GENERAL BRANCHING PROCESSES WITH CONTINUOUS TIME PARAMETER

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1. Introduction

The notion of a general branching process was introduced in ([1] chapter III). The general branching process is a Markov branching process whose state space Ω consists of all nonnegative integral-valued measures concentrated on finite subsets of a given space X. In [1], the discrete time-parameter case is studied in detail.

In the present paper we shall be interested in the continuous time-parameter case, and we shall restrict ourselves to the purely discontinuous Feller type. This restriction, not allowing diffusion of individual particles, is natural for some basic spaces X and generally for those processes where the types of particles change by fission only. In [1] references are given to papers studying general branching processes with a kind of diffusion of individual particles and with a simple fission. The present paper does not include these examples as special cases; on the other hand, it studies the purely discontinuous case in full generality. The axiomatic treatment presents certain existence problems which are solved in section 2. In section 3 we shall provide necessary and sufficient conditions for the degeneration of the process. We may expect that the general case could be studied in a similar way if Feller's pure-discontinuity condition were replaced by a kind of mixed-type condition.

In the whole paper we shall use, with few exceptions, the same notation as in [1]; in particular, X will denote the space of types of particles. We shall assume that X is a σ-compact metric space (that is, a denumerable union of compact subsets), and we shall denote by $\mathfrak{X}$ the corresponding σ-algebra of Borel sets in X. By Ω, we shall denote the set of all nonnegative measures ω on $\mathfrak{X}$, which are concentrated on finite subsets of X and assume integral values. Each element $\omega \in \Omega$ may be characterized by a double vector $(x_1, n_1; \cdots ; x_k, n_k)$ where $\{x_1, \cdots , x_k\}$ is the finite subset of X on which ω is concentrated (that is, $\omega(\Omega - \{x_1, \cdots , x_k\}) = 0$) and $n_i = \omega(\{x_i\})$. According to the definition, n_i is a nonnegative integer. If we denote by $\bar{x}$ the measure concentrated at the point $x \in X$ and which assumes there the value 1, we may express the relation between ω and the corresponding double vector $(x_1, n_1; \cdots ; x_k, n_k)$ by $\omega = \sum_{i=1}^{k} n_i\bar{x}_i$.

We shall denote by $\mathfrak{Y}$ the Kolmogorov σ-algebra in Ω, that is, the least σ-algebra containing all cylinder sets $\{\omega \in \Omega: \omega(\{x\}) = n\}$, $x \in X$, n an integer. The set of all bounded $\mathfrak{X}$-measurable functions on X will be denoted by $\mathfrak{F}$, and the set of all nonnegative or nonpositive functions from $\mathfrak{F}$ will be denoted by $\mathfrak{F}^+$ or $\mathfrak{F}^-$. The symbols 0 or 1 will denote the function $f \equiv 0$ or $f \equiv 1$; the symbol 0 will also denote the measure $\omega \in \Omega$, $\omega(X) = 0$. For $f \in \mathfrak{F}$ and $\omega \in \Omega$ we shall write $[f, \omega] = \int_X f(x)\omega(dx)$. The total variation of a finite generalized measure m on $\mathfrak{X}$ will be denoted by $|m|$, and S_ω will be the shift operator in Ω, namely $S_\omega \Gamma = \{\omega' : \omega' - \omega \in \Gamma\}$. The set indicatrix in an arbitrary space will be denoted by d.

2. Branching processes with continuous time parameter

Any function $P(s, \omega, t, \Gamma)$ defined for all $s \leq t$, $\omega \in \Omega$, and $\Gamma \in \mathcal{Y}$ will be called a branching process with continuous time parameter if it satisfies the following conditions:

(2.1) $$P(s, \cdot, t, \Gamma) \quad \text{is } \mathcal{Y}\text{-measurable};$$

(2.2) $$P(s, \omega, t, \cdot) \quad \text{is a nonnegative measure on } \mathcal{Y};$$

(2.3) $$P(s, \omega, t, \Omega) = 1;$$

(2.4) $$P(t, \omega, t, \Gamma) = d(\omega, \Gamma);$$

(2.5) $$P(t_1, \omega, t_3, \Gamma) = \int_\Omega P(t_2, \omega', t_3, \Gamma) P(t_1, \omega, t_2, d\omega') \quad \text{for all} \quad t_1 \leq t_2 \leq t_3;$$

(2.6) $$P(s, \omega_1 + \omega_2, t, \Gamma) = \int_\Omega \int_\Omega d(\omega_1' + \omega_2', \Gamma) P(s, \omega_1, t, d\omega_1') P(s, \omega_2, t, d\omega_2');$$

(2.7) $$\lim_{t \to s^-} \frac{P(t, \omega, s, \Gamma) - P(s, \omega, s, \Gamma)}{t - s} = \lim_{t \to s^+} \frac{P(s, \omega, t, \Gamma) - P(s, \omega, s, \Gamma)}{t - s}$$
$$= p(s, \omega, \Gamma) \quad \text{exists and is finite for each} \quad s, \omega, \Gamma.$$

Clearly, the following three conditions hold for p:

(2.8) $$p(t, \cdot, \Gamma) \quad \text{is } \mathcal{Y}\text{-measurable};$$

(2.9) $$p(t, \omega, \cdot) \quad \text{is a finite generalized measure on } \mathcal{Y};$$

