ANALYSIS ON HILBERT SPACE WITH REPRODUCING KERNEL ARISING FROM MULTIPLE WIENER INTEGRAL

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1. Introduction

The multiple Wiener integral with respect to an additive process with stationary independent increments plays a fundamental role in the study of the flow derived from that additive process and also in the study of nonlinear prediction theory. Many results on the multiple Wiener integral have been obtained by N. Wiener [14], [15], K. Itô [5], [6], and S. Kakutani [8] by various techniques. The main purpose of our paper is to give an approach to the study of the multiple Wiener integral using reproducing kernel Hilbert space theory.

We will be interested in stationary processes whose sample functions are elements in E^* which is the dual of some nuclear pre-Hilbert function space E. For such processes we introduce a definition of *stationary process* which is convenient for our discussions. This definition, given in detail by section 2, definition 2.1, is a triple $\mathbf{P} = (E^*, \mu, \{T_t\})$, where μ is a probability measure on E^* and $\{T_t\}$ is a flow on the measure space (E^*, μ) derived from shift transformations which shift the arguments of the functions of E.

In order to facilitate the discussion of the Hilbert space $L_2 = L_2(E^*, \mu)$, we shall introduce a transformation τ defined by the following formula:

$$(\tau\varphi)(\xi) = \int_{E^*} e^{i\langle x,\xi\rangle}\varphi(x)\mu(dx) \qquad \text{for} \quad \varphi \in L_2, \tag{1.1}$$

where $\langle\cdot, \cdot\rangle$ denotes the bilinear form of $x \in E^*$ and $\xi \in E$. This transformation τ from L_2 to the space of functionals on E is analogous to the ordinary Fourier transform. By formula (1.1) and a requirement that τ should be a unitary transformation, $\mathfrak{F} \equiv \tau(L_2(E^*, \mu))$ has to be a Hilbert space with reproducing kernel $C(\xi - \eta)$, $(\xi, \eta) \in E \times E$, where C is the *characteristic functional* of the measure μ defined by

$$C(\xi) = \int_{E^*} e^{i\langle x,\xi\rangle}\mu(dx), \qquad \xi \in E. \tag{1.2}$$

The first task in section 2 will be to establish the explicit correspondence between L_2 and $\mathfrak{F}$.

Another concept which we shall introduce in section 2 is a group $\mathbf{G}(\mathbf{P})$ associated with a stationary process $\mathbf{P}$. Consider the set of all linear transformations $\{g\}$ on E satisfying the conditions that

$$(1.3)\qquad \begin{aligned} &\text{(i)}\quad C(g\xi) = C(\xi) \quad \text{for every} \quad \xi \in E, \quad \text{and} \\ &\text{(ii)}\quad \text{that } g \text{ be a homeomorphism on } E. \end{aligned}$$

Obviously the collection $\mathbf{G}(\mathbf{P})$ of all such g's forms a group with respect to the operation $(g_1g_2)\xi = g_1(g_2\xi)$. The collection $\mathbf{G}(\mathbf{P})$ includes not only shift transformations S_h, h real, defined by

$$(1.4)\qquad (S_h\xi)(t) = \xi(t - h),$$

but also some other transformations depending on the form of the characteristic functional.

An interesting subclass of stationary processes is the class of processes with *independent values at every point* (Gelfand and Vilenkin [4]). Sections 4–6 will be devoted to discussions of some typical such processes. Roughly speaking they are the stationary processes obtained by subtracting the mean functions from the derivatives with respect to time of additive processes with independent stationary increments. In these cases the independence at every point can be illustrated rather clearly in the space $\mathfrak{F}$ by using a direct product decomposition of it in the sense of J. von Neumann [10]. Furthermore, because of the particular form of the characteristic functional, we can easily get an infinite direct sum decomposition of $\mathfrak{F}$:

$$(1.5)\qquad \mathfrak{F} = \sum_{n=0}^{\infty} \oplus\, \mathfrak{F}_n.$$

Each $\mathfrak{F}_n$ appearing in the last expression is invariant under every V_g, $g \in \mathbf{G}(\mathbf{P})$ defined by

$$(1.6)\qquad (V_g f)(\xi) = f(g\xi), \qquad f \in \mathfrak{F},$$

that is, $V_g(\mathfrak{F}_n) \subset \mathfrak{F}_n$ for every $g \in \mathbf{G}(\mathbf{P})$. With the aid of these two different kinds of decompositions, we shall investigate $\mathfrak{F}$ and discuss certain applications.

In section 5, we shall consider *Gaussian white noise,* although many of the results are already known. To us, it is the most fundamental example of a stationary process. Here the subspace $\mathfrak{F}_n$ corresponds to the multiple Wiener integral introduced by K. Itô [5] and also to Wiener's homogeneous chaos of degree n. Kakutani [7] expressed the subspace of $L_2(E^*, \mu)$ corresponding to $\mathfrak{F}_n$ in terms of the product of Hermite polynomials in L_2. These results are important in the determination of the spectrum of the flow $\{T_t\}$ of Gaussian white noise. It should be noted that the L_2 space for this process enjoys properties similar to those of $L_2(S^n, \sigma_n)$ over an n-dimensional sphere S^n with the uniform probability measure σ_n. As is mentioned by H. Yosizawa (oral communication), the multiple Wiener integral plays the role of spherical harmonics in

$L_2(S^n, \sigma_n)$; for example, $\{\mathfrak{F}_n, V_g, g \in \mathbf{G}(\mathbf{P})\}$ is an irreducible representation of the group $\mathbf{G}(\mathbf{P})$. This suggests that one should consider certain aspects of the theory which are analogous to the analysis on the finite dimensional sphere. This is done by H. Yosizawa and others (oral communication).

We will also discuss the Hilbert space $\mathfrak{F}$ arising from *Poisson white noise* which is another fundamental example of a stationary process (section 6).

Finally we will show that our approach is applicable to the study of generalized white noise and even to an arbitrary sequence of independent identically distributed random variables. Further discussions, such as the detailed proofs of the theorems stated in this paper and certain of the applications of this work, will appear elsewhere.

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2. Definitions and preparatory considerations

Before defining the term stationary process let us first introduce some notation. Let E be a real nuclear pre-Hilbert space. Denote the inner product by $\langle \cdot, \cdot \rangle$; it determines the norm $\|\cdot\|$. Let H be the completion of E in the norm $\|\cdot\|$ and E^* be the dual of E. Then by the usual identification $H^* = H$ for Hilbert spaces, we have the following relation: $E \subset H \subset E^*$.

Let $\mathcal{B} = \mathcal{B}(E^*)$ be the σ-algebra generated by all cylinder sets in E^*. If $C(\xi)$, $\xi \in E$, is a continuous positive definite functional with $C(0) = 1$, then there exists a unique probability measure μ on the measurable space $(E^*, \mathcal{B})$ such that

$$C(\xi) = \int_{E^*} \exp\,[i\langle x, \xi\rangle]\mu(dx), \qquad \xi \in E \tag{2.1}$$

(cf. Gelfand and Vilenkin [4]).

In what follows we shall deal only with the case in which E is a subset of $R^{\mathbf{T}}$, where R is the field of real numbers and $\mathbf{T}$ is the additive group of real numbers or one of its subgroups. Every element of E then has a coordinate representation $\xi = (\xi(t), t \in \mathbf{T})$. For every h, we consider the point transformation S_h defined by (1.4). Whenever we are concerned with the S_h's, we always assume that E is invariant under all of them. For each S_t we define a transformation T_t on E^* as follows:

$$T_t\colon T_tx \in E^*, \qquad t \in \mathbf{T} \quad \text{with} \quad \langle T_tx, \xi\rangle = \langle x, S_t\xi\rangle \qquad \text{for every } \xi. \tag{2.2}$$

Obviously $\{T_t\colon t \in \mathbf{T}\}$ forms a group satisfying

$$T_tT_s = T_sT_t = T_{t+s}, \qquad s, t \in \mathbf{T}, \tag{2.3}$$

$$T_0 = I \quad \text{(identity)}. \tag{2.4}$$

The group $\{T_t; t \in \mathbf{T}\}$ can be considered as a transformation group acting on E^*.

Let $\mathcal{B}(\mathbf{T})$ be the topological Borel field of $\mathbf{T}$.

DEFINITION 2.1. *The transformation group $\{T_t, t \in \mathbf{T}\}$ is called a group of shift transformations if $T_tx = f(x, t)$ is measurable with respect to $\mathcal{B} \times \mathcal{B}(\mathbf{T})$. The* *triple* $\mathbf{P} = (E^*, \mu, \{T_t\})$ *is called a stationary process if* μ *is invariant under shift transformations* T_t, $t \in \mathbf{T}$.

DEFINITION 2.2. *The functional* $C(\xi)$, $\xi \in E$, *defined by* (2.1) *for a stationary process* $\mathbf{P} = (E^*, \mu, \{T_t\})$, *is called* the characteristic functional *of* $\mathbf{P}$.

Having introduced these definitions, we begin our investigation of stationary processes. For convenience, we assume the following throughout the remainder of the paper.

ASSUMPTION 2.1. *There exists a system* $\{\xi_n^\circ\}_{n=1}^{\infty}$, $\xi_n^\circ \in E$, *which forms a complete orthonormal system in* H.

For the measure space, $(E^*, \mathcal{B}, \mu)$, associated with a stationary process, we can form the Hilbert space $L_2 = L_2(E^*, \mu)$ of all square summable complex-valued functions with the inner product

$$\langle\langle \varphi, \psi \rangle\rangle = \int_{E^*} \varphi(x)\overline{\psi(x)}\mu(dx), \qquad \varphi, \psi \in L_2. \tag{2.5}$$

LEMMA 2.1. *The closed linear subspace of* L_2 *spanned by* $\{e^{i\langle x,\xi\rangle}, \xi \in E\}$ *coincides with the whole space* L_2.

This can be proved by using the uniqueness theorem for Fourier inverse transform (for detailed proof, see Prohorov [13]).

The next lemma is due to Aronszajn ([1], part I, section 2).

LEMMA 2.2. *For any stationary process* $\mathbf{P} = (E^*, \mu, \{T_t\})$ *there always exists a smallest Hilbert space* $\mathfrak{F} = \mathfrak{F}(E, C)$ *of functionals on* E *with reproducing kernel* $C(\xi - \eta)$, $(\xi, \eta) \in E \times E$, *where* $C(\xi)$, $\xi \in E$, *is the characteristic functional of* $\mathbf{P}$.

Let us denote the inner product in $\mathfrak{F}$ by $(\cdot, \cdot)$. We now state some of the properties of $\mathfrak{F}$ obtained by Aronszajn:

(i) for any fixed $\xi \in E$, $C(\cdot - \xi) \in \mathfrak{F}$;
(ii) $(f(\cdot), C(\cdot - \xi)) = f(\xi)$ for any $f \in \mathfrak{F}$;
(iii) $\mathfrak{F}$ is spanned by $\{C(\cdot - \xi), \xi \in E\}$.

From these properties and lemma 2.1 we can prove the following theorem.

THEOREM 2.1. *The transformation* τ *defined by*

$$(\tau\varphi)(\xi) = \int_{E^*} \varphi(x) e^{i\langle x,\xi\rangle}\mu(dx) \tag{2.6}$$

is a unitary operator from L_2 *onto* $\mathfrak{F}$.

In fact, the relation

$$\tau\left(\sum_{j=1}^{n} a_j e^{-i\langle x,\xi_j\rangle}\right)(\cdot) = \sum_{j=1}^{n} a_j C(\cdot - \xi_j) \tag{2.7}$$

shows that τ preserves norm since this relation can be extended to the entire space.

