RANKING LIMIT PROBLEM

MICHEL LOÈVE
UNIVERSITY OF CALIFORNIA, BERKELEY

1. The problem

Let $(\Omega, \mathcal{A}, P)$ be our probability space and let X, with or without affixes, denote a measurable function [a random variable (r.v.) when finite] on this space. $\mathcal{L}(X)$ will represent the (probability) law of X defined by its distribution function (d.f.) F or its characteristic function (ch. f.) f with the same affixes as X, if any. A law degenerate at a is represented by $\mathcal{L}(a)$; if a is finite, it is the law of a r.v. which reduces to a with probability 1; $\mathcal{L}(\infty)$ represents the law of any measurable function which is infinite with probability 1.

Distribution functions and, more generally, monotone functions, say, h on $R = (-\infty, +\infty)$, will be continuous from the left: $h(x-0) = h(x)$, $x \in R$. A sequence h_n of monotone functions, say, nondecreasing ones, converges *weakly* to h on R, and we write $h_n \xrightarrow{w} h$, if $h_n \to h$ on the continuity set of h (it suffices that $h_n \to h$ on a set everywhere dense in R); h_n converges *completely* to h, and we write $h_n \xrightarrow{c} h$, if, moreover, $h_n(\mp\infty) \to h(\mp\infty)$. A sequence of laws $\mathcal{L}(X_n)$ converges weakly or completely to a law $\mathcal{L}(X)$ if $F_n \to F$ weakly or completely, respectively.

Convention I. Throughout this paper, and unless otherwise stated,

(a) To any probability p we make correspond the probability $q = 1 - p$ with the same affixes, if any.

(b) $n = 1, 2, \cdots$; $k = 1, 2, \cdots, k_n$, with $k_n \to \infty$; all limits are taken for $n \to \infty$.

(c) X_{nk} represent r.v.'s independent in k for every fixed n. For every $\omega \in \Omega$, the nondecreasingly ranked numbers $X_{nk}(\omega)$ are denoted by

$$X^*_{n1}(\omega) \leqq X^*_{n2}(\omega) \leqq \cdots \leqq X^*_{nk_n}(\omega); \tag{1}$$

they are values of nondecreasingly *ranked r.v.'s* X^*_{nr}, $r = 1, 2, \cdots, k_n$, of *rank* r and *relative rank* $\rho = r/k_n$ (with the same affixes as r, if any), corresponding to the r.v.'s X_{nk}. The nonincreasingly ranked r.v.'s are denoted by ${}^*X_{ns}$, $s = 1, 2, \cdots, k_n$, of *end rank* s, so that ${}^*X_{ns} = X^*_{n,\, k_n+1-s}$.

Let the X_{nk} be uniformly asymptotically negligible, that is, $\mathcal{L}(X_{nk}) \to \mathcal{L}(0)$ uniformly in k. We know that *if* $\mathcal{L}\left(\sum_k X_{nk}\right) \xrightarrow{c} \mathcal{L}(X)$, *then* $\mathcal{L}(X)$ *is infinitely decomposable*. We recall that a law $\mathcal{L}(X)$ is infinitely decomposable, that is, $f^{1/n}$ is a ch. f. for every n if, and only if, for every $u \in R$

$$\log f(u) = iau - \frac{b^2}{2}u^2 + \int_{-\infty}^{-0}\left(e^{iux} - 1 - \frac{iux}{1+x^2}\right)dL(x) + \int_{+0}^{+\infty}\left(e^{iux} - 1 - \frac{iux}{1+x^2}\right)dM(x), \tag{2}$$

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where $a, b \in R$, and the nondecreasing functions L on $(-\infty, 0)$ and M on $(0, +\infty)$ satisfy the conditions

$$(3) \qquad L(-\infty) = M(+\infty) = 0, \qquad \int_{-1}^{-0} x^2 dL(x) + \int_{+0}^{+1} x^2 dM(x) < \infty .$$

The pair (L, M) is the "P. Lévy function" of the infinitely decomposable law $\mathcal{L}(X)$.

One of the most awaited and beautiful results of probability theory is the normal convergence theorem which, in P. Lévy's form, says that *if* $\mathcal{L}\left(\sum_k X_{nk}\right) \xrightarrow{c} \mathcal{L}(X)$, *then* $\mathcal{L}(X)$ *is normal if, and only if,* $\mathcal{L}(\max_k |X_{nk}|) \to \mathcal{L}(0)$ or, equivalently, $\mathcal{L}(X^*_{n1}) \to \mathcal{L}(0)$ and $\mathcal{L}(^*X_{n1}) \to \mathcal{L}(0)$. Since the infinitely decomposable law $\mathcal{L}(X)$ is normal if, and only if, its P. Lévy function (L, M) vanishes identically, one may expect some general relationship between the asymptotic behavior of $\mathcal{L}\left(\sum_k X_{nk}\right)$ and of $\mathcal{L}(X^*_{n1})$ and $\mathcal{L}(^*X_{n1})$. In fact, the foregoing result is a particular case of the following immediate consequence of the Central Extrema Criterion (see p. 315 in [3]).

If $\mathcal{L}\left(\sum_k X_{nk}\right) \xrightarrow{c} \mathcal{L}(X)$, *then* $\mathcal{L}(X)$ *has P. Lévy function* (L, M) *if, and only if,*

$$(4) \qquad \mathcal{L}(X^*_{n1}) \to \mathcal{L}(X^*), \qquad \mathcal{L}(^*X_{n1}) \to \mathcal{L}(^*X)$$

with

$$(5) \qquad F^*(x) = 1 - e^{-L(x)} \text{ or } 1, \qquad ^*F(x) = 0 \text{ or } e^{M(x)},$$

according as $x < 0$ *or* $x > 0$.

