)$ are in $\mathcal{B}(M; 1)$ and that they are consequently $o(1/xM(x))$ as $x \to \infty$. Therefore, we can write the remainder function of theorem 5 thus:

$$\Lambda(x) = \frac{B(x)}{\mu_1^{(\ell-1)}} + o\left(\frac{1}{xM(x)}\right). \tag{A.9}$$

In particular cases it is straightforward to express $B(x)$ in terms of $F(x)$, $F_{(1)}(x)$, and so on. Thus, in the important case $\ell = 2$, under the conditions of theorem 5 we have

$$H_1(x) = \frac{x}{\mu_1} + \frac{\mu_2}{2\mu_1^2} F_{(2)}(x) + o\left(\frac{1}{xM(x)}\right). \tag{A.10}$$

At this point it is not hard to see that under the conditions of corollary 5.1 we shall have a similar result, the error term being $o(x^{-(\nu-1)})$.

Note added in proof. No changes in our arguments are necessary if, for instance, we replace $M(|x|)$ by $K(x)$, where $K(x) = M(x)$ for $x \geq 0$ and $K(x) = 1$ for $x < 0$. Thus, our theorems also apply to cases where nonsymmetric moment functions are appropriate.

Condition (iv), defining $\mathfrak{M}^*$, can be substantially weakened to the following.

(v) *There is a* $K(x) \in \mathfrak{M}$ *such that*

$$\int_0^\infty \frac{|\log K(x)|}{1 + x^2}\, dx < \infty \tag{1}$$

and, as $x \to \infty$, $M(x) = O(K(\lambda x))$ *for every* $\lambda > 0$.

For example, (v) is satisfied by $M(x) \asymp \exp\{x/(\log x)^2\}$. Condition (iv) is used in only two places. One place is where we show $\alpha_\nu(n)$ of (2.82) to be suitably small, and this use of (iv) can be avoided without difficulty, though we must omit details. The second place is where we show that the S.M.F. based on (2.70) is in $\mathfrak{L}(M; \nu)$ for any $M \in \mathfrak{M}^*$. Suppose we redefine $\mathfrak{M}^*$ using (v) instead of (iv). The following theorem shows that functions exist with which we can construct an S.M.F. in $\mathfrak{L}(M; \nu)$ for any given M in the more general $\mathfrak{M}^*$.

THEOREM. *If* $M \in \mathfrak{M}$ *and satisfies* (v), *then there is a nonnegative* (*and nonnull* $\mathfrak{L}$*-function* $g(x)$ *such that* $g(x) = 0$ *for all sufficiently large* $|x|$ *and, for each* $\lambda > 0$, $g^\dagger(\lambda\theta)(1 + \theta^2)M(|\theta|)$ *is bounded for all* θ.

PROOF. Let $K_r(x)$ belong to $\mathfrak{M}$ and be such that $K_r(x) \asymp K(x)(1 + x^2)^r$, where $K(x)$ is the $\mathfrak{M}$-function of (v). Then, by theorem XII of Paley and Wiener ("*Fourier transforms in the complex domain.*" Amer. Math. Soc. Publications, Vol. XIX, New York, 1934), we can find $f(x) \in L_2$ such that $f(x) = 0$ for all $x < -A$, say, and $|f^\dagger(\theta)| \equiv 1/K_3(|\theta|)$. Assume $\mathfrak{R}f(x)$ is nonnull and set $u(x) = \mathfrak{R}f(x)$ (otherwise use $\mathfrak{I}f(x)$). Since $f(x)$ must be continuous, we may suppose (by moving the origin if needed, a maneuver which leaves $|f^\dagger(\theta)|$ unchanged) that $u(x)$ has no zeros in some neighborhood of the origin. Let $u_2(x) = [u(x)]^2$. Then

$$2\pi u_2^\ddagger(\theta) = \int_{-\infty}^{+\infty} u^\dagger(\theta - \alpha)\, u^\dagger(\alpha) d\alpha, \tag{2}$$

and so

$$u_2^\ddagger(\theta) = O\left(\int_{-\infty}^{+\infty} \frac{d\alpha}{K_2(|\theta - \alpha|)K_2(|\alpha|)(1 + \alpha^2)}\right). \tag{3}$$

From $K_2(|\theta|) \leq K_2(|\theta - \alpha|)K_2(|\alpha|)$ we see that $u_2^\ddagger(\theta) = 0(1/K_2(|\theta|))$. If we now set $g(x) = u_2(x)u_2(-x)$ a similar argument shows that $g^\dagger(\theta) = 0(1/K_1(|\theta|))$. This $g(x)$ has, therefore, all the properties claimed.

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