Note that

$$(\tau 1)(\cdot) = C(\cdot) \tag{2.8}$$

and define

$$\begin{aligned} L_2^* &= \{\varphi; \varphi \in L_2, \langle\langle \varphi, 1\rangle\rangle = 0\}, \\ \mathfrak{F}^* &= \{f; f \in \mathfrak{F}, (f(\cdot), C(\cdot)) = 0\}. \end{aligned} \tag{2.9}$$

Then τ restricted to L_2^* is still a unitary operator from L_2^* onto $\mathfrak{F}^*$. On the other hand, if we define U_t, $t \in \mathbf{T}$, by

$$U_t\colon\ (U_t\varphi)(x) = \varphi(T_t x) \in L_2 \qquad \text{for} \quad \varphi \in L_2, \tag{2.10}$$

then $\{U_t, t \in \mathbf{T}\}$ forms a group of unitary operators on L_2 satisfying

$$\begin{aligned} &U_t U_s = U_s U_t = U_{t+s}, && s, t \in \mathbf{T}, \\ &U_0 = I \quad \text{(identity)}, \\ &U_t \text{ is strongly continuous.} \end{aligned} \tag{2.11}$$

Moreover, we can see that $\{\tilde{U}_t; \tilde{U}_t = \tau \cdot U_t \tau^{-1}, t \in \mathbf{T}\}$ is also a group of unitary operators acting on $\mathfrak{F}$. For simplicity we also use the symbol U_t for $\tilde{U}_t$. Further, we write U_t even when U_t (or $\tilde{U}_t$) is restricted to L_2^* (or $\mathfrak{F}^*$).

In connection with $\mathbf{G}(\mathbf{P})$ we can consider a group $\mathbf{G}^*(\mathbf{P})$ of linear transformations g^* acting on E^* as follows:

$$\mathbf{G}^*(\mathbf{P}) = \{g^*; g^*x \in E^* \text{ for every } x \in E^*, \langle g^*x, \xi\rangle = \langle x, g\xi\rangle \text{ holds for every } x \in E^* \text{ with } g \in \mathbf{G}(\mathbf{P})\}. \tag{2.12}$$

From the definition we can easily prove lemma 2.3.

LEMMA 2.3. *The collection* $\mathbf{G}^*(\mathbf{P})$ *is a group with respect to the operation*

$$(g_1^* g_2^*)x = g_1^*(g_2^* x). \tag{2.13}$$

Also,

$$(g^*)^{-1} = (g^{-1})^*. \tag{2.14}$$

REMARK 2.1. Detailed discussions concerning $\mathbf{G}(\mathbf{P})$ and $\mathbf{G}^*(\mathbf{P})$ will appear elsewhere. For the related topics on such groups we would like to refer to M. G. Kreĭn [9].

By (1.3) (i), it can be proved that every $g^* \in \mathbf{G}^*(\mathbf{P})$ is measure preserving; that is, $\mu(d(g^*x)) = \mu(dx)$. Hence, by the usual method, we can define a unitary operator V_{g^*} acting on $L_2(E^*, \mu)$ by

$$(V_{g^*}\varphi)(x) = \varphi(g^*x), \qquad g^* \in \mathbf{G}^*(\mathbf{P}), \qquad \varphi \in L_2(E^*, \mu). \tag{2.15}$$

Similarly, we define

$$(V_g f)(\xi) = f(g\xi), \qquad g \in \mathbf{G}(\mathbf{P}), \qquad f \in \mathfrak{F}. \tag{2.16}$$

Obviously, $\{V_{g^*}; g^* \in \mathbf{G}^*(\mathbf{P})\}$ and $\{V_g; g \in \mathbf{G}(\mathbf{P})\}$ form groups of unitary operators on $L_2(E^*, \mu)$ and $\mathfrak{F}$, respectively.

We now have the following relation between V_{g^*} and V_g,

$$(\tau(V_{g^*}\varphi))(\xi) = V_{g^{-1}}(\tau\varphi)(\xi), \qquad \xi \in E, \tag{2.17}$$

which is proved by the equations

$$\begin{aligned} \int_{E^*} e^{i\langle x,\xi\rangle}\varphi(g^*x)\mu(dx) &= \int_{E^*} e^{i\langle g^{*-1}x,\xi\rangle}\varphi(x)\mu(dg^{*-1}x) \\ &= \int_{E^*} e^{i\langle x,g^{-1}\xi\rangle}\varphi(x)\mu(dx) = (\tau\varphi)(g^{-1}\xi). \end{aligned} \tag{2.18}$$

Another important concept relating to stationary processes is the purely nondeterministic property. Let $\mathbf{T}_t$ be a set of the form

$$\mathbf{T}_t = \{s; s \in \mathbf{T}, s \leq t\} \tag{2.19}$$

and $\mathcal{B}_t$ be the smallest Borel field generated by all cylinder sets of the form

(2.20)

$$\{x; (\langle x, \xi_1\rangle, \cdots, \langle x, \xi_n\rangle) \in B^n, \xi_k \in E, \mathrm{supp}\,(\xi_k) \subset \mathbf{T}_t, 1 \leq k \leq n, B^n \in \mathcal{B}(R^n)\}.$$

The subspaces of L_2, $L_2(t)$ and $L_2^*(t)$ are defined by

$$\begin{aligned} L_2(t) &= \{\varphi; \varphi \in L_2, \varphi \text{ is } \mathcal{B}_t\text{-measurable}\}, & t \in \mathbf{T}, \\ L_2^*(t) &= \{\varphi; \varphi \in L_2(t), \langle\langle \varphi, 1\rangle\rangle = 0\}, & t \in \mathbf{T}. \end{aligned} \tag{2.21}$$

Corresponding to $L_2(t)$ and $L_2^*(t)$, we can define

$$\begin{aligned} \mathfrak{F}(t) &= \mathfrak{S}\{C(\cdot - \xi); \xi \in E, \mathrm{supp}\,(\xi) \subset \mathbf{T}_t\} \\ \mathfrak{F}^*(t) &= \{f; f \in \mathfrak{F}(t), (f(\cdot), C(\cdot)) = 0\}, \end{aligned} \tag{2.22}$$

where $\mathfrak{S}\{\ \ \}$ denotes the subspace spanned by elements written in the bracket. Then we can easily prove the following proposition.

PROPOSITION 2.1. *For every* $t \in \mathbf{T}$,

$$L_2(t) \cong \mathfrak{F}(t), \qquad \text{(isomorphic)} \tag{2.23}$$

and

$$L_2^*(t) \cong \mathfrak{F}^*(t), \qquad \text{(isomorphic)} \tag{2.24}$$

under the transformation τ *restricted to* $L_2(t)$ *and* $L_2^*(t)$, *respectively.*

DEFINITION 2.5. *If*

$$L_2^*(-\infty) \equiv \bigcap_{s \in \mathbf{T}} L_2^*(s) = \{0\} \tag{2.25}$$

holds, then $\mathbf{P} = (E^*, \mu, \{T_t\})$ *is called purely nondeterministic.*

This definition was given by M. Nisio [12] for the case where E^* is an ordinary function space. By definition and proposition 2.1, $\mathbf{P}$ is purely nondeterministic if and only if

$$\mathfrak{F}^*(-\infty) = \bigcap_{s \in \mathbf{T}} \mathfrak{F}^*(s) = \{0\} \tag{2.26}$$

holds.

We are now in a position to develop certain basic concepts relative to stationary processes. We would like to emphasize the importance of a stationary process with independent values at every point.

DEFINITION 2.6 (Gelfand and Vilenkin [4]). *A stationary process* $\mathbf{P} = (E^*, \mu, \{T_t\})$ *will be called a process with independent values at every point if its characteristic functional* $C(\xi)$, $\xi \in E$, *satisfies*

$$C(\xi_1 + \xi_2) = C(\xi_1)C(\xi_2), \quad \text{whenever} \quad \mathrm{supp}\,(\xi_1) \cap \mathrm{supp}\,(\xi_2) = \varnothing. \tag{2.27}$$

If E is the space $\mathcal{K}$ of C^∞-functions with compact supports introduced by L. Schwartz, this definition coincides with that of Gelfand and Vilenkin. If E

is the space $\mathfrak{s}$ of rapidly decreasing sequences, then we have a sequence of independent random variables with the same distribution.

PROPOSITION 2.2. *If* $\mathbf{P}$ *is a stationary process with independent values at every point, then it is purely nondeterministic.*

PROOF. For $f \in \mathfrak{F}^*(t)$ there exists a sequence $\{f_n\}$ such that $\text{l.i.m.}_{n\to\infty} f_n = f$ and

$$(2.28)\qquad f_n(\cdot) = \sum_{k=1}^{N_n} a_k^{(n)} C(\cdot - \xi_k^{(n)}), \qquad \xi_k^{(n)} \in E, \qquad \text{supp}\,(\xi_k^{(n)}) \subset \mathbf{T}_t.$$

Since $\mathbf{P}$ is a stationary process with independent values at every point, we have

$$(2.29)\qquad f_n(\xi) = \sum_{k=1}^{N_n} a_k^{(n)} C(\xi - \xi_k^{(n)}) = C(\xi) \sum_{k=1}^{N_n} a_k^{(n)} C(-\xi_k^{(n)}) = C(\xi) f_n(0)$$

for any ξ with $\text{supp}\,(\xi) \subset \mathbf{T}_t^c$. However, $f_n(\xi)$ has to be zero since

$$(2.30)\qquad f_n(0) = (f_n(\cdot), C(\cdot - 0)) = 0 \qquad \text{for} \quad f_n \in \mathfrak{F}^*.$$

Thus, $f(\xi) = 0$. If $f \in \bigcap_{t\in\mathbf{T}} \mathfrak{F}^*(t)$, then $f(\xi) = 0$ for every ξ with $\text{supp}\,(\xi) \subset \mathbf{T}_t^c$ for every t. Hence, we have $f(\cdot) = 0$.

Now we can proceed to the analysis of the $L_2(E^*, \mu)$ space arising from a stationary process $\mathbf{P}$ with independent values at every point. First we discuss polynomials on E^*. The function expressed in the form

$$(2.31)\qquad \varphi(x) = P(\langle x, \xi_1\rangle, \cdots, \langle x, \xi_n\rangle), \qquad \xi_1, \cdots, \xi_n \in E, \qquad x \in E^*,$$

where P is a polynomial of n variables with complex coefficients, is called a *polynomial* on E^*. If P is of degree p, we say that $\varphi(x)$ is p-th *degree polynomial*, and if, in particular, P is *homogeneous*, we say the same of φ.

Throughout the remainder of this section we shall assume the following.

ASSUMPTION 2.2. *The following conditions hold:*

(i) $\displaystyle\int_{E^*} |\langle x, \xi\rangle|^p \mu(dx) < \infty$ *for* $\xi \in E$ *and every integer* $p \geq 1$,

(ii) $\displaystyle\int_{E^*} \langle x, \xi\rangle \mu(dx) = 0$ *for every* $\xi \in E$.

With these assumptions we see that the set M of all polynomials on E^* forms a linear manifold of L_2. Consequently, $\tau(M) = \{\tau(\varphi); \varphi \in M\}$ is defined and $\tau(M) \subset \mathfrak{F}$.

DEFINITION 2.7. *An operator* D_ξ, $\xi \in E$, *is defined by*

$$(2.32)\qquad (D_\xi f)(\cdot) = \underset{\epsilon\to 0}{\text{l.i.m.}}\, \frac{1}{\epsilon}\,[f(\cdot + \epsilon\xi) - f(\cdot)]$$

if the limit exists, D_ξ *is called a* differential operator, *and its domain is denoted by* $\mathfrak{D}(D_\xi)$. *We define* $\mathfrak{D}$ *as* $\bigcap_{\xi\in E} \mathfrak{D}(D_\xi)$.