This proposition connects the problem of limit laws of sums $\sum_k X_{nk}$—the Central Limit Problem, with that of limit laws of ranked summands—the Ranking Limit Problem. There exists a huge literature about the last problem (see, for example, the bibliography in [5]). However, the results are not suited to the general Central Limit Problem, for they appear to be confined essentially to the normed and identically distributed case: $X_{nk} = X_k/b_n - a_n$, $k_n = n$, *the r.v.'s* X_k *having a common and fixed d.f.* F, so that $F_{nk}(x) = F(a_n + b_n x)$. Thus, the situation is similar to that of the Central Limit Problem in 1925, when it was confined to the same case.

The foregoing connections and similarities lead to the research of limit laws and of convergence conditions for general ranked r.v.'s. However, as stated, the problem is so wide that it has no content. For, any law $\mathcal{L}(X)$ of a r.v. may be such a limit law. For example, take, for every n, $\mathcal{L}(X_{n1}) = \mathcal{L}(X)$ and $\mathcal{L}(X_{nk}) = \mathcal{L}(n)$ for $k > 1$; then clearly, $\mathcal{L}(X^*_{n1}) \to \mathcal{L}(X)$. This indicates the need for some "natural" restriction. Uniform asymptotic negligibility seems to be a reasonable one. Yet, a look at its "raison d'être" shows it to be unnaturally restrictive. It has been introduced into the problem of limit laws of sums for a double purpose. On the one hand, these limit laws are to be essentially characterized as being those of an infinitely increasing number of summands so that no one of them may play a privileged role in the formation of the limit laws. This is ensured by their uniform asymptotic behavior. On the other hand, these limit laws are to be laws of r.v.'s, that is, no positive probability may escape to infinity. Together with the preceding requirement this leads to uniform asymptotic negligibility. However, in the ranking problem the no escape requirement does not lead to asymptotic degeneracy of the X_{nk}. Furthermore, we shall find that in this problem it is more convenient not to impose *a priori* that the limit laws be those of r.v.'s.

This leads to *uniformly asymptotically distributed r.v.'s* X_{nk}: there exist d.f.'s F_n such that $F_{nk} - F_n \to 0$ uniformly in k; observe that we may then take $F_n = (1/k_n)\sum_k F_{nk}$. Uniform asymptotic negligibility is obtained by adding the condition that $F_n(x) \to 0$ or 1 according as $x < 0$ or $x > 0$. Finally, we may state our problem as follows.

Ranking Limit Problem.

(H) *Let* X^*_{nr} *be ranked r.v.'s corresponding to independent and uniformly asymptotically distributed r.v.'s* X_{nk}.

(i) *Find the class of all possible limit laws for the* X^*_{nr}.

(ii) *Find conditions for convergence to any specified element of the class.*

Convention II. Throughout this paper, we set

$$L_n = \sum_k F_{nk}, \qquad M_n = \sum_k \{F_{nk} - 1\}; \tag{6}$$

$$g_{r_n} = \frac{r_n - \sum_k F_{nk}}{\sqrt{\sum_k F_{nk}(1 - F_{nk})}}, \qquad X^*_{r_n} = X^*_{nr_n}, \qquad {}^*X_{s_n} = {}^*X_{ns_n}; \tag{7}$$

$$I_n = \sum_k I_{nk}, \qquad I^c_n = \sum_k I^c_{nk}, \qquad \bar{I}_n = \frac{I_n - EI_n}{\sigma I_n}, \tag{8}$$

with

$$I_{nk}(x) = I_{[X_{nk} < x]}, \qquad I^c_{nk}(x) = I_{[X_{nk} \geqq x]}, \tag{9}$$

where the right-hand sides are indicators of the subscript events.

The obvious relation

$$[X^*_{nr} < x] = [I_n(x) \geqq r] \tag{10}$$

will play a basic role in this study (observe that it shows that the functions X^*_{nr} on Ω as defined in convention I are r.v.'s). This relation defines a type of inversion for random functions (r.f.'s) I_n on R. An investigation per se of such inversions for large classes of r.f.'s will be presented elsewhere. The present inversion problem is characterized by the fact that the r.f.'s I_n are sums of independent indicator functions I_{nk} which are nondecreasing in $x \in R$ and whose (random) values are uniformly asymptotically distributed for every fixed x.

2. The individual ranking limit problem

The limiting behavior of individual ranked r.v.'s of fixed or variable ranks will be deduced from the corresponding behavior of $P\{X^*_{nr} < x\} = P\{I_n(x) \geqq r\}$ for fixed values x of the argument. We require the following elementary lemma.

LEMMA 1. *Let* $J_n = \sum_k J_{nk}$ *be sums of independent indicators for every fixed n. Set* $p_{nk} = EJ_{nk} = P\{J_{nk} = 1\}$, *so that*

$$EJ_n = \sum_k p_{nk}, \qquad \sigma^2 J_n = \sum_k p_{nk} q_{nk}, \tag{11}$$

and set $\bar{J}_n = (J_n - EJ_n)/\sigma J_n$.

If $\sigma^2 J_n \to \infty$, *then* $\mathcal{L}[(J_n - EJ_n)/\sigma J_n] \to \mathcal{N}(0, 1)$.