(2.10) $$p(s, \omega, \{\omega\}) \leq 0, \qquad p(s, \omega, \Gamma) \geq 0$$
$$\text{for} \quad \Gamma \subset \Omega - \{\omega\} \quad \text{and} \quad p(s, \omega, \Omega) = 0.$$

Let us denote by Φ the Laplace functional of $P(s, \omega, t, \cdot)$, that is, $\Phi(s, \omega, t, f) = \int_\Omega e^{[f,\omega']} P(s, \omega, t, d\omega')$ and let us write $\Psi(s, \omega, t, f) = \log \Phi(s, \omega, t, f)$ and $\varphi(s, \omega, f) = \int_\Omega e^{[f,\omega']} p(s, \omega, d\omega')$ and $\psi(s, \omega, f) = e^{-[f,\omega]}\varphi(s, \omega, f)$. By (2.6), $\Phi(s, \omega_1 + \omega_2, t, f) = \Phi(s, \omega_1, t, f)\Phi(s, \omega_2, t, f)$ and, by (2.7), $(\partial/\partial s)\Phi(s, \omega, t, f)]_{s=t^-} = \varphi(t, \omega, f)$. Hence,

(2.11) $$\varphi(t, \omega_1 + \omega_2, f) = \varphi(t, \omega_1, f)e^{[f,\omega_2]} + \varphi(t, \omega_2, f)e^{[f,\omega_1]}$$

and dividing by exp $[f, \omega_1 + \omega_2]$ we obtain

(2.12) $$\psi(t, \omega_1 + \omega_2, f) = \psi(t, \omega_1, f) + \psi(t, \omega_2, f).$$

In the theory of branching processes, the transition probabilities, starting with one particle of a certain type, are fundamental. We introduce a special notation for them writing $\bar{P}(s, x, t, \Gamma) = P(s, \bar{x}, t, \Gamma)$. We shall apply the same convention to all other functions p, Φ, Ψ, φ, ψ, and so on. Particularly, the function $\bar{p}(t, x, \Gamma) = p(t, \bar{x}, \Gamma)$ satisfies the following three conditions:

(2.13) $\bar{p}(t, \cdot, \Gamma)$ is $\mathfrak{X}$-measurable;

(2.14) $\bar{p}(t, x, \cdot)$ is a finite generalized measure on $\mathfrak{Y}$;

(2.15) $\bar{p}(t, x, \{\bar{x}\}) \leq 0$ and $\bar{p}(t, x, \Gamma) \geq 0$ for $\Gamma \subset \Omega - \{\bar{x}\}$,

$$\bar{p}(t, x, \Omega) = 0.$$

The condition (2.13) follows from the fact that the mapping $x \to \bar{x}$ is $\mathfrak{X}$-$\mathfrak{Y}$-measurable. Using this notation we may rewrite (2.12) in an equivalent form; namely $\psi(t, \omega, f) = [\bar{\psi}(t, \cdot, f), \omega]$ which implies

$$p(t, \omega, \Gamma) = \sum_{i=1}^{k} n_i \bar{p}(t, x_i, S_{\bar{x}_i - \omega}\Gamma) \tag{2.16}$$

for $\omega = \sum_{i=1}^{k} n_i \bar{x}_i$.

The existence problem may be formulated now as follows. Given a function $\bar{p}$ on $T \times X \times \mathfrak{Y}$ satisfying (2.13)–(2.15), does there exist a function P for which (2.1)–(2.7) hold with p defined by (2.16)? We shall solve this problem under the assumption that $\bar{p}(t, x, \Gamma)$ is continuous with respect to t. This assumption is supposed to hold in the rest of section 2.

Suppose a function $\bar{p}$ on $T \times X \times \mathfrak{Y}$ satisfying (2.13)–(2.15) and continuous with respect to t is given, and let us define a function p on $T \times \Omega \times \mathfrak{Y}$ by the relation (2.16). It is easily seen that it satisfies (2.8)–(2.10) and it is continuous with respect to t. Hence, we may construct the fundamental Feller solution (see [2])

$$P(s, \omega, t, \Gamma) = \sum_{k=0}^{\infty} P^{(k)}(s, \omega, t, \Gamma), \tag{2.17}$$

where

$$P^{(0)}(s, \omega, t, \Gamma) = d(\omega, \Gamma) \exp [J(s, t, \cdot), \omega],$$

$$P^{(k)}(s, \omega, t, \Gamma)$$

$$= \int_s^t \exp [J(s, t, \cdot), \omega] \int_\Omega P^{(k-1)}(s', \omega', t, \Gamma) p_1(s', \omega, d\omega')\, ds',$$

$$J(s, t, x) = \int_s^t \bar{q}(s', x)\, ds', \tag{2.18}$$

$$q(s, \omega) = p(s, \omega, \{\omega\}),$$

$$\bar{q}(s, x) = q(s, \bar{x}),$$

$$p_1(s, \omega, \Gamma) = p(s, \omega, \Gamma - \{\omega\}).$$

It is well known from [2] that $P(s, \omega, t, \Gamma)$ satisfies (2.1), (2.2), (2.4), (2.5), (2.7). Using (2.2) we could also prove that (2.6) holds. We shall omit the proof which would be similar to that of theorem 3.3 in [3]. The only problem that remains is to find under what conditions the process P is "honest," that is (2.3) holds. To the author's best knowledge no simple necessary and sufficient conditions are known even when the set X is finite and the process homogeneous. We shall prove, however, that under a simple and not too restrictive condition on first moments the process is honest.