LEMMA 2.4. *If* $\mathbf{P}$ *satisfies assumption* 2.2, *we have the following:*

(i) $C(\cdot - \xi) \in \mathfrak{D}$ *for every* $\xi \in E$, *and* $\prod_{j=1}^n D_{\xi_j} C(\cdot - \xi)$ *belongs to* $\mathfrak{D}$ *for any* n *and* $\xi_1, \cdots, \xi_n \in E$;

(ii) $\tau(M) \subset \mathfrak{D}$;

(iii) *For any* $\xi_1, \cdots, \xi_n \in E$ *and any choice of positive integers* $k_1, \cdots, k_n$, *we have*

$$\tau^{-1}\left\{\left(\prod_{j=1}^{n} D_{\xi_j}^{k_j}\right) C(\cdot)\right\}(x) = (i)^{\sum k_j} \prod_{j=1}^{n} \langle x, \xi_j\rangle^{k_j}, \qquad x \in E^*; \tag{2.33}$$

(iv) *$(i)^{-1}D_\xi$ is a self-adjoint operator, the domain of which includes $\{C(\cdot - \xi); \xi \in E\} \cup \tau(M)$;*

(v) *for any $f \in \tau(M)$ and $\xi_1, \xi_2 \in E$, we have*

$$D_{\xi_1}D_{\xi_2}f(\cdot) = D_{\xi_2}D_{\xi_1}f(\cdot). \tag{2.34}$$

PROOF. By assumption 2.2,

$$\underset{\epsilon\to 0}{\text{l.i.m.}} \left\{\frac{1}{\epsilon}(e^{i\epsilon\langle x,\xi\rangle} - 1)e^{i\langle x,\eta\rangle} - i\langle x, \xi\rangle e^{i\langle x,\eta\rangle}\right\} = 0. \tag{2.35}$$

Using τ, the above relation proves that

$$\underset{\epsilon\to 0}{\text{l.i.m.}} \frac{1}{\epsilon}\{C(\cdot + \epsilon\xi + \eta) + C(\cdot + \eta)\} \tag{2.36}$$

exists and is equal to $\tau\{i\langle x, \xi\rangle e^{i\langle x,\eta\rangle}\}$.

In a similar way, we can prove the second assertion using assumption 2.2. (ii). By assumption 2.2, $\exp\{i \sum_{j=1}^{n} t_j\langle x, \xi_j\rangle\}$ is differentiable infinitely many times (in L_2-norm) with respect to $t_1, \cdots, t_{n-1}$ and t_n, and we have

$$(i)^{-\sum k_j}\left(\partial^{\sum k_j}/\partial t_1^{k_1}\cdots\partial t_n^{k_n}\right) \exp\left\{i \sum_{j=1}^{n} t_j\langle x, \xi_j\rangle\right\}\Big|_{t_1=t_2=\cdots=t_n=0} = \prod_{j=1}^{n} \langle x, \xi_j\rangle^{k_j}. \tag{2.37}$$

The right-hand side belongs to L_2, and mapping by τ, we have (2.33).

For assertion (iv), if $f \in \tau(M)$, we have

$$\begin{aligned}(D_\xi f)(\eta) &= (D_\xi f(\cdot), C(\cdot - \eta)) = \left(\underset{\epsilon\to 0}{\text{l.i.m.}} \frac{1}{\epsilon}[f(\cdot + \epsilon\xi) - f(\cdot)], C(\cdot - \eta)\right)\\ &= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\{f(\eta + \epsilon\xi) - f(\eta)\},\\ (f(\cdot), D_\xi C(\cdot - \eta)) &= \lim_{\epsilon\to 0} \frac{1}{\epsilon}(f(\eta - \epsilon\xi) - f(\eta)) = -(D_\xi f)(\eta),\end{aligned} \tag{2.38}$$

which prove that $(i)^{-1}D_\xi$ is self-adjoint.

For (v) consider $(f(\eta + \epsilon\xi_1 + \epsilon\xi_2) - f(\eta + \epsilon\xi_2) - f(\eta + \epsilon\xi_1) + f(\eta))/\epsilon^2$. Arguments like the above prove that D_{ξ_1} and D_{ξ_2} are commutative.

N. Wiener [15] discussed the following decomposition of $\mathfrak{F}(L_2)$ for the case of Gaussian white noise. Consider a system K of elements of $\mathfrak{F}$ defined by

$$\begin{aligned}K_n &= \left\{\prod_{j=1}^{n} D_{\xi_{k_j}} C(\cdot); k_1, \cdots, k_n = 1, 2, \cdots\right\}, \qquad n \geq 1,\\ K_0 &= \{C(\cdot)\}, \quad \text{and} \quad K = \bigcup_{n=0}^{\infty} K_n\end{aligned} \tag{2.39}$$

where $\{\xi_n^0\}_{n=1}^{\infty}$ is the system appearing in assumption 2.1. Since the system K_n

forms a countable set, we can arrange all the elements in linear order. We shall denote them by $\{g_k^{(n)}(\cdot)\}$. Let $\mathbf{P}$ be a stationary process satisfying assumption 2.2 and let $\mathfrak{F}_0$ be the one-dimensional space spanned by $C(\cdot)$, that is, $\mathfrak{F}_0 = K_0$. Put $f_k^{(1)} = g_k^{(1)}$ and define

$$\mathfrak{F}_1 = \mathfrak{S}\{f_k^{(1)}; k = 1, 2, \cdots\}. \tag{2.40}$$

Obviously, $\mathfrak{F}_0$ and $\mathfrak{F}_1$ are mutually orthogonal. Suppose that $\{\mathfrak{F}_j\}_{j=0}^{n-1}$ are defined and mutually orthogonal, consider

$$f_k^{(n)} = P_{\left(\sum_{j=0}^{n-1} \oplus \mathfrak{F}_j\right)^{\perp}} g_k^{(n)}, \qquad k = 1, 2, \cdots, \tag{2.41}$$

where $P_{(\cdot)}$ denotes the projection on $(\cdot)$. Then $\mathfrak{F}_n$ is defined by

$$\mathfrak{F}_n = \mathfrak{S}\{f_k^{(n)}; k = 1, 2, \cdots\}, \tag{2.42}$$

and it is orthogonal to $\sum_{j=0}^{n-1} \oplus \mathfrak{F}_j$. This procedure can be continued until there are no more elements $g_k^{(n+1)}$ not belonging to $\sum_{j=0}^{n} \oplus \mathfrak{F}_j$. Finally

$$\overline{K} \ (= \text{the closure of } K \text{ in } \mathfrak{F}) = \sum_{j \geq 0} \oplus \mathfrak{F}_j. \tag{2.43}$$

DEFINITION 2.8. *The direct sum decomposition* (2.43) *is called Wiener's direct sum decomposition.*

THEOREM 2.2. *Let* $\mathbf{P}$ *be a stationary process satisfying assumptions* 2.1 *and* 2.2. *If*

$$\sum_{n=0}^{\infty} g_\xi^{(n)}(\cdot)/n!, \qquad g_\xi^{(n)}(\cdot) = (D_\xi)^n C(\cdot), \tag{2.44}$$

converges for every ξ, *then Wiener's direct sum decomposition satisfies the following properties:*

$$\text{(i)} \quad \sum_{n=0}^{\infty} \oplus \mathfrak{F}_n = \mathfrak{F}, \qquad \text{(ii)} \quad U_t(\mathfrak{F}_n) = \mathfrak{F}_n. \tag{2.45}$$

PROOF. By assumption,

$$\sum_{n=0}^{\infty} (i)^n \langle x, \xi\rangle^n/n! = \tau^{-1}\left(\sum_{n=0}^{\infty} g_\xi^{(n)}(\cdot)/n!\right), \qquad (g_\xi^{(0)}(\cdot) = C(\cdot)), \tag{2.46}$$

converges and the sum is equal to $\exp\{i\langle x, \xi\rangle\}$. Hence,

$$C(\cdot - \xi) = \sum_{n=0}^{\infty} \frac{g_\xi^{(n)}(\cdot)}{n}. \tag{2.47}$$

On the other hand, by the construction of the $\mathfrak{F}_n$'s we can prove

$$g_\xi^{(n)} \in \sum_{j=0}^{\infty} \oplus \mathfrak{F}_j. \tag{2.48}$$

Therefore, $C(\cdot - \xi) \in \sum_{j=0}^{\infty} \oplus \mathfrak{F}_j$ which proves

$$\mathfrak{F} \subset \sum_{n=0}^{\infty} \oplus \mathfrak{F}_n (\subset \mathfrak{F}), \tag{2.49}$$

since $\{C(\cdot - \xi)\}$ spans the entire $\mathfrak{F}$.

The second assertion is easily proved noting that

$$U_t g_\xi^{(n)}(\cdot) = g_{S_t\xi}^{(n)}(\cdot) \tag{2.50}$$

and

$$\mathfrak{S}(U_t K_n) = \mathfrak{S}(K_n). \tag{2.51}$$

Let us return to the group $\mathbf{G}(\mathbf{P})$. If

$$V_g(\mathfrak{F}_n) \subset \mathfrak{F}_n \qquad \text{for every} \quad g \in \mathbf{G}(\mathbf{P}), \tag{2.52}$$

we call the decomposition $\mathfrak{F} = \sum_{n=0}^{\infty} \oplus \mathfrak{F}_n$ *invariant* with respect to $\mathbf{G}(\mathbf{P})$. This concept is important in connection with the Wiener's direct sum decomposition. We shall discuss this topic in the later sections (4–6).

3. Orthogonal polynomials and reproducing kernels

From now on we shall deal with the decomposition of $L_2(E^*, \mu)$ and $\mathfrak{F}(E, C)$ associated with a stationary process with independent values at every point. First we consider, in this section, the simple case where E is a finite dimensional space. We can find a relation between the space with reproducing kernel and Rodrigues' formula for classical orthogonal polynomials. Such considerations will aid us in considering the case where E is an infinite dimensional nuclear space and will be preparation for later discussions.

Let ν be a probability measure (distribution) on R^1 and $\tilde{C}$ be its Fourier-Stieltjes transform (characteristic function); that is,

$$\tilde{C}(\lambda) = \int_{R^1} e^{i\lambda x} \nu(dx), \qquad \lambda \in R^1. \tag{3.1}$$

Appealing to Aronszajn's results [1] stated in lemma 2.2, we obtain the smallest Hilbert space $\tilde{\mathfrak{F}} = \mathfrak{F}(R^1, \tilde{C})$, the reproducing kernel of which is $\tilde{C}(\lambda - \mu)$, $\lambda, \mu \in R^1$. By theorem 2.1, there exists an isomorphism $\tilde{\tau}$ which maps $\tilde{L}_2 = L_2(\nu; R^1) = \{f; \int_{R^1} |f(x)|^2 \nu(dx)\}$ onto $\tilde{\mathfrak{F}}$ in the following way:

$$(\tilde{\tau} f)(\lambda) = \int_{R^1} e^{i\lambda x} f(x) \nu(dx). \tag{3.2}$$

We shall examine this isomorphism $\tilde{\tau}$ in detail in the following examples. It is more interesting to discuss the analysis on $\tilde{\mathfrak{F}}$ rather than on $\tilde{L}_2$, since, for one thing, the development of functions belonging to $\tilde{L}_2$ in terms of orthogonal polynomials turns out to be the power series expansion in $\tilde{\mathfrak{F}}$.