If $p_{nk} \to p$ *uniformly in* k, *then*

(i) $EJ_n \to \lambda \Leftrightarrow \mathcal{L}(J_n) \to \mathcal{L}(X)$; *more precisely, either*

$$EJ_n \to \lambda < \infty \Leftrightarrow \mathcal{L}(J_n) \to \mathcal{P}(\lambda)\,, \tag{12}$$

and $p = 0$, *or*

$$EJ_n \to \infty \Leftrightarrow \mathcal{L}(J_n) \to \mathcal{L}(\infty)\,. \tag{13}$$

(ii) $\sigma^2 J_n \to \sigma^2 \Leftrightarrow \mathcal{L}(J_n) \to \mathcal{L}(X')$; *more precisely,*

$$\sigma^2 J_n \to \infty \Leftrightarrow \mathcal{L}(J_n) \to \mathcal{N}(0, 1)\,, \tag{14}$$

$$\sigma^2 J_n \to \sigma^2 < \infty \Leftrightarrow \mathcal{L}(J_n) \to \mathcal{L}(X') \tag{15}$$

of Poisson or degenerate (at $+1$, -1, *or infinity) type.*

PROOF. (a) The two propositions below are immediate consequences of, say, the Central Limit Theorem, but the direct proof is so simple that we give it.

If $\sigma^2 J_n \to \infty$, then, for n sufficiently large,

$$\begin{aligned} \log E e^{iu\bar{J}_n} &= \log\left[1 + \left(p_{nk} e^{iq_{nk}/\sigma J_{nk}} + q_{nk} e^{-ip_{nk}/\sigma J_{nk}} - 1\right)\right] \\ &= -\left[1 + o(1)\right] \frac{p_{nk} q_{nk}}{\sigma^2 J_n} \cdot \frac{u^2}{2}\,, \end{aligned} \tag{16}$$

where $o(1) = O(1/\sigma J_n) \to 0$ uniformly in k, so that $\log E \exp(iu\bar{J}_n) = -[1 + o(1)]u^2/2$ and, hence, $\mathcal{L}(J_n) \to \mathcal{N}(0, 1)$.

If $p_{nk} \to 0$ uniformly in k, then, for n sufficiently large,

$$\log E e^{iuJ_{nk}} = \log\left[1 + p_{nk}(e^{iu} - 1)\right] = \left[1 + o(1)\right] p_{nk}(e^{iu} - 1) \tag{17}$$

where $o(1) = O(p_{nk}) \to 0$ uniformly in k, so that $\log E \exp(iuJ_n) = [1 + o(1)]EJ_n [\exp(iu) - 1]$ and, hence, $EJ_n \to \lambda < \infty \Leftrightarrow \mathcal{L}(J_n) \to \mathcal{P}(\lambda)$.

(b) Let $p_{nk} \to p$ uniformly in k. If $p = 1$, then $EJ_n \sim k_n \to \infty$ and, for every fixed integer m,

$$P\{J_n \geqq m\} \geqq \prod_{h=1}^{m} p_{nh} \to 1\,, \tag{18}$$

so that $\mathcal{L}(J_n) \to \mathcal{L}(\infty)$. If $p < 1$ and $EJ_n \to \infty$ (necessarily if $0 < p < 1$), then $\sum_k p_{nk} q_{nk} \sim q \sum_k p_{nk} \to \infty$, so that for any given pair $a, b \in R$,

$$\frac{a - EJ_n}{\sigma J_n} \sim -\sqrt{qEJ_n} \to \infty\,, \tag{19}$$

and, for n sufficiently large,

$$P\{J_n \geqq a\} = P\left\{\bar{J}_n \geqq \frac{a - EJ_n}{\sigma J_n}\right\} \geqq P\{\bar{J}_n \geqq b\} \to \frac{1}{\sqrt{2\pi}} \int_b^\infty e^{-t^2/2} dt\,. \tag{20}$$

By letting $b \to \infty$, it follows that $\mathcal{L}(J_n) \to \mathcal{L}(\infty)$.

What precedes, together with (a), proves assertion (i) provided we can show that $\mathcal{L}(J_n) \to \mathcal{L}(X)$ implies $EJ_n \to \lambda$. This is proved upon taking sequences $\{m\}$ and $\{m'\}$ of integers such that $\lambda = \lim_m EJ_m = \liminf_n EJ_n$ and $\lambda' = \lim_{m'} EJ_{m'} = \limsup EJ_{m'}$. For, either $\lambda = \infty$ so that $\lambda' = \infty$ and hence $EJ_n \to \infty$, or $\lambda < \infty$ while $\lambda' \leqq \infty$ so that $\mathcal{L}(X) = \mathcal{P}(\lambda)$ and also $\mathcal{L}(X) = \mathcal{P}(\lambda')$ or $\mathcal{L}(\infty)$ leading to a contradiction unless $\lambda' = \lambda$ and hence $EJ_n \to \lambda < \infty$. Thus, the proof of assertion (i) is complete.

(c) Let $p_{nk} \to p$ uniformly in k and let $\sigma^2 J_n \to \sigma^2 < \infty$, so that $p = 0$ or 1. If $p = 0$, we have $EJ_n \sim \sigma^2 J_n \to \sigma^2 < \infty$. Then, either $\sigma^2 > 0$ and hence, by (i), $\mathcal{L}(X')$ is of Poisson type, or $\sigma^2 = 0$ and, given $\epsilon > 0$, for n sufficiently large

$$P\{J_n < -1 - \epsilon\} = P\{J_n < 0\} = 0 \tag{21}$$

while

$$P\{J_n < -1 + \epsilon\} = P\{J_n = 0\} = \prod_k (1 - p_{nk}) = e^{-[1+o(1)]EJ_n} \to 1 \tag{22}$$

and hence $\mathcal{L}(X') = \mathcal{L}(-1)$. Similarly if $p = 1$. Proceeding as above or reducing to what precedes upon replacing J_{nk} by $1 - J_{nk}$, we find that $\mathcal{L}(X')$ is either of Poisson type or is $\mathcal{L}(+1)$. What precedes, together with (a), proves assertion (ii) provided it can be shown that $\mathcal{L}(\bar{J}_n) \to \mathcal{L}(X')$ implies $\sigma^2 J_n \to \sigma^2$, and this is proved as the corresponding assertion in (i). This terminates the proof.