We shall write $m(\omega, t, Y) = \int_\Omega \omega'(Y)p(\omega, t, d\omega')$ and $m_1(\omega, t, Y) = \int_\Omega \omega'(Y)p_1(\omega, t, d\omega')$. Clearly,

$$\begin{aligned} m(\omega, t, Y) &= \omega(Y)q(t, \omega) + m_1(\omega, t, Y), \\ |m|(\omega, t, Y) &\leq -\omega(Y)q(t, \omega) + m_1(\omega, t, Y). \end{aligned} \tag{2.19}$$

We shall also use the notation $\overline{m}$ and $\overline{m}_1$ according to the rule stated above. It is easily seen from (2.16) that $m(\omega, t, Y) = [\overline{m}(\cdot, t, Y), \omega]$.

Theorem 1. *For each $t > 0$, let*

$$\sup_{0 \leq s \leq t, x \in X} |\overline{q}(s, x)| < \infty, \qquad \sup_{0 \leq s \leq t, x \in X} \overline{m}_1(s, x, X) < \infty. \tag{2.20}$$

Then there exists exactly one process P satisfying all the conditions (2.1)–(2.7). The corresponding logarithmic functional $\overline{\Psi}$ is the only nonpositive bounded solution o the infinite system of differential equations

$$\frac{\partial}{\partial s} \overline{\Psi}(s, x, t, f) = -\overline{\psi}(s, x, \overline{\Psi}(s, \cdot, t, f)), \quad (x \in X, 0 \leq s \leq t) \tag{2.21}$$

with the initial condition $\overline{\Psi}(t, x, t, f) = f(x)$.

Proof. Let us denote by P the Feller fundamental solution and by Φ, Ψ the corresponding Laplace and logarithmic functionals. We shall first show that the corresponding functional $\overline{\Psi}$ satisfies (2.21). According to (2.17),

$$\begin{aligned} \Phi(s, \omega, t, f) = {} & \exp\,[f + J(s, t, \cdot), \omega] \\ & + \int_s^t \int_\Omega \exp\,[J(s, s', \cdot), \omega]\Phi(s', \omega', t, f)p_1(s', \omega, d\omega')\, ds'. \end{aligned} \tag{2.22}$$

Taking derivatives with respect to s we obtain

$$\begin{aligned} \frac{\partial}{\partial s} \Phi(s, \omega, t, f) = {} & -[\overline{q}(s, \cdot), \omega] \exp\,[f + (s, t, \cdot), \omega] \\ & - \int_\Omega \Phi(s, \omega', t, f)p_1(s, \omega, d\omega') \\ & - [\overline{q}(s, \cdot), \omega] \int_s^t \int_\Omega \exp\,[J(s, s', \cdot), \omega]\Phi(s', \omega', t, f)p_1(s', \omega, d\omega')\, ds' \\ = {} & - \int_\Omega \Phi(s, \omega', t, f)p(s, \omega, d\omega') \\ = {} & -\Phi(s, \omega, t, f) \int_\Omega \exp\,[\overline{\Psi}(s, \cdot, t, f), \omega' - \omega]p(s, \omega, d\omega'). \end{aligned} \tag{2.23}$$

Dividing by $\Phi(s, \omega, t, f)$ we get

$$\frac{\partial}{\partial s} \Psi(s, \omega, t, f) = -\psi(s, \omega, \overline{\Psi}(s, \cdot, t, f)) \tag{2.24}$$

which proves (2.21).

For each $t > 0$ and $f \in \mathfrak{F}^-$ there exists a constant $k_0 < 0$ such that $k_0 \leq$

$\Psi(s, x, t, f) \leq 0$ for all $x \in X$ and $0 \leq s \leq t$. This follows from (2.20) and the inequality

$$\bar{P}(s, x, t, \Gamma) \geq d(\bar{x}, \Gamma) \exp \int_s^t \bar{q}(s', x)\, ds' \tag{2.25}$$

which implies

$$\begin{aligned} \Phi(s, x, t, f) &\geq \exp \{f(x) + \int_s^t \bar{q}(s', x)\, ds'\} \\ &\geq \exp \{\inf_x f(x) + t \inf_{s', x} \bar{q}(s', x)\} > 0. \end{aligned} \tag{2.26}$$

We shall now prove that $\Psi(s, x, t, f)$ is, for each $f \in \mathfrak{F}^-$ and $t > 0$, the only bounded nonpositive solution of (2.21) with respect to $s, 0 \leq s \leq t$ with the initial condition $f(x)$ for $s = t$. Let us suppose that this is not true for some t and f and let us write, for this t and f, $y_0(s, x)$ instead of $\Psi(s, x, t, f)$. According to the assumption, there exists another solution $y_1(s, x)$ and a constant $k_1 < 0$ such that $k_i \leq y_i(s, x) \leq 0$ for $i = 0, 1, 0 \leq s \leq t$ and $x \in X$, $y_0(t, x) = y_1(t, x)$ for all x and $y_0(s, x) \neq y_1(s, x)$ for some couple (s, x). Let s_0 be the greatest lower bound of all those s for which $y_0(s, x) = y_1(s, x)$ for all $x \in X$. Clearly, s_0 is finite and $y_0(s_0, x) = y_1(s_0, x)$ for all x because of the continuity with respect to s. On the other hand, to each $\epsilon > 0$ there exist $x' \in X$ and s', $s_0 - \epsilon < s' < s_0$ such that $y_0(s', x') \neq y_1(s', x')$. Put $k_2 = \max \{|k_0|, |k_1|\}$,