3.1. *Gaussian distribution.* Consider the case where

$$\nu(dx) = \nu(x; \sigma^2)\, dx = (2\pi\sigma^2)^{-1/2} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} dx, \tag{3.3}$$

then

$$\tilde{C}(\lambda, \sigma^2) = \int_{R^1} e^{i\lambda x} \nu(x; \sigma^2)\, dx = \exp\left(-\frac{\sigma^2}{2} \lambda^2\right). \tag{3.4}$$

Choose Hermite polynomials

$$H_n(x;\sigma^2) = \frac{(-1)^n\sigma^{2n}}{n!}\frac{1}{\nu(x;\sigma^2)}\frac{d^n}{dx^n}\nu(x;\sigma^2), \qquad n = 1, 2, \cdots \tag{3.5}$$

(Rodrigues' formula), which form a complete orthonormal system in $\tilde{L}_2$. The isomorphism $\tilde{\tau}$ maps $H_n(x)$ to the n-th degree monomial of λ times $\tilde{C}$. In fact,

$$(\tilde{\tau}H_n(\cdot\,;\sigma^2))(\lambda) = \sigma_n\lambda^n\tilde{C}(\lambda:\sigma^2),\ \sigma_n = \frac{\sigma^{2n}i^n}{n!}. \tag{3.6}$$

The proof of the formula (3.6) is as follows:

$$\begin{aligned}(\tilde{\tau}H_n(\cdot,\sigma^2))(\lambda) &= \frac{(-1)^n\sigma^{2n}}{n!}\int_{R^1}\{\exp(i\lambda x)\}\left\{\nu(x;\sigma^2)^{-1}\frac{d^n}{dx^n}\nu(x;\sigma^2)\right\}\nu(x;\sigma^2)\,dx\\ &= \frac{(-1)^n\sigma^{2n}}{n!}\int_{R^1}\exp(i\lambda x)\left\{\frac{d^n}{dx^n}\nu(x;\sigma^2)\right\}dx\\ &= \frac{\sigma^{2n}}{n!}i^n\lambda^n\tilde{C}(\lambda;\sigma^2).\end{aligned} \tag{3.7}$$

More generally, we have

$$\left\{\tilde{\tau}\left(\sum_{n=0}^{\infty}a_nH_n(\cdot\,;\sigma^2)\right)\right\}(\lambda) = \left(\sum_{n=0}^{\infty}a_n\sigma_n\lambda^n\right)\tilde{C}(\lambda:\sigma^2). \tag{3.8}$$

3.2. *Poisson distribution.* Let $\nu(dx)$ be given by

$$\nu(dx) = \nu(x,c)\delta_{S_c}(dx) = \frac{c^{x+c}}{\Gamma(x+c+1)}e^{-c}, \qquad x \in S_c, \tag{3.9}$$

where $S_c = \{-c, 1-c, 2-c, \cdots\}$. We obtain orthogonal polynomials with respect to the measure $\nu(x,c)\delta_{S_c}(dx)$, which are called *generalized Charlier polynomials*, by the following generalized Rodrigues' formula (cf. Bateman and others ([2], p. 222, and p. 227)):

$$p_n(x,c) = (-c)^n(\nu(x,c))^{-1}\Delta_x^n\,\nu(x-n,c) = L_n^{x+c-n}(c)n!, \qquad x \in S_c, \tag{3.10}$$

where Δ_x^n denotes the n-th order difference operator acting on functions of x. The relation

$$\begin{aligned}\tilde{C}(\lambda:c) &= \int_{R^1}e^{i\lambda x}\nu(x;c)\delta_{S_c}(dx) = \sum_{x\in S_c}e^{i\lambda x}\nu(x,c)\\ &= \exp\{e^{i\lambda c} - 1 - i\lambda c\}\end{aligned} \tag{3.11}$$

is easily obtained. Now put

$$Q_n(x,c) = \frac{1}{n!c^n}P_n(x,c); \tag{3.12}$$

then we get the orthogonality relation for Q_n:

$$\sum_{x\in S_c}Q_n(x,c)Q_m(x,c)\nu(x,c) = \delta_{n,m}, \qquad n, m = 1, 2, \cdots. \tag{3.13}$$

Every P_n, of course, belongs to $L^2(\nu, R^1)$, and it is transformed by $\tilde{\tau}$ into

$$(\tilde{\tau}P_n(\cdot, c))(\lambda) = c^n(e^{i\lambda} - 1)^n\tilde{C}(\lambda: c). \tag{3.14}$$

This is proved as follows:

$$\begin{aligned}(\tilde{\tau}P_n(\cdot, c))(\lambda) &= \int_{R^1} e^{i\lambda x}(-c)^n((\nu(x, c))^{-1}\Delta_x^n\nu(x - n, c))\nu(x, c)\delta_{S_c}(dx) \\ &= (-c)^n(-1)^n \sum_{x\in S_c} (\Delta_x^n e^{i\lambda x})\nu(x, c) \\ &= c^n \sum_{x\in S_c} \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} e^{im\lambda}e^{ix\lambda}\nu(x, c) \\ &= c^n(e^{i\lambda} - 1)^n\tilde{C}(\lambda: c).\end{aligned} \tag{3.15}$$

Note that the last expression is of the monomial form of $(e^{i\lambda} - 1)$ times $\tilde{C}$.

4. Stationary process with independent values at every point

This section is devoted to the study of the general theory for a certain class of stationary processes with independent values at every point. Let $\mathbf{P} = (E^*, \mu, \{T_t\})$ be a stationary process, where $\mathbf{T}$ is a set of real numbers and E is the function space $\mathcal{S}$ in the sense of L. Schwartz, and let its characteristic function be given by

$$\begin{aligned}C(\xi) &= \exp \int_{-\infty}^{\infty} \alpha(\xi(t))\, dt, \\ \alpha(x) &= (-\sigma^2x^2)/2 + \int_{-\infty}^{\infty} \left(e^{ixu} - 1 - \frac{ixu}{1 + u^2}\right)\frac{1 + u^2}{u^2}\, d\beta(u).\end{aligned} \tag{4.1}$$

Here $0 \leq \sigma^2 < \infty$ and $d\beta(u)$ is a bounded measure on $(-\infty, \infty)$ such that $d\beta\,(\{0\}) = 0$. Obviously, $\mathbf{P}$ satisfies (2.27); that is, it is a stationary process with independent values at every point.

For the moment let us turn our attention from the flow $\{U_t, t \text{ real}\}$ to the direct sum decomposition of $\mathfrak{F} = \mathfrak{F}(\mathcal{S}, C)$ mentioned in section 1. Define

$$\begin{aligned}K_0(\xi, \eta) &= \exp\left(\int \alpha(\xi(t))\, dt\right) \exp\left(\int \alpha(-\eta(t))\, dt\right) = C(\xi)C(-\eta), \\ K_1(\xi, \eta) &= \int \alpha(\xi(t) - \eta(t))\, dt - \int \alpha(\xi(t))\, dt - \int \alpha(-\eta(t))\, dt, \\ K_p(\xi, \eta) &= \frac{1}{p!}\,(K_1(\xi, \eta))^p, \qquad p \geq 2, \\ k_p(\xi, \eta) &= \frac{1}{p!}\,K_0(\xi, \eta)(K_1(\xi, \eta))^p, \qquad p \geq 0, \qquad \xi,\eta \in \mathcal{S}.\end{aligned} \tag{4.2}$$

Note that $C(\xi - \eta) = \sum_{p=0}^{\infty} k_p(\xi, \eta)$. We then prove the following lemma using the fact that $\alpha(x - y)$ is conditionally positive definite (cf. Gelfand and Vilenkin [4], chapter 3).

LEMMA 4.1. *The functionals* $K_0(\xi, \eta)$, $K_p(\xi, \eta)$, *and* $k_p(\xi, \eta)$, $p = 0, 1, \cdots$, $(\xi, \eta) \in \mathcal{S} \times \mathcal{S}$, *are all positive definite and continuous.*

Again appealing to the Aronszajn's theorem (lemma 2.2), we obtain the Hilbert spaces $\mathfrak{F}$, $\hat{\mathfrak{F}}_p$ and $\mathfrak{F}_p$, $p = 0, 1, 2, \cdots$ with reproducing kernels C, K_p, and k_p respectively. Consider subspaces $\hat{\mathfrak{F}}_p$ and $\mathfrak{F}_p$. We use the symbol $\otimes^*$ to express the direct product of subspaces in the sense of Aronszajn [1]. Hereafter we use subscripts to distinguish the various norms.

LEMMA 4.2. *The space $\hat{\mathfrak{F}}_p$ is the class of all restrictions of functionals belonging to $\hat{\mathfrak{F}}_1 \otimes^* \hat{\mathfrak{F}}_1 \otimes^* \cdots \otimes^* \hat{\mathfrak{F}}_1$ (p times) to the diagonal set $\mathfrak{S}_p = \{(\xi, \cdots, \xi); \xi \in \mathfrak{S}\}$. The norm $\|\cdot\|_{\hat{\mathfrak{F}}_p}$ in $\hat{\mathfrak{F}}_p$ can be expressed in the form*

$$\|f\|_{\hat{\mathfrak{F}}_p} = \inf_{\substack{f' \in \hat{\mathfrak{F}}_1 \otimes^* \cdots \otimes^* \hat{\mathfrak{F}}_1 \\ f = f'/\mathfrak{S}_p}} \|f'\|_{\hat{\mathfrak{F}}_1 \otimes^* \cdots \otimes^* \hat{\mathfrak{F}}_1} \tag{4.3}$$

where $f'/\mathfrak{S}_p$ denotes the restriction of f' to $\mathfrak{S}_p$.

PROOF. By the definition of K_p and by Aronszajn ([1], section 8, theorem II) the assertion is easily proved.

LEMMA 4.3. *The space $\mathfrak{F}_p$ is the class of all restrictions of functionals belonging to $\hat{\mathfrak{F}}_0 \otimes^* \hat{\mathfrak{F}}_p$ to the diagonal set $\mathfrak{S}_2 = \{(\xi, \xi); \xi \in \mathfrak{S}\}$. The norm $\|\cdot\|_{\mathfrak{F}_p}$ can be expressed in the form*

$$\|f\|_{\mathfrak{F}_p} = \inf_{\substack{f' \in \hat{\mathfrak{F}}_0 \otimes^* \hat{\mathfrak{F}}_p \\ f = f'/\mathfrak{S}_2}} \|f'\|_{\hat{\mathfrak{F}}_0 \otimes^* \hat{\mathfrak{F}}_p}. \tag{4.4}$$

LEMMA 4.4. *The space $\mathfrak{F}_p$, $p = 1, 2, \cdots$, are mutually orthogonal subspaces of $\mathfrak{F}$.*

PROOF. Put

$$K_{p,q}(\xi, \eta) = K_p(\xi, \eta) + K_q(\xi, \eta), \qquad p \neq q. \tag{4.5}$$

Then the Hilbert space $\hat{\mathfrak{F}}_{p,q}$ with reproducing kernel $K_{p,q}(\xi, \eta)$ is expressible in the form

$$\hat{\mathfrak{F}}_{p,q} = \hat{\mathfrak{F}}_p \oplus \hat{\mathfrak{F}}_q. \tag{4.6}$$

To prove this assertion we first show that

$$\hat{\mathfrak{F}}_p \cap \hat{\mathfrak{F}}_q = \{0\}. \tag{4.7}$$

Suppose $p > q$; then

$$K_p(\xi, \eta) = K_q(\xi, \eta) \frac{q!(p-q)!}{p!} K_{p-q}(\xi, \eta). \tag{4.8}$$

Consequently, $\hat{\mathfrak{F}}_p$ is the class of all restrictions of functionals belonging to $\hat{\mathfrak{F}}_q \otimes^* \hat{\mathfrak{F}}_{p-q}$ to the diagonal set $\mathfrak{S}_2$. Now suppose $f \in \hat{\mathfrak{F}}_p \cap \hat{\mathfrak{F}}_q$ and let $\{f_k^{(q)}\}$ be a complete orthonormal system in $\hat{\mathfrak{F}}_q$. Since $f \in \hat{\mathfrak{F}}_p$, it can be expressed in the form

$$f(\xi) = \sum_{k=1}^{\infty} g_k(\xi) f_k^{(q)}(\xi), \qquad g \in \hat{\mathfrak{F}}_{p-q} \tag{4.9}$$

(remark attached to theorem II, Aronszajn ([1], p. 361)). But by assumption, f belongs to $\hat{\mathfrak{F}}_q$. Consequently, all the g_k's must be zero, which implies $f = 0$.

Let us recall the discussion of Aronszajn ([1], part I, section 6). By (4.7), if $f_p \in \hat{\mathfrak{F}}_p$, $f_q \in \hat{\mathfrak{F}}_q$, then

$$\begin{aligned}(4.10)\qquad &\|f_p + f_q\|^2_{\mathfrak{F}_{p,q}} = \|f_p\|^2_{\mathfrak{F}_p} + \|f_q\|^2_{\mathfrak{F}_q},\\ &\|f_p - f_q\|^2_{\mathfrak{F}_{p,q}} = \|f_p\|^2_{\mathfrak{F}_p} + \|-f_q\|^2_{\mathfrak{F}_q} = \|f_p\|^2_{\mathfrak{F}_p} + \|f_q\|^2_{\mathfrak{F}_q},\end{aligned}$$

which imply $\mathrm{Re}\,(f_p, f_q) = 0$. Similarly, we have $\mathrm{Im}\,(f_p, f_q) = 0$. Thus we have proved (4.6) and the lemma.