The following "simplifying argument" will be used, without further comment, to prove various statements under supplementary assumptions which do not, in fact, restrict the generality.

Let $\{s_n\}$ be an arbitrary numerical sequence. Since we accept infinite limits, this sequence is compact. Thus, to prove that $s_n \to s$, it suffices to show that *every* convergent subsequence $\{s_{n'}\} \subset \{s_n\}$ converges to s. In turn, this will follow if we show that *some* subsequence $\{s_{n''}\} \subset \{s_{n'}\}$ converges to s.

THEOREM 1. *Let the r.v.'s X_{nk} be such that $F_{nk}(x) - F_n(x) \to 0$ for a given $x \in R$, uniformly in k.*

(i) *For fixed ranks r,*

$$\Delta_n = P\{X^*_{nr} < x\} - \int_0^{L_n(x)} \frac{t^{r-1}}{(r-1)!} e^{-t} dt \to 0, \tag{23}$$

and

$$L_n(x) \to L(x) \Leftrightarrow P\{X^*_{nr} < x\} \to \int_0^{L(x)} \frac{t^{r-1}}{(r-1)!} e^{-t} dt. \tag{24}$$

If $\limsup L_n(x) < \infty$, *then $P\{X^*_{r_n} < x\} \to 0$ for variable ranks $r_n \to \infty$.*

(i′) *Similarly for end-ranks s and s_n, upon replacing X^*, r, L, $\int_0^L$ by X, S, $-M$, $\int_{-M}^{+\infty}$, respectively.*

(ii) *For variable ranks $r_n \to \infty$ with $s_n = k_n + 1 - r_n \to \infty$,*

$$\bar{\Delta}_n = P\{X^*_{r_n} < x\} - \frac{1}{\sqrt{2\pi}} \int_{g_{r_n}(x)}^{+\infty} e^{-t^2/2} dt \to 0, \tag{25}$$

and

$$g_{r_n}(x) \to g(x) \Leftrightarrow P\{X^*_{r_n} < x\} \to \frac{1}{\sqrt{2\pi}} \int_{g(x)}^{+\infty} e^{-t^2/2} dt. \tag{26}$$

If $\limsup |g_{r_n}(x)| < \infty$, *then $\sigma^2 I_n(x) \to \infty$ and*

$$P\{X^*_{nr} < x\} \to 0, \quad P\{{}^*X_{ns} < x\} \to 1 \tag{27}$$

for fixed ranks r and end ranks s.

PROOF. (a) We have

$$P\{X^*_{nr} < x\} = P\{I_n(x) \geq r\}. \tag{28}$$

To prove the first assertion in (i), we can assume, because of the simplifying argument, that $\Delta_n \to \Delta$, $L_n(x) \to L(x)$, and $F_n(x) \to F(x)$. Then lemma 1 applies to $I_n(x)$ since $F_{nk}(x) = P\{I_{nk}(x) = 1\} \to F(x)$ uniformly in k and $EI_n(x) = L_n(x) \to L(x)$. Therefore, for $L(x) < \infty$,

$$(29) \qquad P\{X^*_{nr} < x\} \to e^{-L} \sum_{h=r}^{\infty} \frac{L^h}{h!} = \int_0^L \frac{t^{r-1}}{(r-1)!} e^{-t} dt$$

and, for $L(x) = +\infty$,

$$(30) \qquad P\{X^*_{nr} < x\} \to 1 = \int_0^{+\infty} \frac{t^{r-1}}{(r-1)!} e^{-t} dt.$$

Thus $\Delta = 0$, the first assertion in (i) is proved, and the second follows from it.

Similarly for the last assertion in (i): We can assume that $P\{X^*_{r_n} < x\} \to p$ and $L_n(x) \to L(x)$ finite [since $\lim\sup L_n(x) < \infty$]. Then, any fixed rank $r < r_n$ from some n on, and hence

$$(31) \qquad P\{X^*_{r_n} < x\} \leqq P\{X^*_{nr} < x\} \to \int_0^{L(x)} \frac{t^{r-1}}{(r-1)!} e^{-t} dt \leqq \frac{L^r(x)}{r!}.$$

By letting $r \to \infty$, we find that $p = 0$ and the assertion follows.

(b) We have

$$(32) \qquad P\{X^*_{r_n} < x\} = P\{I_n(x) \geqq r_n\} = P\{\bar{I}_n(x) \geqq g_{r_n}(x)\}.$$

To prove the first assertion in (ii), we can assume, because of the simplifying argument, that $\bar{\Delta}_n \to \bar{\Delta}$, $g_{r_n}(x) \to g(x)$, $F_n(x) \to F(x)$, and $\sigma^2 I_n(x) \to \sigma^2$.