$$k_3 = \sup_{0 \leq s \leq t, x \in X} \{\bar{m}_1(s, x, X) - \bar{q}(s, x)\}, \tag{2.27}$$

and $\epsilon = (1/3k_3)\, e^{-k_2}$ and $\theta = \sup_{s_0 - \epsilon \leq s \leq s_0, x \in X} |y_0(s, x) - y_1(s, x)|$. According to the assumptions, $0 < \theta < \infty$ and $|y_0(s_1, x_1) - y_1(s_1, x_1)| > (\theta/2)$ for some $s_1, s_0 - \epsilon < s_1 < s_0$ and $x_1 \in X$. Integrating the equation in $s_1 \leq s \leq s_0$ we get

$$\begin{aligned} y_0(s_1, x_1) - y_1(s_1, x_1) &= \int_{s_1}^{s_0} (\bar{\psi}(s, x_1, y_0(s, \cdot)) - \bar{\psi}(s, x_1, y_1(s, \cdot)))\, ds \\ &= \int_{s_1}^{s_0} \int_\Omega (\exp [y_0(s, \cdot), \omega - \bar{x}_1] - \exp [y_1(s, \cdot), \omega - \bar{x}_1])\bar{p}_1(s, x_1, d\omega)\, ds. \end{aligned} \tag{2.28}$$

Using the relation $e^u - e^v = e^w(u - v)$ (with w lying between u and v) and the inequalities

$$\begin{aligned} &[y_i(s, \cdot), \omega - \bar{x}_1] \leq -y_i(s, x_1) \leq k_2, \\ &\left| \int_\Omega [y_0(s, \cdot) - y_1(s, \cdot), \omega - \bar{x}_1]\bar{p}_1(s, x_1, d\omega) \right| \leq \theta k_3, \end{aligned} \tag{2.29}$$

we finally obtain

$$\frac{\theta}{2} < |y_0(s_1, x_1) - y_1(s_1, x_1)| \leq (s_0 - s_1)\theta k_3 e^{k_2} \tag{2.30}$$

which implies the contradiction $1 \leq 2k_3 e^{k_2}\epsilon < 1$.

Since $y(s, x) \equiv 0$, $(s \leq t, x \in X)$ is a solution of (2.21), and since there is no other nonpositive bounded solution with $y(t, x) \equiv 0$, we have $\Psi(s, x, t, 0) \equiv 0$ which implies $\bar{P}(s, x, t, \Omega) = 1$ for all x and $s \leq t$. Hence, the fundamental Feller solution is "honest" and it is well known from the general theory that it represents then the only solution of (2.1)–(2.7).

We shall now introduce the first moments of the transition probabilities. Let us write $M(s, \omega, t, Y) = \int_\Omega \omega'(Y)P(s, \omega, t, d\omega')$ and $\overline{M}(s, x, t, Y) = M(s, \overline{x}, t, Y)$. Clearly,

$$(2.31)\qquad M(s, \omega, t, Y) = \frac{\partial}{\partial\sigma}\Phi(s, \omega, t, \sigma d(\cdot, Y))]_{\sigma=0}$$
$$= \frac{\partial}{\partial\sigma}\Psi(s, \omega, t, \sigma d(\cdot, Y))]_{\sigma=0} = [\overline{M}(s, \cdot, t, Y), \omega].$$

THEOREM 2. *Let (2.20) hold for each $t > 0$ and let $\overline{m}_1(s, x, Y)$ be continuous with respect to s. Then all first moments are finite and*

$$(2.32)\qquad \overline{M}(s, x, t, Y)$$
$$= d(x, Y) + \sum_{k=1}^{\infty}\int_{\langle s,t\rangle_k}\int_{X^{k-1}}\overline{m}(s_1, x, dx_1)\cdots\overline{m}(s_k, x_{k-1}, Y)d(s_1, \cdots, s_k)$$

where $\langle s, t\rangle_k$ denotes the set $s \le s_1 \le \cdots \le s_k \le t$.

PROOF. The formula (2.32) is a formal consequence of (2.21) and (2.31). However, since we do not know a priori whether the first moments are finite, we must proceed more carefully. Let us write, for a fixed $t > 0$ and $Y \in \mathfrak{X}$, $\overline{\Phi}_0(s, x, \sigma) = \overline{\Phi}(s, x, t, \sigma d(\cdot, Y))(\sigma \le 0)$ and $\Psi_0(s, x, \sigma) = \log\Phi_0(s, x, \sigma)$. Further, choose arbitrary σ_1 and σ_2 such that $\sigma_1 < \sigma_2 < 0$. From the proof of theorem 1 we know that there is a constant $K_1 > 0$ such that $\Phi_0(s, x, \sigma) \ge K_1$ for all $0 \le s \le t, x \in X$ and $\sigma_1 \le \sigma \le 0$. Since $\omega(Y)e^{\sigma\omega(Y)}$ is bounded in $\sigma \le \sigma_2$ and $\omega \in \Omega$, there exists $K_2 < \infty$ such that