Further, by the proof of lemma 4.4, we can show the following. If $\mathfrak{F}_{p,q}$ is the reproducing kernel Hilbert space with kernel $k_p(\xi, \eta) + k_q(\xi, \eta)$, then

$$(4.11)\qquad \mathfrak{F}_{p,q} = \mathfrak{F}_p \oplus \mathfrak{F}_q, \qquad p \neq q,$$

and if $f \in \mathfrak{F}_{p,q}$, then

$$(4.12)\qquad (f(\cdot), k_p(\cdot, \xi)) \equiv f_p(\xi)$$

is the projection of f on $\mathfrak{F}_p$.

LEMMA 4.5. *The kernel $K_p(\xi, \eta)$ and $k_p(\xi, \eta)$, $p \geq 0$, are $\mathbf{G}(\mathbf{P})$-invariant; that is,*

$$(4.13)\qquad K_p(g\xi, g\eta) = K_p(\xi, \eta), \qquad k_p(g\xi, g\eta) = k_p(\xi, \eta)$$

for every $g \in \mathbf{G}(\mathbf{P})$.

PROOF. It is sufficient to prove that K_0 and K_1 are $\mathbf{G}(\mathbf{P})$-invariant. For K_0 this is easily proved by (4.2) and the definition of $\mathbf{G}(\mathbf{P})$. Concerning K_1, we have

$$\begin{aligned}(4.14)\qquad K_1(g\xi, g\eta) &= \int_{-\infty}^{\infty} \alpha((g\xi)(t) - (g\eta)(t))\,dt - \int_{-\infty}^{\infty} \alpha((g\xi)(t))\,dt\\ &- \int_{-\infty}^{\infty} \alpha(-(g\eta)(t))\,dt = \int_{-\infty}^{\infty} \alpha(g(\xi - \eta)(t))\,dt - \int_{-\infty}^{\infty} \alpha((g\xi)(t))\,dt\\ &- \int \alpha(-(g\eta)(t))\,dt = K_1(\xi, \eta),\end{aligned}$$

since every $g \in \mathbf{G}(\mathbf{P})$ keeps the integral $\int_{-\infty}^{\infty} \alpha(\xi(t))\,dt$ invariant.

Now we shall state one of our main results.

THEOREM 4.1. *The space $\mathfrak{F}$ has the direct sum decomposition*

$$(4.15)\qquad \mathfrak{F} = \sum_{p=0}^{\infty} \oplus \mathfrak{F}_p,$$

and it is $\mathbf{G}(\mathbf{P})$-invariant. The kernel $k_p(\cdot, \xi)$ is a projection operator in the following sense:

$$(4.16)\qquad (f(\cdot), k_p(\cdot, \xi)) \equiv f_p(\xi)$$

is the projection of f on $\mathfrak{F}_p$.

PROOF. By lemma 4.4,

$$(4.17)\qquad \sum_{p=0}^{n} k_p(\xi, \eta), \qquad (\xi, \eta) \in \mathcal{S} \times \mathcal{S},$$

will be the reproducing kernel of the subspace $\sum_{p=0}^{n} \oplus \mathfrak{F}_p$. Noting that

$$(4.18)\qquad C(\xi, \eta) = \sum_{p=0}^{\infty} k_p(\xi, \eta),$$

and

(4.19) $$\left\| \sum_{p=n}^{m} k_p(\cdot, \eta) \right\|_{\mathfrak{F}}^2 = \sum_{p=n}^{m} k_p(\eta, \eta),$$

we conclude

(4.20) $$\mathfrak{F} = \sum_{p=0}^{\infty} \oplus \mathfrak{F}_p$$

(cf. Aronszajn [1], part I, section 9). The **G**(**P**)-invariantness of $\mathfrak{F}_p$ comes from the definition of $\mathfrak{F}_p$ and lemma 4.5. By (4.12) and the above discussions, we have the last assertion.

Coming back to L_2 space, we have the following decomposition:

(4.21) $$L_2 = \sum_{p=0}^{\infty} \oplus L_2^{(p)} \qquad \text{with} \quad \tau(L_2^{(p)}) = \mathfrak{F}_p.$$

5. Gaussian white noise

In the following three sections we shall discuss some typical stationary processes with independent values at every point. First we deal with Gaussian white noise, the characteristic functional of which is

(5.1) $$C(\xi) = \exp\left\{-\tfrac{1}{2}\int_{-\infty}^{\infty} \xi(t)^2\, dt\right\}, \qquad \xi \in \mathcal{S}'$$

namely the particular case where $\alpha(x) = -\frac{1}{2}x^2$ in the formula (4.1). Consequently, $K_p(\xi, \eta)$ is of the form

(5.2) $$K_p(\xi, \eta) = \frac{1}{p!}\langle \xi, \eta\rangle^p = \frac{1}{p!}\left(\int_{-\infty}^{\infty} \xi(t)\eta(t)\, dt\right)^p.$$

Now put

(5.3) $$\hat{L}_2(R^p) = \{F; F \in L_2(R^p), F \text{ is symmetric}\},$$
$$\hat{F}(t_1, \cdots, t_p) = \frac{1}{p!}\sum_{\pi} F(t_{\pi(1)}, \cdots, t_{\pi(p)}), \qquad \text{(symmetrization)}$$

where π denotes the permutation of integers $1, 2, \cdots, p$. Define $I_p^*(\xi; F)$ by

(5.4) $$I_p^*(\xi; F) = \int_{-\infty}^{\infty}\!\!\cdots\!\int \xi(t_1)\cdots\xi(t_p)F(t_1, \cdots, t_p)\, dt_1, \cdots, dt_p, \qquad \xi \in \mathcal{S},$$

then we have

(5.5) $$I_p^*(\xi; F) = I_p^*(\xi; \hat{F}), \qquad \text{for every } \xi \in \mathcal{S} \text{ and } F \in L_2(R^p).$$

THEOREM 5.1. *For Gaussian white noise we have the following properties:*

(i) $$\mathfrak{F}_p = \{f(\cdot); f(\xi) = I_p^*(\xi; F),\ F \in L_2(R^p)\},$$

(ii) $$(I_p^*(\cdot\,; F), I_p^*(\cdot\,; G))_{\mathfrak{F}_p} = p!\int_{-\infty}^{\infty}\!\!\cdots\!\int \hat{F}(t_1, \cdots, t_p)\overline{G(t_1, \cdots, t_p)}\, dt_1\cdots dt_p.$$

PROOF. Define $\tilde{L}_2(R^p)$ and $\tilde{\mathfrak{F}}_p$ by

(5.6) $$\tilde{L}_2(R^p) = \left\{F; F(t_1, \cdots, t_p) = \frac{1}{p!}\sum_{k=1}^{n} a_k\xi_k(t_1)\cdots\xi_k(t_p),\right.$$
$$\left. a_k \text{ complex}, \xi_1, \cdots, \xi_n \in \mathcal{S}\right\}$$

and

$$\mathfrak{F}_p = \{f(\cdot); f(\xi) = I_p^*(\xi; F),\ F \in \tilde{L}_2(R^p)\}. \tag{5.7}$$

Then we can prove $\mathfrak{F}_p \subseteq \mathfrak{F}_p$. If F and G are elements of $\tilde{L}_2(R^p)$ of the form

$$F = \frac{1}{p!}\sum_{k=1}^{n} a_k \xi_k(t_1)\cdots\xi_k(t_p), \qquad G = \frac{1}{p!}\sum_{k=1}^{n} b_k \eta_k(t_1)\cdots\eta_k(t_p), \tag{5.8}$$

where a_k's and b_k's are complex numbers and $\xi_k, \eta_k \in \mathcal{S}$, we have

$$\begin{aligned}(I_p^*(\cdot\,; F), I_p^*(\cdot\,; G))_{\mathfrak{F}_p} &= \sum_{k=1}^{n}\sum_{j=1}^{n} a_k \overline{b_j}\frac{1}{p!}\int\overset{\infty}{\underset{-\infty}{\cdots}}\int \xi_k(t_1)\cdots\xi_k(t_p)\eta_j(t_1)\cdots\eta_j(t_p)\,dt_1\cdots dt_p \\ &= p!\int\overset{\infty}{\underset{-\infty}{\cdots}}\int F(t_1, \cdots, t_p)\overline{G(t_1, \cdots, t_p)}\,dt_1\cdots dt_p.\end{aligned} \tag{5.9}$$

Since $\tilde{L}_2(R^p)$ is dense in $\hat{L}_2(R^p)$, we can prove that $\mathfrak{F}_p$ is also dense in $\mathfrak{F}_p$. Indeed,

$$\mathfrak{F}_p = \{f(\cdot); f(\xi) = I_p^*(\xi; F),\ F \in \hat{L}_2(R^p)\}. \tag{5.10}$$

Thus, by (5.5), we get (i).

The second assertion is easily verified using (5.5) and (5.9).

Take a complete orthonormal system $\{\mathring{\xi}_j\}_{j=1}^{\infty}$ in $L_2(R^1)$ such that all the $\mathring{\xi}_j$'s belong to $\mathcal{S}$ (cf. assumption 2.1).

COROLLARY 5.1 (M. G. Kreĭn [9]). *Define the functional*

$$\Phi_{(k_1,\cdots,k_q)}^{(j_1,\cdots,j_q)}(\xi) = \frac{\sqrt{p!}}{\sqrt{k_1!\cdots k_q!}}\prod_{n=1}^{q}\langle\xi, \mathring{\xi}_{j_n}\rangle^{k_n}; \tag{5.11}$$

then

$$\left\{\Phi_{(k_1,\cdots,k_q)}^{(j_1,\cdots,j_q)} : \begin{array}{ll} j_1, \cdots, j_q & \text{different positive integers,} \\ k_1, \cdots, k_q & \text{different positive integers such that } \sum_1^q k_j = p\end{array}\right\} \tag{5.12}$$

forms a complete orthonormal system in $\mathfrak{F}_p$.

PROOF. The set of functionals on $\mathcal{S} \times \cdots \times \mathcal{S}$ (p times) of the form

$$\left\{\prod_{n=1}^{p}\langle\xi, \mathring{\xi}_{j_n}\rangle; j_1, \cdots, j_p = 1, 2, \cdots\right\} \tag{5.13}$$

is a complete orthonormal system in $\mathfrak{F}_1 \otimes^* \cdots \otimes^* \mathfrak{F}_1$ (p times).

On the other hand, we have

$$\begin{aligned}(\Phi_{(k_1,\cdots,k_q)}^{(j_1,\cdots,j_q)}, \Phi_{(k_1',\cdots,k_q')}^{(j_1',\cdots,j_q')})_{\mathfrak{F}_p} &= p!\frac{\sqrt{p!}}{\sqrt{k_1!\cdots k_q!}}\frac{\sqrt{p!}}{\sqrt{k_1'!\cdots k_q'!}}\left(\frac{1}{p!}\right)^2 \\ &\quad \sum_{\pi}\sum_{\pi'}\int\overset{\infty}{\underset{-\infty}{\cdots}}\int \mathring{\xi}_{j_1}(t_{\pi(1)})\cdots\mathring{\xi}_{j_1}(t_{\pi(k_1)})\mathring{\xi}_{j_2}(t_{\pi(k_1+1)})\cdots \\ &\quad \times \mathring{\xi}_{j_2}(t_{\pi(k_1+k_2)})\cdots\mathring{\xi}_{j_q}(t_{\pi(p)})\cdots\mathring{\xi}_{j_1'}(t_{\pi'(k_1')})\cdots\mathring{\xi}_{j_q'}(t_{\pi'(p)})\,dt_1\cdots dt_p.\end{aligned} \tag{5.14}$$

The right-hand side vanishes if $((j_1, k_1), \cdots, (j_q, k_q)) \neq ((j'_1, k'_1), \cdots, (j'_q, k'_q))$ (as sets) and is equal to

$$\frac{1}{k_1!\cdots k_q!}\left[\sum_{\pi(k_1)}\left(\int \mathring{\xi}_{j_1}(t)^2\,dt\right)^{k_1}\right]\cdots\left[\sum_{\pi(k_q)}\left(\int \mathring{\xi}_{j_q}(t)^2\,dt\right)^{k_q}\right] = 1 \tag{5.15}$$

otherwise, where $\pi^{(k)}$ denotes the permutation of k integers. Moreover, if $q \neq q'$, then (5.14) obviously vanishes. Thus we have proved the corollary.