Let $\sigma^2 = \infty$. Then, by lemma 1, for $g(x)$ finite

$$(33) \qquad P\{X^*_{r_n} < x\} \to \frac{1}{\sqrt{2\pi}} \int_{g(x)}^{+\infty} e^{-t^2/2} dt,$$

while for $g(x) = +\infty$ or $-\infty$ (so that the foregoing integral is 0 or 1), given $a \in R$, for sufficiently large n

$$(34) \qquad P\{X^*_{r_n} < x\} \leqq P\{\bar{I}_n(x) \geqq a\} \text{ or } \geqq P\{\bar{I}_n(x) \geqq a\},$$

so that, by letting $n \to \infty$ and then $a \to +\infty$ or $-\infty$, we find that $P\{X^*_{r_n} < x\} \to 0$ or 1. It follows that $\bar{\Delta} = 0$, and the first assertion in (ii) is proved when $\sigma^2 = +\infty$.

Let $\sigma^2 < \infty$. Then $F(x) = 0$ or 1, for otherwise $\sigma^2 I_n(x) \sim k_n F_n(x)[1 - F_n(x)] \to \infty$. If $F(x) = 0$, then $L_n(x) \sim \sigma^2 I_n(x) \to \sigma^2 < \infty$ and $g_{r_n}(x) = [r_n(x) - L_n(x)]/\sigma I_n(x) \to +\infty$, while if $F(x) = 1$, then $M_n(x) \sim -\sigma^2 I_n(x) \to -\sigma^2 > -\infty$ and $g_{r_n}(x) = [-M_n(x) - S_n]/\sigma I_n(x) \to -\infty$. But $L(x) < \infty$ or $M(x) > -\infty$ implies, by the last assertion in (i), that $P\{X^*_{r_n} < x\} \to 0$ or 1, while $g(x) = +\infty$ or $-\infty$ implies that $\int_{g(x)}^{+\infty} \exp(-t^2/2) dt = 0$ or 1. It follows that $\bar{\Delta} = 0$ and the first assertion in (ii) is proved also when $\sigma^2 < \infty$.

The next assertion in (ii) follows from the first one. The last but one assertion results, *ab contrario*, from the foregoing discussion of the case $\sigma^2 < \infty$, and the last one in (ii) results from the last one in (i) by $L_n(x) \geqq \sigma^2 I_n(x) \to \infty$ and by the interchange in (i′).

THEOREM 2. (*Individual ranking limit theorem.*) *Under* (H)

I. *For fixed ranks* r

(i) *The class of limit laws of ranked r.v.'s* X^*_{nr} *is that of laws* $\mathcal{L}(X^*_r)$ *with d.f.'s*

$$(35) \qquad F^L_r = \int_0^L \frac{t^{r-1}}{(r-1)!} e^{-t} dt$$

where the functions L on R are nondecreasing, nonnegative, and not necessarily finite.
These laws are laws of r.v.'s if, and only if, $L(-\infty) = 0, L(+\infty) = +\infty$.

$$(36)\quad \text{(ii)} \qquad F^*_{nr} \xrightarrow[c]{w} F^L_r \Leftrightarrow L_n \xrightarrow[c]{w} L.$$

I′. *For fixed end ranks s*

(i′) *The class of limit laws of ranked r.v.'s $^*X_{ns}$ is that of laws $\mathcal{L}(^*X_s)$ with d.f.'s*

$$(37)\qquad {}^MF_s = \int_{-M}^{+\infty} \frac{t^{s-1}}{(s-1)!} e^{-t}dt,$$

where the functions M on R are nondecreasing, nonpositive, and not necessarily finite.
These laws are laws of r.v.'s if, and only if, $M(-\infty) = -\infty, M(+\infty) = 0$.

$$(38)\quad \text{(ii′)} \qquad {}^*F_{ns} \xrightarrow[c]{w} {}^MF_s \Leftrightarrow M_n \xrightarrow[c]{w} M.$$

II. *For variable ranks $r_n \to \infty$ with $s_n = k_n + 1 - r_n \to \infty$*

(i) *The class of limit laws of ranked r.v.'s $X^*_{r_n}$ is that of laws with d.f.'s*

$$(39)\qquad F^g = \frac{1}{\sqrt{2\pi}} \int_g^{+\infty} e^{-t^2/2}dt$$

where the functions g on R are nonincreasing, and not necessarily finite.
These laws are those of r.v.'s if, and only if, $g(-\infty) = +\infty, g(+\infty) = -\infty$.

$$(40)\quad \text{(ii)} \qquad F^*_{r_n} \xrightarrow[c]{w} F^g \Leftrightarrow g_{r_n} \xrightarrow[c]{w} g.$$

PROOF. The second part of assertions (i) follows from the first one. Assertions (ii) and the fact that limit laws are of the form stated in assertions (i) follow from theorem 1. It remains only to show that given L or given g and r_n, there exist d.f.'s F_{nk} of r.v.'s such that $L_n \xrightarrow{w} L$ or $g_{r_n} \xrightarrow{w} g$. Take $F_{nk} = F_n$ with $k = 1, \cdots, k_n \to \infty$.

Given L, if there exists an $a \in R$ such that $L(x)$ is finite or infinite according as $x < a$ or $x > a$, then, for n sufficiently large, it suffices to select F_n such that $F_n(x) = 0$ for $x < -n$, $= L(x)/k_n$ for $x \in (-n, a)$, $= 1$ for $x > a$. If $L(-\infty) = +\infty$, and hence $L = +\infty$, it suffices to select F_n such that $F_n(x) = 0$ for $x < -n$, 1 for $x > -n$.