$$(2.33)\qquad \frac{\partial}{\partial\sigma}\Psi_0(s, x, \sigma) = \frac{\int\omega(Y)e^{\sigma\omega(Y)}\overline{P}(s, x, t, d\omega)}{\Phi_0(s, x, \sigma)} \le K_2$$

for all $0 \le s \le t$, $x \in X$, $\sigma_1 \le \sigma \le \sigma_2$. Clearly,

$$(2.34)\qquad \overline{\psi}(s, x, \Psi_0(s, \cdot, \sigma)) = \int_\Omega \exp[\Psi_0(s, \cdot, \sigma), \omega - \overline{x}]\overline{p}(s, x, d\omega)$$

and

$$(2.35)\qquad \frac{\partial}{\partial\sigma}\exp[\Psi_0(s, \cdot, \sigma), \omega - \overline{x}]$$
$$= \exp[\Psi_0(s, \cdot, \sigma), \omega - \overline{x}]\frac{\partial}{\partial\sigma}[\Psi_0(s, \cdot, \sigma), \omega - \overline{x}]$$
$$\le \Phi_0^{-1}(s, x, \sigma)K_2\omega(X) \le K_1^{-1}K_2\omega(X),$$

and we see by (2.20) that $(\partial/\partial\sigma)\exp[\Psi_0(s, \cdot, \sigma), \omega - \overline{x}]$ is integrable according to the measure $\overline{p}(s, x, \cdot)$, uniformly with respect to $\sigma_1 \le \sigma \le \sigma_2$. Hence,

$$(2.36)\qquad \frac{\partial^2}{\partial\sigma\, ds}\Psi_0(s, x, \sigma) = -\frac{\partial}{\partial\sigma}\overline{\psi}(s, x, \Psi_0(s, \cdot, \sigma))$$
$$= -\int_X\frac{\partial}{\partial\sigma}\Psi_0(s, y, \sigma)h(s, x, \sigma, dy)$$

where $h(s, x, \sigma, \cdot)$ is a finite measure on $\mathfrak{X}$ defined by

$$h(s, x, \sigma, Y) = \int_\Omega (\omega(Y) - \bar{x}(Y)) \exp [\Psi_0(s, \cdot, \sigma), \omega - \bar{x}]\bar{p}_1(s, x, d\omega). \tag{2.37}$$

The measure $\bar{p}_1$ is nonnegative, finite, and continuous with respect to s for each $\Gamma \subset \mathcal{Y}$ and $\int_\Omega \omega(Y)\bar{p}_1(s, x, d\omega)$ is continuous by the assumptions of the theorem. Then, according to a well-known theorem, $\int_\Gamma \omega(Y)\bar{p}_1(s, x, d\omega)$ is continuous with respect to s for all $\Gamma \subset \mathcal{Y}$ and, since $\exp [\Psi_0(s, \cdot, \sigma), \omega - \bar{x}] \leq \Phi_0^{-1}(s, x, \sigma) \leq K_1^{-1} < \infty$, the measure $h(s, x, \sigma, Y)$ is continuous with respect to (s, σ). Then, according to (2.33) and (2.36), $(\partial^2/\partial\sigma\, \partial s)\Psi_0$ is continuous with respect to (s, σ). On the other hand,

$$\frac{\partial^2}{\partial s\, \partial\sigma} \Psi_0 = \Phi_0^{-2} \left(\Phi_0 \frac{\partial^2}{\partial s\, \partial\sigma} \Phi_0 - \left(\frac{\partial}{\partial\sigma} \Phi_0\right)^2\right). \tag{2.38}$$

Clearly, the derivative $(\partial/\partial s)P(s, x, t, \Gamma)$ exists and is finite and continuous with respect to s for each $\Gamma \in \mathcal{Y}$, and the function $\omega(Y)e^{\sigma\omega(Y)}$ is bounded in $\Omega \times (\sigma_1, \sigma_2)$. This is sufficient for the following two formulas to hold:

$$\frac{\partial^2}{\partial s\, \partial\sigma} \Phi_0(s, x, \sigma) = \int_\Omega \omega(Y)e^{\sigma\omega(Y)} \frac{\partial}{\partial s} P(s, x, t, d\omega), \tag{2.39}$$

$$\frac{\partial}{\partial s} \Phi_0(s, x, \sigma) = \int_\Omega e^{\sigma\omega(Y)} \frac{\partial}{\partial s} P(s, x, t, d\omega), \tag{2.40}$$

and we see that both functions are continuous with respect to (s, σ). Hence, $(\partial^2/\partial s\, \partial\sigma)\Psi_0$ is also continuous with respect to (s, σ) which implies $(\partial^2/\partial\sigma\, \partial s)\Psi_0 = (\partial^2/\partial s\, \partial\sigma)\Psi_0$. This proves, according to (2.36) that $(\partial/\partial\sigma)\Psi_0$ satisfies for each $\sigma < 0$ the system of differential equations

$$\frac{\partial}{\partial s}\left(\frac{\partial}{\partial\sigma} \Psi_0(s, x, \sigma)\right) = -\int_X \frac{\partial}{\partial\sigma} \Psi_0(s, y, \sigma)h(s, x, \sigma, dy) \tag{2.41}$$

in $0 \leq s \leq t$. For each $\sigma < 0$, $\sup_{0 \leq s \leq t, x \in X} (\partial/\partial\sigma)\Psi_0(s, x, \sigma) < \infty$ because of (2.33). Further, for each $\sigma_1 < 0$

$$\sup_{0 \leq s \leq t, \sigma_1 \leq \sigma \leq 0, x \in X} |h|(s, x, \sigma, Y) \leq K_3 \tag{2.42}$$

where $K_3 = K_1^{-1} \sup_{0 \leq s \leq t, x \in X} (\overline{m},(s, x, X) + \bar{q}(s, x)) < \infty$.