REMARK 5.1. The above result has already been proved by M. G. Kreĭn ([9], section 4), although it is stated in a somewhat different form.

COROLLARY 5.2. *If $H_n(x; 1)$ denotes the Hermite polynomial defined by* (3.5) *with $\sigma = 1$, then*

$$\tau^{-1}\{\Phi^{(j_1,\cdots,j_q)}_{(k_1,\cdots,k_q)}(\cdot)C(\cdot)\}(x) = \sqrt{p!}\sqrt{k_1!\cdots k_q!}(i)^{-p}\prod_{m=1}^{q} H_{k_m}(\langle x, \mathring{\xi}_{j_m}\rangle; 1). \tag{5.16}$$

PROOF. The formula

$$\begin{aligned}&\int_{\mathcal{S}^*} e^{i\langle x,\xi\rangle}\prod_{m=1}^{q} H_{k_m}(\langle x, \mathring{\xi}_{j_m}\rangle; 1)\mu(dx) \\ &= \int_{\mathcal{S}^*}\prod_{m=1}^{q}[H_{k_m}(\langle x, \mathring{\xi}_{j_m}\rangle; 1)e^{i\langle x,\xi^0_{j_m}\rangle\langle\xi,\xi^0_{j_m}\rangle}]\mu(dx)\int_{\mathcal{S}^*} e^{i\sum_{l\notin(j_1,\cdots,j_q)}\langle x,\xi^0_l\rangle\langle\xi,\xi^0_l\rangle}\mu(dx) \\ &= \prod_{m=1}^{q}\int_{-\infty}^{\infty} e^{i\langle\xi,\xi^0_{j_m}\rangle x}H_{k_m}(x; 1)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx\; C\left(\sum_{l\notin(j_1,\cdots,j_q)}\langle\xi, \mathring{\xi}_l\rangle\,\mathring{\xi}_l\right)\end{aligned} \tag{5.17}$$

becomes

$$\frac{(i)^p}{k_1!\cdots k_q!}\prod_{m=1}^{q}\langle\xi, \mathring{\xi}_{j_m}\rangle^{k_m}C(\xi). \tag{5.18}$$

This proves (5.16).

REMARK 5.2. From theorem 4.1 and the above result, we get the orthogonal development of the elements of L_2 due to Cameron and Martin [3].

In the above discussion we use an important property of Gaussian white noise, that is, the equivalence of independence and orthogonality. For other cases discussed here, the multiple Wiener integral due to K. Itô [6] plays an important role.

Let $\{I_j\}_{j=1}^{n}$ be a finite partition of $\mathbf{T}$. Then we have

$$C(\xi) = \prod_{j=1}^{n} C(\xi\chi_{I_j}), \qquad \xi \in \mathcal{S}, \tag{5.19}$$

where χ_{I_j} is the indicator function of I_j. Note that $C(\xi\chi_{I_j})$ has meaning even though $\xi\chi_I$ may not be in $\mathcal{S}$.

Now if we consider the restriction of $C(\xi)$ to $\mathcal{K}(\bar{I}_j)$; then

$$C_{I_j}(\xi) = C(\xi), \qquad \xi \in \mathcal{K}(\bar{I}_j), \tag{5.20}$$

is a continuous positive definite functional. Therefore, we can follow exactly the same arguments as we did for $C(\xi)$. Let us use the symbols $\mathfrak{F}(I_j)$, $\hat{\mathfrak{F}}_p(I_j)$, and $\mathfrak{F}_p(I_j)$ to denote the Hilbert spaces corresponding to $\mathfrak{F}$, $\hat{\mathfrak{F}}_p$, and $\mathfrak{F}_p$ defined for $C(\xi)$. Then we have

$$\mathfrak{F} = \prod_{j=1}^{n} \otimes^* \mathfrak{F}(I_j) \tag{5.21}$$

by the formula (5.19). We can also prove

$$\prod_{j=1}^{n} \otimes^* \mathfrak{F}(I_j) \cong \prod_{j=1}^{n} \otimes \mathfrak{F}(I_j), \qquad \text{(isomorphic)} \tag{5.22}$$

by J. von Neumann's theory [10].

Let $\mathfrak{B}(I_j)$ be the smallest Borel field generated by sets of the form

$$\{x; \langle x, \xi\rangle \in B\} \ \xi \in \mathcal{K}(\bar{I}),\ B \text{ is a one-dimensional Borel set}, \tag{5.23}$$

and let $L_2(I_j)$ be the Hilbert space defined by

$$L_2(I_j) = \{\varphi; \varphi \in L_2, \varphi \text{ is } \mathfrak{B}(I_j)\text{-measurable}\}. \tag{5.24}$$

Then by (5.21),

$$L_2 = \prod_{j=1}^{n} \otimes^* L_2(I_j). \tag{5.25}$$

Because of the particular form of $C(\xi)$, we can prove that

$$\underset{q\to\infty}{\text{l.i.m.}}\ \langle x, \xi_q\rangle \tag{5.26}$$

exists if ξ_q tends to χ_{I_j} in $L_2(R^1)$ as $q \to \infty$. We denote the above limit by $\langle x, \chi_{I_j}\rangle$.

We are now in a position to define the multiple Wiener integral of K. Itô. Let $F(t_1, \cdots, t_p)$ be a special elementary function (see K. Itô [5], p. 160) defined as follows:

$$F(t_1, \cdots, t_p) = \begin{cases} a_{i_1,\cdots,i_p}, & \text{for } (t_1, \cdots, t_p) \in T_{i_1} \times \cdots \times T_{i_p}, \\ 0, & \text{otherwise}, \end{cases} \tag{5.27}$$

where the T_i's are mutually disjoint finite intervals. For such F, $I_p(x; F)$ is defined by

$$I_p(x; F) = \sum_{i_1,\cdots,i_p} a_{i_1,\cdots,i_p} \prod_{j=1}^{p} \langle x, \chi_{T_{i_j}}\rangle. \tag{5.28}$$

This function satisfies the following properties (5.29)–(5.32): for any two special elementary functions F and G,

$$I_p(x; F + G) = I_p(x; F) + I_p(x; G), \tag{5.29}$$

$$I_p(x; F) = I_p(x; \hat{F}), \tag{5.30}$$

where $\hat{F}$ is the symmetrization of F;

$$I_p(x; F) \in L_2 \quad \text{for any } p \text{ and any special elementary function } F, \tag{5.31}$$

and

$$\begin{aligned} \langle\langle I_p(x; F), I_p(x; G)\rangle\rangle &= p!(\hat{F}, \hat{G})_{L_2(R^p)}, \\ \langle\langle I_p(x; F), I_q(x; G)\rangle\rangle &= 0, \qquad \text{if } p \neq q. \end{aligned} \tag{5.32}$$

The map I_p can be extended to a bounded linear operator from $L_2(R^p)$ to L_2, which will be denoted by the same symbol I_p. The integral $I_p(x; F)$ is called *the multiple Wiener integral.* It is essentially the same as that of K. Itô except that we can consider complex $L_2(R^p)$ functions as integrands.

THEOREM 5.2. *For every $F \in L_2(R^p)$, we have*

$$I_p(x; F) = (i)^p \tau^{-1}\{I_p^*(\cdot\,; F)C(\cdot)\}(x). \tag{5.33}$$

PROOF. If F is a special elementary function, (5.33) is obvious by the definition of I^* and τ. In fact, if F is defined by (5.27),

$$I_p^*(\xi; F)C(\xi) = \sum_{i_1,\cdots,i_p} a_{i_1,\cdots,i_p} \prod_{j=1}^{p} \langle \xi, \chi_{T_{i_j}}\rangle. \tag{5.34}$$

Hence, we have

$$\tau^{-1}\{I_p^*(\cdot\,; F)C(\cdot)\}(x) = \sum_{i_1,\cdots,i_p} a_{i_1,\cdots,i_p} \prod_{j=1}^{p} [\tau^{-1}\{\langle \cdot, \chi_{T_{i_j}}\rangle C(\cdot)\}]. \tag{5.35}$$

Since $\tau^{-1}\{(\langle \cdot, \chi_{T_i}\rangle C(\cdot)\}(x) = (i)^{-1}\langle x, \chi_{T_i}\rangle$, the above formula is equal to

$$(i)^{-p} \sum_{i_1,\cdots,i_p} a_{i_1,\cdots,i_p} \prod_{j=1}^{p} \langle x, \chi_{T_{i_j}}\rangle. \tag{5.36}$$

Such a relation can be extended to the case of general F.

S. Kakutani [7] also gave a direct sum decomposition of L_2 using the addition formula for Hermite polynomials. It is known that Kakutani's decomposition is the same as that obtained by using multiple Wiener integrals. Conversely, this addition formula can be illustrated by using the decomposition of $\mathfrak{F}$. This was shown by N. Kôno (private communication) in the following way.

Let I, I_1, and I_2 be finite intervals such that $I = I_1 + I_2$; then

$$\frac{1}{n!}\langle \cdot, \chi_I\rangle^n C(\cdot) = \sum_{k=0}^{n} \frac{1}{k!}\langle \cdot, \chi_{I_1}\rangle^k C(\cdot\chi_{I_1}) \frac{1}{(n-k)!}\langle \cdot, \chi_{I_2}\rangle^{n-k} C(\cdot\chi_{I_2}) C(\cdot\chi_{I^c}). \tag{5.37}$$

Noting that

$$\begin{aligned} &\frac{1}{k!}\langle \cdot, \chi_{I_j}\rangle^k C(\cdot\chi_{I_j}) \in \mathfrak{F}(I_j), \qquad j = 1, 2, \\ &C(\cdot\chi_{I^c}) \in \mathfrak{F}(I^c) \end{aligned} \tag{5.38}$$

and that (5.25) holds, we have

$$\begin{aligned} \tau^{-1}&\left\{\frac{1}{k!}\langle \cdot, \chi_{I_1}\rangle^k C(\cdot\chi_{I_1}) \frac{1}{(n-k)!}\langle \cdot, \chi_{I_2}\rangle^{n-k} C(\cdot\chi_{I_2}) C(\cdot\chi_{I^c})\right\}(x) \\ &= \tau_{I_1}^{-1}\left\{\frac{1}{k!}\langle \cdot, \chi_{I_1}\rangle^k C(\cdot\chi_{I_1})\right\}(x) \\ &\quad \cdot \tau_{I_2}^{-1}\left\{\frac{1}{(n-k)!}\langle \cdot, \chi_{I_2}\rangle^{n-k} C(\cdot\chi_{I_2})\right\}(x) \cdot \tau_{I^c}^{-1}\{C(\cdot\chi_{I^c})\}(x), \end{aligned} \tag{5.39}$$

where τ_I denotes the mapping from $L_2(I)$ to $\mathfrak{F}(I)$ which is similar to τ. Here each factor of the right-hand side is expressed in the form

$$\tau_{I_1}^{-1}\left\{\frac{1}{k!}\langle \cdot, \chi_{I_1}\rangle^k C(\cdot\chi_{I_1})\right\}(x) = (i)^{-k} H_k(\langle x, \chi_{I_1}\rangle; |I_1|) \in L_2(I_1), \tag{5.40}$$

$$\begin{aligned} \tau_{I_2}^{-1}&\left\{\frac{1}{(n-k)!}\langle \cdot, \chi_{I_2}\rangle^{n-k} C(\cdot\chi_{I_2})\right\}(x) \\ &= (i)^{-n+k} H_{n-k}(\langle x, \chi_{I_2}\rangle; |I_2|) \in L_2(I_2), \end{aligned} \tag{5.41}$$

$$\tau_{I^c}^{-1}\{C(\cdot\chi_{I^c})\}(x) = 1 \in L_2(I^c), \tag{5.42}$$

where $|I|$ denotes the length of the interval I. On the other hand, since

$$\tau^{-1}\left(\frac{1}{n!}\langle\cdot,\chi_I\rangle^n C(\cdot)\right)(x) = H_n(\langle x,\chi_I\rangle;|I|), \tag{5.43}$$

we get

$$H_n(\langle x,\chi_I\rangle;|I|) = \sum_{k=0}^{n} H_k(\langle x,\chi_{I_1}\rangle;|I_1|)H_{n-k}(\langle x,\chi_{I_2}\rangle;|I_2|). \tag{5.44}$$

Therefore,

$$H_n(x+y;|I_1|+|I_2|) = \sum_{k=0}^{n} H_k(x;|I_1|)H_{n-k}(y;|I_2|) \tag{5.45}$$

for almost all $(x, y) \in R^2$ with respect to the Gaussian measure,

$$\frac{1}{\sqrt{2\pi|I_1|\,|I_2|}}\exp\left[-\frac{x^2}{2|I_1|}+\frac{y^2}{2|I_2|}\right]dx\,dy. \tag{5.46}$$

Since $H_n(x;\sigma^2)$ is a continuous function of x, (5.45) is true for all $(x, y) \in R^2$. Indeed, (5.45) is the addition formula obtained by S. Kakutani [7].