Given g and $r_n \to \infty$, take k_n such that $s_n = k_n + 1 - r_n \to \infty$, and observe that the relation

$$(41)\qquad \frac{\frac{r_n}{k_n} - F'_n}{\sqrt{F'_n(1-F'_n)}} \sqrt{k_n} = g$$

determines F'_n uniquely and that $0 < F'_n(x) < 1$ for $g(x)$ finite while $F'_n(x) = 0$ or 1 according as $g(x) = +\infty$ or $-\infty$. It suffices to select $F_n(x)$ such that $F_n(x) = F'_n(x)$ for $x \in (-n, +n)$, $= 0$ for $x < -n$, and $=1$ for $x > n$. This completes the proof.

Remarks. 1. The foregoing proof shows that all our limit laws are also limit laws of ranked r.v.'s corresponding to r.v.'s X_{nk} with common d.f. F_n. However, for general functions L, M, g, we are not in the normed identically distributed case, that is, with $F_n(x) = F(a_n + b_nx), x \in R$, where F is some fixed d.f. In fact, various authors, and especially Gnedenko [2] and Smirnov [6], determined the *particular functions* L and g which may occur in this special case. It would be of interest to determine the class of functions L and g which may occur in the general normed case $F_{nk}(x) = F_k(a_n + b_nx)$.

2. The foregoing theorems solve the problem of limit laws of ranked r.v.'s whatever be the asymptotic behavior of the ranks. In fact, set $r_n + s_n = k_n + 1$ and observe that if, say, $r_n \to r$ finite, then $r_n = r$ from some n on while $s_n \to \infty$. Thus, the individual theorem solves the problem in the case $r_n \to r \leqq \infty$ with $s_n \to s \leqq \infty$. In the case $r_n \not\to r$ finite or not, the sequence r_n has distinct lower and upper limits $r' < r''$. If $r' < \infty$ and $r'' < \infty$, then by I(i) the limit d.f. (if it exists) is such that

$$\int_0^L \frac{t^{r'-1}}{(r'-1)!} e^{-t}dt = \int_0^L \frac{t^{r''-1}}{(r''-1)!} e^{-t}dt, \qquad r' \neq r'', \tag{42}$$

so that L can take at most two values 0 and $+\infty$ and the limit law is degenerate. If $r' < \infty$ and $r'' = +\infty$, then, by part (i) of theorem 1, for every x for which $L(x) < \infty$,

$$\int_0^{L(x)} \frac{t^{r'-1}}{(r'-1)!} e^{-t}dt = 0, \tag{43}$$

so that the same conclusion holds. Thus, in all cases not covered by the individual theorem the limit laws are degenerate.

3. According to the last assertion in part (i) of theorem 1, if L on R is finite [or, equivalently, if, for a fixed r, $\mathcal{L}(X^*_{nr}) \to \mathcal{L}(X)$ not a.s. bounded on the right], then $\mathcal{L}(X^*_{r_n}) \to \mathcal{L}(+\infty)$ for $r_n \to \infty$. Similarly, if M on R is finite [or, equivalently, if, for a fixed s, $\mathcal{L}(^*X_{ns}) \to \mathcal{L}(X)$ not a.s. bounded on the left], then $\mathcal{L}(^*X_{s_n}) \to \mathcal{L}(-\infty)$ for $s_n \to \infty$. Also, according to the last assertion in part (ii) of theorem 1, if g on R is finite [or, equivalently, for an $r_n \to \infty$ with $s_n = k_n + 1 - r_n \to \infty$, $\mathcal{L}(X^*_{r_n}) \to \mathcal{L}(X)$ with X a.s. bounded neither on the right nor on the left], then $\mathcal{L}(X^*_{nr}) \to \mathcal{L}(-\infty)$ and $\mathcal{L}(^*X_{ns}) \to \mathcal{L}(+\infty)$ for fixed r and s.

4. The limit laws of "ρ_n-quantiles" are those of ranked r.v.'s with $\rho_n = r_n/k_n \to \rho$, $0 < \rho < 1$. The following propositions are immediate consequences of II.

If $\rho_n \to \rho$ with $0 < \rho < 1$, then

$$F^*_{r_n} \xrightarrow[c]{w} F^g \Leftrightarrow \frac{\rho_n - F_n}{\sqrt{\rho(1-\rho)}} \sqrt{k_n} \xrightarrow[c]{w} g, \tag{44}$$

and then $F_n(x) = L_n(x)/n \to \rho$ whenever $g(x)$ is finite.

If, moreover, $\rho_n' \to \rho$, then nondegenerate limit laws of ρ_n-quantiles and ρ_n'-quantiles are the same if, and only if, $(\rho_n - \rho_n')\sqrt{k_n} \to 0$.

From II follows also that (here we do not assume that $\rho_n \to \rho$ with $0 < \rho < 1$)

If, for some $a \in R$ and every $\epsilon > 0$,

$$(\mathrm{H}_a) \quad \liminf\,[\rho_n - F_n(a-\epsilon)] > 0, \qquad \limsup\,[\rho_n - F_n(a+\epsilon)] < 0, \tag{45}$$

then $X^*_{r_n} \xrightarrow{P} a$.

It suffices to observe that under (H_a) $r_n \to \infty$, $s_n = k_n + 1 - r_n \to \infty$, and $g_{r_n}(x) \to -\infty$ or $+\infty$ according as $x < a$ or $x > a$.

The same proposition can be proved directly and, at the same time, completed as follows.

Under (H_a), if $\sum_n 1/k_n^2 < \infty$, then $X_{r_n} \xrightarrow{a.s.} a$.