Using (2.42) we could prove in the same way as in theorem 1 that $(\partial/\partial\sigma)\Psi_0$ is the only bounded and nonnegative solution of (2.41) with the initial condition $(\partial/\partial\sigma)\Psi_0(t, x, \sigma) = d(x, Y)$. Let us consider the series

(2.43)

$$d(x, X) + \sum_{k=1}^{\infty} \int_{(s,t)_k} \int_{X^{k-1}} h(s_1, x, \sigma, dx_1) \cdots h(s_k, x_{k-1}, \sigma, Y)d(s_1, \cdots, s_k).$$

According to (2.42), the k-th term of this series is less than $(K_3^k t^k/k!)$ uniformly with respect to $0 \leq s \leq t$, $\sigma_1 \leq \sigma < 0$, $x \in X$. Hence, the series is convergent and, for the same reason, it is term-by-term differentiable. It is also easily seen that is satisfies (2.41) and, consequently, $(\partial/\partial\sigma)\Psi_0(s, x, \sigma)$ is equal to (2.43). Since the series is convergent uniformly with respect to $\sigma_1 \leq \sigma < 0$, we may apply the term-by-term limit procedure $\sigma \to 0$ and (2.32) results.

3. The homogeneous case

We shall suppose in this section, that the process is homogeneous and we shall write $P(\omega, t, \Gamma)$ instead of $P(0, \omega, t, \Gamma) = P(s, \omega, s + t, \Gamma)$, and similarly for all related functions. The infinitesimal functions p, m, and so on, do not depend on s in this case. The main assumptions of theorems 1 and 2 are

$$\sup_{x \in X} \overline{m}_1(x, X) < \infty, \qquad \sup_{x \in X} |\overline{q}(x)| < \infty. \tag{3.1}$$

The system (2.21) of differential equations assumes the form

$$\frac{d}{dt} \overline{\Psi}(x, t, f) = \overline{\psi}(x, \overline{\Psi}(\cdot, t, f)) \tag{3.2}$$

and

$$\overline{M}(x, t, Y) = d(x, Y) + \sum_{k=1}^{\infty} \frac{\overline{m}^{(k)}(x, Y)}{k!} \tag{3.3}$$

where $\overline{m}^{(k)}(x, Y) = \int_X \overline{m}^{(k-1)}(y, Y)\overline{m}(x, dy)$. Similar expressions can be obtained for second moments.

Let us write

$$\begin{aligned} V(\omega, t, Y, Z) &= \int_\Omega (\omega'(Y) - M(\omega, t, Y))(\omega'(Z) - M(\omega, t, Z))P(\omega, t, d\omega'), \\ v(\omega, Y, Z) &= \int_\Omega (\omega'(Y) - \omega(Y))(\omega'(Z) - \omega(Z))p(\omega, d\omega'), \\ v_1(\omega, Y, Z) &= \int_\Omega \omega'(Y)\omega'(Z)p_1(\omega, d\omega'). \end{aligned} \tag{3.4}$$

Clearly,

$$\begin{aligned} V(\omega, t, Y, Z) &= \frac{\partial^2}{\partial\sigma_1\, \partial\sigma_2} \Psi(\omega, t, \sigma_1 d(\cdot, Y) + \sigma_2 d(\cdot, Z))]_{\sigma_1=\sigma_2=0}, \\ v(\omega, Y, Z) &= \frac{\partial^2}{\partial\sigma_1\, \partial\sigma_2} \psi(\omega, \sigma_1 d(\cdot, Y) + \sigma_2 d(\cdot, Z))]_{\sigma_1=\sigma_2=0}, \\ v(\omega, Y, Z) &= v_1(\omega, Y, Z) - \omega(Y)m_1(\omega, Z) - \omega(Z)m_1(\omega, Y) - \omega(Y \cap Z)q(\omega). \end{aligned} \tag{3.5}$$

We shall write again $\overline{V}(x, t, Y, Z)$ instead of $V(\overline{x}, t, Y, Z)$ and similarly for v and v_1.

THEOREM 3. *Let*

$$\sup_{x \in X} \overline{v}_1(x, X, X) < \infty, \qquad \sup_{x \in X} |\overline{q}(x)| < \infty. \tag{3.6}$$

Then all $V(\omega, t, Y, Z)$ are finite

$$V(\omega, t, Y, Z) = [\overline{V}(x, t, Y, Z), \omega] \tag{3.7}$$

and

$$\overline{V}(x, t, Y, Z) = \int_0^t \int_X W(y, s, Y, Z)\overline{M}(x, t - s, dy)\, ds \tag{3.8}$$

where

$$W(y, s, Y, Z) = \int_{X\times X} \overline{M}(y, t, Y)\overline{M}(z, t, Z)\overline{v}(x, d(y, z)). \tag{3.9}$$