Let us further note that N. Kōno has shown that (5.45) can also be proved by using the *Gauss transform* defined by

$$\tilde{\varphi}(y) = \int_{S^*}\varphi(x+iy)\mu(dx), \qquad \varphi \in L_2(S^*,\mu), \qquad y \in S^*. \tag{5.47}$$

This transformation is well-defined for polynomials. Since the transformation is bounded and linear, and since polynomials form a dense set in $L_2(S^*,\mu)$, we can extend (5.47) to all of $L_2(S^*,\mu)$.

Let $\check{\varphi}$ be the Gauss inverse transform of φ. We can then introduce an operation $\circ$ from $L_2(S^*,\mu)\times L_2(S^*,\mu)$ to $L_2(S^*,\mu)$:

$$(\varphi\circ\psi)(y) = \widetilde{(\check{\varphi}\circ\check{\psi})}(y) \qquad \text{for} \quad \varphi,\psi \in L_2(S^*,\mu) \quad \text{and} \quad y \in S^*. \tag{5.48}$$

By simple computations we can prove the following: if $\varphi(x) = H_n(\langle x,\xi_1\rangle, 1)$ and $\psi(x) = H_m(\langle x,\xi_2\rangle, 1)$,

$$(\varphi\circ\psi)(y) = \begin{cases} c_{n,m}H_{n+m}(\langle y,\xi\rangle, 1) & \text{for} \quad \xi_1 = \xi_2 = \xi, \\ c'_{n,m}\varphi(y)\psi(y) & \text{for} \quad \langle\xi_1,\xi_2\rangle = 0. \end{cases} \tag{5.49}$$

More generally, we can prove that if $\varphi \in L_2^{(n)}$ and $\psi \in L_2^{(m)}$, then

$$\varphi\circ\psi \in L_2^{(n+m)}. \tag{5.50}$$

This operation becomes simpler when it is considered in $\mathfrak{F}$. We shall use the same symbol $\circ$ to express the corresponding operation, namely,

$$f\circ g = \tau((\tau^{-1}f)\circ(\tau^{-1}g)) \qquad \text{for} \quad f, g \in \mathfrak{F}. \tag{5.51}$$

Recalling that the $\mathfrak{F}_n$ appearing in Wiener's direct sum decomposition of $\mathfrak{F}$ is $\tau(L_2^{(n)})$, we have the following proposition.

PROPOSITION 6.1. *The spaces* $\{\mathfrak{F}_n\}_{n=0}^{\infty}$ *form a graded ring with respect to the operation* $\circ$. (*For definition, see Zariski and Samuel* [17], p. 150).

6. Poisson white noise

In this section we shall deal with Poisson white noise, which is another typical stationary process with independent values at every point. Our goal is to find the explicit expressions for $\mathfrak{F}$, $\hat{\mathfrak{F}}_p$, and $\mathfrak{F}_p$ and also to look for relations between the multiple Wiener integrals and the Charier polynomials. Since Poisson white noise enjoys many properties similar to those of Gaussian white noise, we shall sometimes skip the detailed proofs except when there is an interesting difference from Gaussian case.

The characteristic functional of Poisson white noise **P** is given by

$$C(\xi) = \exp\left\{\int_{-\infty}^{\infty} (e^{i\xi(t)} - 1 - i\xi(t))\, dt\right\}, \qquad \xi \in \mathcal{S}, \tag{6.1}$$

that is, $\alpha(x) = (e^{ix} - 1 - ix)$ in the expression (4.1). Hence, $K_p(\xi, \eta)$ is expressed in the form

$$K_p(\xi, \eta) = \frac{1}{p!}\left(\int_{-\infty}^{\infty} P(\xi(t))P(\eta(t))\, dt\right)^p, \qquad \xi, \eta \in \mathcal{S}, \tag{6.2}$$

where $P(x) = e^{ix} - 1$. For $F \in L_2(R^p)$ we define $J_p^*(\xi; F)$ by

$$J_p^*(\xi; F) = \int \overset{\infty}{\underset{-\infty}{\cdots}} \int P(\xi(t_1))\cdots P(\xi(t_p))F(t_1, \cdots, t_p)\, dt_1\cdots dt_p. \tag{6.3}$$

Obviously,

$$J_p^*(\xi; F) = J_p^*(\xi; \hat{F}), \qquad \xi \in \mathcal{S}, \tag{6.4}$$

still holds (cf. (5.30)).

THEOREM 6.1. *For Poisson white noise, we have*

(i) $\hat{\mathfrak{F}}_p = \{f(\cdot); f(\xi) = J_p^*(\xi; F), F \in L_2(R^p)\}$

(ii) $(J_p^*(\cdot\,; F), J_p^*(\cdot\,; G))_{\hat{\mathfrak{F}}_p} = p! \int \overset{\infty}{\underset{-\infty}{\cdots}} \int \hat{F}(t_1, \cdots, t_p)\overline{\hat{G}(t_1, \cdots, t_p)}\, dt_1\cdots dt_p$

for any $F, G \in L_2(R^p)$.

PROOF. The proof is nearly the same as that of theorem 5.1. Thus, we shall just point out the necessary changes. The spaces $\tilde{L}_2(R^p)$ and $\tilde{\mathfrak{F}}_p$ have to be defined in the following way:

$$\tilde{L}_2(R^p) = \left\{F; F(t_1, \cdots, t_p) = \frac{1}{p!}\sum_{k=1}^{n} a_k P(\xi_k(t_1))\cdots P(\xi_k(t_p));\ a_k \text{ complex}, \xi_k \in \mathcal{S}\right\}, \tag{6.5}$$
$$\tilde{\mathfrak{F}}_p = \{f(\cdot); f(\xi) = J_p^*(\xi; F), f \in \tilde{L}_2(R^p)\}.$$

If we prove that $\tilde{L}_2(R^p)$ is dense in $\hat{L}_2(R^p)$, then the rest of the proof is exactly the same as that of theorem 5.1. To do this, note that the totality of all linear combinations of functions such as $\chi_{T_1}(t_1)\cdots\chi_{T_p}(t_p)$ with disjoint finite intervals $\{T_j\}, {}_{j=1}^{p}$ is dense in $L_2(R^p)$, and also note that the fact that

$$\int \overset{\infty}{\underset{-\infty}{\cdots}} \int \sum_k a_k P(\xi_k(t_1))\cdots P(\xi_k(t_1))\chi_{T_1}(t_1)\cdots\chi_{T_p}(t_p)\, dt_1\cdots dt_p = 0 \tag{6.6}$$

for any choice of $\{a_k\}$ and ξ_k's in $\mathcal{S}$ implies that

$$\chi_{T_1}(t_1)\cdots\chi_{T_p}(t_p) = 0, \qquad \text{a.e.} \tag{6.7}$$

We can therefore prove that $\tilde{L}_2(R^p)$ is dense in $\hat{L}_2(R^p)$.

The direct product decomposition of $\mathfrak{F}$ and L_2 is the same as in section 5. For any finite partition $\{I_j\}_{j=1}^n$ of $\mathbf{T}$,

$$C(\xi) = \prod_{j=1}^{n} C(\xi\chi_{I_j}), \qquad \xi \in \mathcal{S}, \tag{6.8}$$

still holds. Therefore we have, using the same notation,

$$\mathfrak{F} = \prod_{j=1}^{n} \otimes^* \mathfrak{F}(I_j), \tag{6.9}$$

$$L_2 = \prod_{j=1}^{n} \otimes^* L_2(I_j). \tag{6.10}$$

Moreover, we can define the multiple Wiener integral with respect to Poisson white noise similarly. First note that $\langle x, \chi_I\rangle$ is defined as an element of L_2. If F is a special elementary function given by (5.27), then $J_p(x; F)$ is defined by

$$J_p(x; F) = \sum_{i_1, \cdots, i_p} a_{i_1, \cdots, i_p} \prod_{j=1}^{p} \langle x, \chi_{T_{i_j}}\rangle. \tag{6.11}$$

The map J_p can be extended to a bounded linear operator from $L_2(R^p)$ to L_2 as was done in section 5 (cf. K. Itô [6], section 3).

THEOREM 6.2. *For every* $F \in L_2(R^p)$,

$$J_p(x; F) = \tau^{-1}\{J_p^*(\cdot\,; F)C(\cdot)\}(x). \tag{6.12}$$

PROOF. This proof is also the same as that of theorem 5.2, except for the following relation:

$$\tau(\langle x, \chi_{T_1}\rangle\cdots\langle x, \chi_{T_p}\rangle)(\xi) = \prod_{j=1}^{p} \int_{T_j} P(\xi(t_j))\, dt_j C(\xi). \tag{6.13}$$

From the last theorem we can show that

$$\mathfrak{F} = \sum_{p=0}^{\infty} \oplus \mathfrak{F}_p \tag{6.14}$$

is nothing but Wiener's direct sum decomposition. This fact can also be proved using a certain addition formula for a one-parameter family of generalized Charier polynomials: let $\nu(x, c)$ be given by

$$\nu(x, c) = \frac{c^{x+c}}{\Gamma(x + c + 1)} e^{-c}, \qquad x = -c, 1 - c, 2 - c, \cdots, \tag{6.15}$$

and let

$$P_n(x, c) = n! P_n^*(x, c) = (-c)^n(\nu(x, c))^{-1}\Delta_x^n \nu(x - n, c), \tag{6.16}$$

where Δ_x^n is a difference operator of order n; then the formula is

$$P_n^*(x + y, c) = \sum_{k=0}^{n} P_k^*(x, c_1)P_{n-k}^*(y, c_2), \qquad c_1, c_2 > 0,\ c = c_1 + c_2. \tag{6.17}$$

7. Concluding remarks

The theorems given in sections 5 and 6 extend to generalized white noise. Furthermore, we shall show that a sequence of independent identically distributed random variables can be dealt with in our scheme. We do not take up detailed discussions but summarize some of their properties.