For, take a for the origin of values of x and observe that (H_0) implies existence of a $\delta = \delta_\epsilon > 0$ such that, for $n \geqq n_\epsilon$ sufficiently large,

$$\rho_n - F_n(-\epsilon) > \delta, \qquad \rho_n - F_n(+\epsilon) < -\delta. \tag{46}$$

Then, by the Markov inequality, for any $a > 0$,

$$(47)\quad P\{|X^*_{r_n}| > \epsilon\} \leqq P\{X^*_{r_n} < -\epsilon\} + P\{X^*_{r_n} \geqq -\epsilon\}$$
$$= P\{I_n(-\epsilon) \geqq r_n\} + P\{I_n(+\epsilon) < r_n\}$$
$$\leqq P\left\{\frac{I_n(-\epsilon) - EI_n(-\epsilon)}{k_n} \geqq \delta\right\} + P\left\{\frac{I_n(+\epsilon) - EI_n(+\epsilon)}{k_n} \leqq -\delta\right\}$$
$$\leqq \frac{1}{\delta^a k_n^a}[E|I_n(-\epsilon) - EI_n(-\epsilon)|^a + E|I_n(+\epsilon) - EI_n(+\epsilon)|^a].$$

Since, for every $x \in R$, $I_n(x) = \sum_k I_{nk}(x)$ where the summands are independent indicators, it follows at once that there exist finite constants c_2 and c_4 such that the above bracket is bounded by $c_2 k_n$ and by $c_4 k_n^2$ for $a = 2$ and 4, respectively. Therefore, for every $\epsilon > 0$,

$$(48)\qquad P\{|X^*_{r_n}| > \epsilon\} \leqq \frac{c_2}{\delta_\epsilon^2 k_n} \to 0$$

so that $X^*_{r_n} \xrightarrow{P} 0$, and, if $\sum_n 1/k_n^2 < \infty$, then

$$(49)\qquad \sum_n P\{|X^*_{r_n}| > \epsilon\} \leqq \frac{c_4}{\delta_\epsilon^4}\sum_n \frac{1}{k_n^2} < \infty$$

so that $X^*_{r_n} \xrightarrow{\text{a.s.}} 0$.

3. The joint problem

We examine now the joint asymptotic behavior of ranked r.v.'s. The method of attack is the same as for the individual problem. To avoid trivial exceptions, we include within the Poisson and the normal type, laws which are degenerate at a finite or infinite number. We say that a random function (r.f.) on $S \subset R$ is *L-Poisson*, where L is a function on S, if its values at every $x \in S$ are Poisson r.v.'s with parameter $L(x)$. We say that a sequence of laws $\mathcal{L}(X_S^{(n)})$ of r.f.'s $X_S^{(n)}$ on S converges to the law $\mathcal{L}(X_S)$ of a r.f. X on S, and write $\mathcal{L}(X_S^{(n)}) \to \mathcal{L}(X_S)$, if $\mathcal{L}(X_{S'}^{(n)}) \to \mathcal{L}(X_{S'})$ for restrictions $X_{S'}^{(n)}$ and $X_{S'}$ of these r.f.'s on every finite subset $S' \subset S$.

We require the following elementary lemma.

LEMMA 2. *Let $J_n = \sum_k J_{nk}$ be sums of indicator functions J_{nk} on $S \subset R$, independent in k and nondecreasing in $x \in S$, such that $EJ_{nk} = p_{nk} \to p$ on S uniformly in k. Let $J_n^c = \sum_k J_{nk}^c$ where $J_{nk}^c = 1 - J_{nk}$, and let $\bar{J}_n = (J_n - EJ_n)/\sigma S_n$ with covariance γ_n on $S \times S$.*

$$(50)\quad \text{(i)}\qquad EJ_n \to \lambda \;\; on \;\; S \Leftrightarrow \mathcal{L}(J_n) \to \mathcal{L}(J)$$

where J on S is an L-Poisson r.f. with independent increments; similarly for J_n^c.

The r.f.'s J and J^c are independent (when they exist).

(ii) *If $\sigma^2 J_n \to \infty$, then $\mathcal{L}(\bar{J}_n) \sim \mathcal{L}(J_n^N)$ where the J_n^N on S are normal r.f.'s with covariance γ_n,*

$$(51)\qquad \sigma^2 J_n \to \infty, \quad \gamma_n \to \gamma \Leftrightarrow \mathcal{L}(\bar{J}_n) \to \mathcal{L}(\bar{J})$$

where $\bar{J}$ on S is a normal r.f. with covariance γ.

The r.f.'s J and J or J^c are independent; in fact, the three r.f.'s J, J, J^c are independent (when they exist).

Observe that independence of, say, J and J^c means asymptotic independence of J_n *and* J_n^c.

PROOF. To avoid repetition, it is to be understood that asymptotic relations are written for n sufficiently large. Also we use the fact that degeneracy implies independence from any family of r.v.'s. Furthermore, according to the definition of convergence of laws of r.f.'s, we may and shall take their domain S to be a finite set. The method of proof being the same whatever be this finite set S, to simplify the writing, we take S to be composed of two points $x < x'$, drop "(x)," and replace "(x')" by "$'$". Thus, we have two sums of indicators $J_n = \sum_k J_{nk}$, $J_n' = \sum_k J_{nk}'$, with $J_{nk} \leqq J_{nk}'$. Hence $EJ_{nk} = p_{nk} \leqq p_{nk}' = EJ_{nk}'$, with $p_{nk} \to p$, $p_{nk}' \to p'$, uniformly in k.