PROOF. Since we do not know a priori whether the second moments are finite, the complete proof should follow the method used in theorem 2. We shall omit these details, which would show, in an analogous way, that all $\overline{V}$ are finite and uniformly bounded in x and that formal differentiation of (3.2) is correct. Hence

$$\frac{d}{dt}\frac{\partial^2}{\partial\sigma_1\,\partial\sigma_2}\Psi(x, t, \sigma_1 d(\cdot, Y) + \sigma_2 d(\cdot, Z)) = \frac{d}{dt}\overline{V}(x, t, Y, Z) \tag{3.10}$$

$$= \int_\Omega \{[\overline{V}(\cdot, t, Y, Z), \omega - \overline{x}] + [\overline{M}(\cdot, t, Y), \omega - \overline{x}][\overline{M}(\cdot, t, Z), \omega - \overline{x}]\}\overline{p}(x, d\omega)$$

which yields the following system of differential equations:

$$\frac{d}{dt}\overline{V}(x, t, Y, Z) = \int_X \overline{V}(y, t, Y, Z)\overline{m}(x, dy) + W(x, t, Y, Z). \tag{3.11}$$

As in theorem 2 we could prove that there exists exactly one bounded solution, and it is easily seen that (3.8) satisfies (3.11).

We shall now prove three theorems on the degeneration of the process. We shall have to impose further restrictions on the process. The main assumption will be that the first moments are compact and strictly positive operators on an appropriate subspace $\mathcal{G}$ of $\mathcal{F}$. From several natural possibilities we shall choose $\mathcal{G}$ equal to the class of all continuous bounded functions on X. Let $\mathbf{m}$ be the linear operator on $\mathcal{F}$ defined by $\mathbf{m}f = \int_X f(y)\overline{m}(\cdot, dy)$ and let $\mathbf{m}_1$ be the linear operator defined by $\overline{m}_1(x, Y)$ in a similar way. Put $\mathbf{m}_2 = \mathbf{m} + k\mathbf{I}$ where $k = \sup_{x\in X} |q(x)|$ and $\mathbf{I}$ is the identity operator.

We shall suppose in the rest of this section that m_2 is an operator on $\mathcal{G}$, that is $\mathbf{m}_2 f \in \mathcal{G}$ if $f \in \mathcal{G}$, and that it is compact and strictly positive with respect to the cone $\mathcal{G}^+ = \mathcal{F}^+ \cap \mathcal{G}$. Let ρ be the spectral radius of $\mathbf{m}_2$. It is well known from the theory of strictly positive compact operators (see [4], for example) that there exists exactly one function $l \in \mathcal{G}^+$ and exactly one finite and nonnegative measure λ on $\mathfrak{X}$ such that $\mathbf{m}_2 l = \rho l$, $0 < \alpha \le l(x) \le \beta < \infty$ for all $x \in X$, $\int_X m_2(x, Y)\lambda(dx) = \rho\lambda(Y)$, $\lambda(X) = 1$, $[l, \lambda] = 1$. Clearly, the operator $\mathbf{m}$ has the same pair l, λ of eigenvectors corresponding to the largest eigenvalue $r = \rho - k$. The operator $\mathbf{M}^t$, induced by $\overline{M}(x, t, Y)$ is equal to $\exp \mathbf{m}t = e^{-kt} \exp \mathbf{m}_2 t$. It is also compact and strictly positive on $\mathcal{G}$ with the same pair l, λ of eigenvectors and with the largest eigenvalue equal to e^{rt}.

The following formula (3.12) will be useful. Set for a fixed x and t, $F(\tau) = \Psi(x, t, \tau f)$ for $0 \le \tau \le 1$. Using the Taylor expansion $F(1) = F(0) + F'(0) + \frac{1}{2}F''(\tau)$ for some $0 < \tau < 1$, we easily see that

$$(3.12)\qquad \Psi(x, t, f) = \int_\Omega [f, \omega]\overline{P}(x, t, d\omega) + \Phi^{-2}(x, t, \tau f) \times \left\{\Phi \int_\Omega [f, \omega]^2 e^{\tau[f,\omega]}\overline{P}(x, t, d\omega) - \left(\int_\Omega [f, \omega]e^{\tau[f,\omega]}\overline{P}(x, t, d\omega)\right)^2\right\} = \int_X f(y)\overline{M}(x, t, dy) + \Phi^{-1}(x, t, \tau f) \int_\Omega ([f, \omega] - \Psi'(x, t, \tau f))^2 e^{\tau[f,\omega]}\overline{P}(x, t, d\omega)$$

where $\Psi'(x, t, \tau f) = \Phi^{-1}(x, t, \tau f) \int_\Omega [f, \omega]e^{\tau[f,\omega]}\overline{P}(x, t, d\omega)$. Since the last term in (3.12) is nonnegative,

$$(3.13)\qquad 0 \geq \Psi(x, t, f) \geq \int_X f(y)\overline{M}(x, t, dy).$$

We shall call the process degenerate, if $\overline{P}(x, t, \{0\}) \rightarrow 1$ for all $x \in X$.