7.1. *Generalized white noise.* We now discuss the stationary process with the characteristic functional

$$
\begin{aligned}
C(\xi) &= \exp\left\{\int_{-\infty}^{\infty} \alpha(\xi(t))\, dt\right\}, \qquad \xi \in \mathcal{S},\\
\alpha(x) &= \int_{-\infty}^{\infty} \left(e^{ixu} - 1 - \frac{ixu}{1+u^2}\right)\frac{1+u^2}{u^2}\, d\beta(u),
\end{aligned} \tag{7.1}
$$

which is the one obtained from (4.1) by eliminating the Gaussian part $-(\sigma^2/2)x^2$. Then $K_p(\xi, \eta)$ is expressible as follows:

$$
K_p(\xi, \eta) = \frac{1}{p!}\left(\iint_{-\infty}^{\infty} P(\xi(t)u)P(\eta(t)u)\, d\nu(t, u)\right) \tag{7.2}
$$

where $d\nu(t, u) = dt\, d\beta(u)$. We introduce the following notations:

$$
\begin{aligned}
&D_p = R^{2p}, \qquad dm = u^2\, d\nu, \qquad dm_p = dm \times \cdots \times dm \ (p \text{ times}),\\
&L_2(D_p; m_p) = \left\{F; F \text{ is square summable with respect to } \frac{1}{p!}\, dm_p\right\},\\
&\hat{L}_2(D_p; m_p) = \{F; F \in L_2(D_p; m_p), F((t_1, u_1), \cdots, (t_p, u_p))\\
&\qquad = F((t_{\pi(1)}, u_{\pi(1)}), \cdots, (t_{\pi(p)}, u_{\pi(p)})) \text{ for any permutation } \pi\}.
\end{aligned} \tag{7.3}
$$

Define $M_p^*(\xi; F)$ by

$$
\begin{aligned}
M_p^*(\xi; F) = \int \overset{\infty}{\underset{-\infty}{\cdots}} \int & P(\xi(t_1)u_1)\cdots P(\xi(t_p)u_p)F((t_1, u_1), \cdots, (t_p, u_p))\\
&\times u_1\cdots u_p\, d\nu(t_1, u_1)\cdots d\nu(t_p, u_p)
\end{aligned} \tag{7.4}
$$

using the same technique as in sections 5 and 6. Then we have

$$
M_p^*(\xi; F) = M_p^*(\xi; \hat{F}), \qquad \xi \in \mathcal{S}, \tag{7.5}
$$

where

$$
\hat{F}((t_1, u_1), \cdots, (t_p, u_p)) = \frac{1}{p!}\sum_{\pi} F((t_{\pi(1)}, u_{\pi(1)}), \cdots, (t_{\pi(p)}, u_{\pi(p)})). \tag{7.6}
$$

For generalized white noise with characteristic functional (7.1), we have the following results:

(i)
$$
\begin{aligned}
\mathfrak{F}_p &= \{f(\cdot); f(\xi) = M_p^*(\xi; F), F \in L_2(D_p, m_p)\}\\
M_p^*(\xi; F) &= M_p^*(\xi; \hat{F});
\end{aligned}
$$

(ii)
$$
\begin{aligned}
(M_p^*(\cdot\,; F), M_p^*(\cdot\,; G))_{\mathfrak{F}_p} = p! \iint_{-\infty}^{\infty} & \hat{F}((t_1, u_1), \cdots, (t_p, u_p))\\
&\times \overline{\hat{G}((t_1, u_1), \cdots, (t_p, u_p))}\, dm_p((t_1, u_1), \cdots, (t_p, u_p)).
\end{aligned}
$$

Let us emphasize some of the important differences from Gaussian or Poisson white noise. First we cannot expect that the decomposition $\mathfrak{F} = \sum_{p=0}^{\infty} \oplus \mathfrak{F}_p$, where $\mathfrak{F}_p$ corresponds to the $\mathfrak{F}_p$ appearing in (i), will be the Wiener's direct sum decomposition. However, $\tau^{-1}(\mathfrak{F}_n)$ coincides with the multiple Wiener integral introduced by K. Itô [6]. The next remarkable thing concerns the direct product decomposition.

Let $\{I_j\}_{j=1}^{n}$ and $\{J_k\}_{k=1}^{m}$ be finite partitions of $\mathbf{T}$ and R^1, respectively, and define

$$C(\xi; I_j \times J_k) = \exp\left\{\int_{I_j}\int_{J_k}\left(e^{i\xi(t)u} - 1 - \frac{i\xi(t)u}{1+u^2}\right)\frac{1+u^2}{u^2}\,d\beta(u)\,dt\right\}. \tag{7.7}$$

Recall $C(\xi; I_j \times J_k)$ defines the subspaces $\mathfrak{F}(I_j \times J_k)$, $\mathfrak{F}_p(I_j \times J_k)$ and $\mathfrak{F}_p(I_j \times J_k)$. Since

$$C(\xi) = \prod_{j=1}^{n}\prod_{k=1}^{m} C(\xi; I_j \times J_k), \tag{7.8}$$

we have

$$\mathfrak{F} = \prod_{j=1}^{n}\prod_{k=1}^{m} \otimes^{*}\, \mathfrak{F}(I_j \times J_k). \tag{7.9}$$

Now we note a connection with K. Itô's multiple Wiener integral. It seems to be difficult to start in the same way as in sections 5 and 6 by introducing $\langle x, \chi_I\rangle$ in L_2. However, if we consider $\mathfrak{F}$, we can proceed by defining for finite intervals I and J,

$$M^{*}(\xi; I \times J) = \int_I \int_J P(\xi(t)u)u\, d\nu(t, u), \tag{7.10}$$

$$M(x; I \times J) = \tau^{-1}(M^{*}(\cdot\,; I \times J)C(\cdot)). \tag{7.11}$$

$M(x; \cdot)$ can be considered as a random measure as in K. Itô ([6], section 3) and using it, we can define the multiple Wiener integral $I_p(F)$. Let us denote it by $M_p(x; F)$. Then, for every $F \in L_2(D_p, m_p)$ we can easily prove that

$$\tau^{-1}(M_p^{*}(\cdot\,; F)C(\cdot)) = M_p(x; F). \tag{7.12}$$

Rather than discuss the group $\mathbf{G(P)}$ in detail, we shall just give a simple example.

EXAMPLE. Consider the case where

$$\alpha(x) = |x|^{\theta}, \qquad 0 < \theta < 2. \tag{7.13}$$

This corresponds to the symmetric stable distribution with exponent θ. A transformation g on E belongs to $\mathbf{G(P)}$, that is,

$$C(g\xi) = C(\xi), \qquad \xi \in E \tag{7.14}$$

if and only if

$$\int_{-\infty}^{\infty} |(g\xi)(t)|^{\theta}\, dt = \int_{-\infty}^{\infty} |\xi(t)|^{\theta}\, dt. \tag{7.15}$$

Then g,

$$(g\xi)(t) = c\xi(c^{\theta}t), \qquad c > 0, \tag{7.16}$$

is an example satisfying (7.15).

7.2. *A sequence of independent random variables.* Consider a stationary process $\mathbf{P} = (E^*, \mu, \{T_t\})$ with independent values at every point, where $E = \mathcal{s} = (\xi = \{\xi^k\}_{k=-\infty}^{\infty};\ \xi^k$ real) is the space of rapidly decreasing sequences and $\mathbf{T}$ is the additive group of integers. A system of independent identically distributed random variables arises in the following way.

Take a sequence $\{\xi_n\}_{n=-\infty}^{\infty}$ of sequences,

$$\xi_n = \{\xi_n^k\}_{k=-\infty}^{\infty} \in \mathcal{s},\ \xi_n^k = \delta_{n,k}. \tag{7.17}$$

Since the ξ_n's have disjoint supports, the $\langle x, \xi_n\rangle \equiv X_n(x)$, $-\infty < n < \infty$, are mutually independent.

Further, we have

$$U_t X_n(x) = X_n(T_t x) = X_{n+t}(x). \tag{7.18}$$

In view of the above, $T = T_1$ is called *a Bernoulli automorphism* and $\mathbf{P}$ is called *a stationary process with a Bernoulli automorphism T.*

If $\tilde{C}(z)$ is the characteristic function of $X_n(x)$, that is,

$$\tilde{C}(z) = \int_{\mathcal{s}^*} e^{izX_n(x)}\mu(dx), \tag{7.19}$$

then the characteristic functional C of $\mathbf{P}$ is expressible in the form

$$C(\xi) = \prod_{k=-\infty}^{\infty} \tilde{C}(\xi^k), \qquad \xi = \{\xi^k\}_{k=-\infty}^{\infty} \in \mathcal{s}. \tag{7.20}$$

We can now form the Hilbert space $\mathfrak{F} = \mathfrak{F}(\mathcal{s}, C)$ with reproducing kernel C given by (7.20).

The direct product decomposition and the direct sum decomposition of $\mathfrak{F}$ can be done in section 4. We would like to mention two particular cases of stationary processes with Bernoulli automorphisms.

(a) *The Gaussian case.* Let $\mathbf{P} = (\mathcal{s}^*, \mu, \{T_t\})$ be the stationary process with a Bernoulli automorphism. Suppose that the characteristic functional of P is given by

$$C(\xi) = \exp\{-\tfrac{1}{2}\|\xi\|^2\}, \qquad \xi \in \mathcal{s} \tag{7.21}$$

where $\|\xi\|^2 = \sum_{k=-\infty}^{\infty} (\xi^k)^2$. In this case, the subspace $\mathfrak{F}_p$ of $\mathfrak{F}(\mathcal{s}, C)$ turns out to be the following:

$$\mathfrak{F}_p = \Big\{f(\cdot); f(\eta) = \sum_{j_1,\cdots,j_p=-\infty}^{\infty} a(j_1, \cdots, j_p)\eta^{j_1}\cdots\eta^{j_p}C(\eta), \\ \eta = \{\eta^n\}_{n=-\infty}^{\infty} \in \mathcal{s},\ a(j_1, \cdots, j_p) \in \hat{\ell}_2(R^p)\Big\}, \tag{7.22}$$

where $\hat{\ell}_2(R^p)$ is defined by

$$\hat{\ell}_2(R^p) = \Big\{a(j_1, \cdots, j_p); \sum_{j_1,\cdots,j_p=-\infty}^{\infty} |a(j_1, \cdots, j_p)|^2 < \infty,\ a(j_1, \cdots, j_p) \\ \text{is symmetric with respect to } j_k\text{'s}\Big\}. \tag{7.23}$$

If the $f_k(\cdot)$, $k = 1, 2$, in $\mathfrak{F}$ are given by

$$(7.24)\qquad f_k(\eta) = \sum_{j_1,\cdots,j_p=-\infty}^{\infty} a_k(j_1, \cdots, j_p)\eta^{j_1}\cdots\eta^{j_p}C(\eta), \qquad k = 1, 2,$$

then we have the following:

$$(7.25)\qquad (f_1, f_2)_{\mathfrak{F}} = p! \sum_{j_1,\cdots,j_p=-\infty}^{\infty} a_1(j_1, \cdots, j_p)\overline{a_2(j_1, \cdots, j_p)}.$$

Actually $\mathfrak{F} = \sum_{p=0}^{\infty} \oplus \mathfrak{F}_p$ is the Wiener's direct sum decomposition. Thus, the subspace $L_2^{(p)}$ of $L_2(\mathcal{S}^*, \mu)$, corresponding to $\mathfrak{F}_p$, is expressed as

$$(7.26)\qquad L_2^{(p)} = \mathfrak{S}\left\{\prod_{k=1}^{n} H_{p_k}(X_{q_k}(x); 1);\ \{q_k\} \text{ different}, \sum_1^n p_k = p\right\}.$$

(b) *Poisson case.* Consider the stationary process $\mathbf{P} = (\mathcal{S}^*, \mu, \{T_t\})$ whose characteristic functional is

$$(7.27)\qquad C(\xi) = \exp\left\{\sum_{j=-\infty}^{\infty} (e^{i\xi^j} - 1 - i\xi^j)\right\}, \qquad \xi = \{\xi^j\}_{j=-\infty}^{\infty} \in \mathcal{S}.$$

Of course, $\mathbf{P}$ is a stationary process with a Bernoulli automorphism. The interesting thing is that $L_2^{(p)}$ is spanned by elements of the form

$$(7.28)\qquad \sum_{p_1,\cdots,p_n} a_{p_1,\cdots,p_n} \prod_{k=1}^{n} Q_{p_k}(X_{q_k}(x); 1), \sum_{k=1}^{n} p_k = p,$$

where Q_n is the function defined by (3.12). Note that the Q_n's form a complete orthonormal system in $L_2(S_1, d\nu(x, 1))$ (for notation, see (3.9)).

We can also prove that

$$(7.29)\qquad \mathfrak{F}_p = \left\{f(\cdot); f(\eta) = \sum_{j_1,\cdots,j_p=-\infty}^{\infty} a(j_1, \cdots, j_p) \prod_{k=1}^{p} (e^{i\eta^{j_k}} - 1)C(\eta),\right.$$
$$\left. a(j_1, \cdots, j_p) \in \hat{\ell}_2(R^p)\right\}.$$

Although the expression of $f(\cdot)$ in (7.29) is quite different from that in (7.22), we still have the same formula for the inner product, that is, (7.25).

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