THEOREM 4. *If $r < 0$, then the process is degenerate.*

PROOF. By (3.13)

$$(3.14)\qquad |\Psi(x, t, -l)| \leq \int_X l(y)\overline{M}(x, t, dy) = e^{rt}l(x) \underset{t}{\rightarrow} 0.$$

On the other hand,

$$(3.15)\qquad \overline{P}(x, t, \{\omega: \omega(X) \geq 1\}) \leq \frac{1 - \Phi(x, t, -l)}{1 - e^{-\alpha}} \underset{t}{\rightarrow} 0.$$

THEOREM 5. *Let $r = 0$ and let $\psi(x, f) \in \mathcal{G}$ for each $f \in \mathfrak{F}^-$. Then the process is not degenerate if and only if*

$$(3.16)\qquad \overline{p}(x, \{\omega: \omega(X) = 1\}) = 0 \qquad \text{for all} \quad x \in X.$$

PROOF. Suppose first that (3.16) does not hold. By (3.13)

$$(3.17)\qquad \Psi(x, t, -l) \geq - \int_X l(y)\overline{M}(x, t, dy) = -l(x) \qquad \text{for all} \quad x, t,$$

and from the fundamental relation

$$(3.18)\qquad \Psi(x, s + t, f) = \Psi(x, s, \Psi(\cdot, t, f)),$$

we see that $\Psi(x, s + t, -l) \geq \Psi(x, s, -l)$. Hence, $\lim_{t\rightarrow\infty} \Psi(x, t, -l) = f_0(x)$ exists for all x, and by (3.18) we have

$$(3.19)\qquad \Psi(x, s, f_0) = f_0(x) \qquad \text{for all} \quad x \text{ and } s.$$

By the assumption of the theorem, $\psi(x, f)$ is continuous for each $f \in \mathfrak{F}^-$. It is easily seen from the construction of the process that $\Psi(\cdot, s, f) \in \mathcal{G}$ for each s and $f \in \mathfrak{F}^-$. Hence, $f_0(\cdot) = \Psi(\cdot, s, f_0) \in \mathcal{G}$ and, at the same time $f_0(x) = \Psi(x, s, f_0) \geq \int_X f_0(y)M(x, s, dy)$. If $f_0 \not\equiv 0$, then (by [4], theorem 8) $f_0 = \gamma l$ for some $\gamma < 0$ and $\Psi(x, s, f_0) = \int_X f_0(y)\overline{M}(x, s, dy)$ which implies that the last term in (3.12) is equal to zero for all x and some $0 < \tau < 1$. Then

$$(3.20)\qquad \overline{P}(x, s, \{\omega: [f_0, \omega] = \Psi'(x, s, \tau, f_0)\}) = 1$$

for all x and s. Integrating with respect to the measure $\overline{P}(x, s, \cdot)$, we obtain $f_0(x) = \int_X f_0(y)\overline{M}(x, s, dy) = \Psi'(x, s, \tau f_0)$, or, $\overline{P}(x, s, \{\omega: [l, \omega] = l(x)\}) = 1$. Since $l(x) \geq \alpha > 0$ for all $x \in X$, this is possible only if $\overline{P}(x, s, \{\omega: \omega(X) = 1\}) =$

1 for all x and s, or, if $\bar{p}(x, \{\omega: \omega(X) = 1\}) = 0$ for all x. But we have supposed the contrary and, consequently, $f_0(x) \equiv 0$ which proves that $\Psi(x, t, -l) \underset{t}{\to} 0$ for all x. The fact that the process is degenerate follows now in the same way as in the preceding theorem. On the other hand, if (3.16) holds for all x, then

$$P(x, t, \{\omega: \omega(X) = 1\}) = 1 \qquad \text{for all} \quad x, t \tag{3.21}$$

and the process is not degenerate.

THEOREM 6. *Let* $r > 0$ *and* $K = \sup_{x \in X} \bar{v}(x, X, X) < \infty$. *Then the process is not degenerate.*

PROOF. Let $\xi(t)$ be the sample functions of the process. For each t, $\xi(t) \in \Omega$, and we may therefore define a new random variable

$$\theta(t) = e^{-rt} \int_X l(y) \xi(t, dy). \tag{3.22}$$

We shall first estimate $E_x(\theta(t))$ and $E_x(\theta^2(t))$, where E_x means the expectation with respect to the initial distribution concentrated at $\{x\}$:

$$\begin{aligned} E_x(\theta(t)) &= e^{-rt} \int_\Omega [l, \omega] \bar{P}(x, t, d\omega) \\ &= e^{-rt} \int_X l(y) \bar{M}(x, t, dy) = l(x) \geq \alpha > 0. \end{aligned} \tag{3.23}$$

By (3.8),

(3.24)

$$\begin{aligned} E_x(\theta^2(t)) &= e^{-2rt} \int_\Omega [l, \omega]^2 \bar{P}(x, t, d\omega) \\ &= e^{-2rt} \left\{ \int_{X \times X} l(y) l(z) \bar{V}(x, t, d(y, z)) + \left(\int_X l(y) \bar{M}(x, t, dy) \right)^2 \right\} \\ &= e^{-2rt} \left\{ \int_0^t \int_X e^{2rs} l(y) l(z) \bar{v}(u, d(y, z)) \bar{M}(x, t - s, du)\, ds + e^{2rt} l^2(x) \right\} \\ &\leq \beta^2 \left\{ K \alpha^{-1} e^{-rt} \int_0^t e^{rs}\, ds + 1 \right\} \leq \beta^2 (K (\alpha r)^{-1} + 1) < \infty. \end{aligned}$$

If the process were degenerate, then

$$E_x(\theta(t)) \leq (E_x(\theta^2(t)) \bar{P}(x, t, \{\omega: \omega(X) \geq 1\}))^{1/2} \underset{t}{\to} 0 \tag{3.25}$$

in contradiction to (3.23).

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