(a) We prove assertion (i). Let $\mathcal{L}(J_n)$ and $\mathcal{L}(J_n')$ converge so that, by lemma 1, $EJ_n \to \lambda$, $EJ_n' \to \lambda'$. Conversely, let $EJ_n \to \lambda$ and $EJ_n' \to \lambda'$ so that, by the same lemma, $\mathcal{L}(J_n) \to \mathcal{P}(\lambda)$ and $\mathcal{L}(J_n') \to \mathcal{P}(\lambda')$. If λ and/or λ' are infinite, then the corresponding limit laws are degenerate and convergence of the joint law follows from that of the individual ones, and conversely.

Thus, let λ and λ' be finite and hence $p = p' = 0$. Since $J_{nk}(J_{nk}' - J_{nk}) = 0$, it follows that

$$(52)\quad \begin{aligned} &\log E \exp [iu_1 J_{nk} + iu_2 (J_{nk}' - J_{nk})] \\ &\qquad = \log [1 + p_{nk} (e^{iu_1} - 1) + (p_{nk}' - p_{nk}) (e^{iu_2} - 1)] \\ &\qquad = [1 + o(1)] [p_{nk} (e^{iu_1} - 1) + (p_{nk}' - p_{nk}) (e^{iu_2} - 1)] \end{aligned}$$

where $o(1) \to 0$ uniformly in k. Therefore

$$(53)\quad \begin{aligned} &\log E \exp [iu_1 J_n + iu_2 (J_n' - J_n)] \\ &\qquad = [1 + o(1)] [EJ_n (e^{iu_1} - 1) + (EJ_n' - EJ_n) (e^{iu_2} - 1)] \\ &\qquad \to \lambda (e^{iu_1} - 1) + (\lambda' - \lambda) (e^{iu_2} - 1) \end{aligned}$$

and hence

$$(54)\quad \begin{aligned} \log E \exp (iuJ_n + iu'J_n') &= \log E \exp [i (u + u') J_n + iu' (J_n' - J_n)] \\ &\to \lambda (e^{iu} - 1) + \lambda' (e^{iu'} - 1) . \end{aligned}$$

This completes the proof of the if, and only if, assertion in (i). Similarly for $(J_n'^c, J_n^c)$.

As for the last assertion in (i), from

$$(55)\qquad EJ_n + EJ_n^c = EJ_n' + EJ_n = k_n \to \infty ,$$

it follows that individual limit laws such as $\mathcal{L}(J_n)$ and $\mathcal{L}(J_n^c)$ cannot both be nondegenerate. Thus it suffices to prove asymptotic independence for those r.v.'s whose limit laws may be nondegenerate, say, J_n and $J_n'^c$. Because of the relation $J_{nk} J_{nk}'^c = 0$, relation (53) holds with $J_n'^c$ and $\lim EJ_n'^c$ in lieu of $J_n' - J_n$ and $\lambda' - \lambda$, and the assertion follows.

(b) We prove assertion (ii). Dropping momentarily the subscripts "nk," we have

$$(56)\quad \begin{aligned} E \exp [iu (J - p) + iu (J' - p')] = {} & [1 + (p e^{iuq} + q e^{-iup} - 1) e^{iu'q'} \\ & + (p' e^{iu'q'} + q' e^{-iu'p'} - 1) e^{-iup} + (1 - e^{-iup}) (e^{iu'q'} - 1)] . \end{aligned}$$

If $\sigma^2 J_n \to \infty$ and $\sigma^2 J'_n \to \infty$, then, restoring the subscripts, summing over k, replacing u and u' by $u/\sigma J_n$ and $u'/\sigma J'_n$, and expanding, we find that

$$\log E \exp (iuJ_n + iu'J'_n) = -[1 + o(1)]\left(\frac{u^2}{2} + \gamma_n uu' + \frac{u'^2}{2}\right) \tag{57}$$

where $\gamma_n = EJ_nJ'_n = \sigma J_n/\sigma J'_n$. This proves the first assertion in (ii). Furthermore, the right-hand side converges necessarily to $-(u^2/2 + \gamma uu' + u'^2/2)$ if, and only if, $\gamma_n \to \gamma$. But convergence of $\mathcal{L}(J_n, J'_n)$ to $\mathcal{L}(J, J')$ where J and J' are normal r.v.'s implies that individual laws converge to normal ones and hence, by lemma 1, that $\sigma^2 J_n \to \infty$, $\sigma^2 J'_n \to \infty$. This completes the proof of the if, and only if, assertion in (ii).

As for the last assertion, let, say, J and $\bar{J}'$ be nondegenerate. From $\sigma^2 J'_n \to \infty$ and (56) with $p = 0$, it follows that

$$\log E \exp (iuJ_n + iu'J'_n) = [1 + o(1)]\left[EJ_n(e^{iu} - 1) - \frac{u'^2}{2}\right], \tag{58}$$

and hence J_n and J'_n are asymptotically independent. Similar but longer computations show that, say, the three r.v.'s J_n, $\bar{J}_n$, J'^c_n are asymptotically independent. This concludes the proof.

In what follows we use the same simplifying argument as in section 2.

THEOREM 3. *Let the r.v.'s X_{nk} be such that $F_{nk} - F_n \to 0$ on $S = \{x_j, j = 1, \cdots, h\}$, uniformly in k.*

(i) *For fixed ranks r_j,*

$$\Delta_n = P \bigcap_j [X^*_{nr_j} < x_j] - P \bigcap_j [I^P_n(x_j) \geqq r_j] \to 0 \tag{59}$$

where I^P on S are L_n-Poisson r.f.'s with independent increments.

If lim sup $L_n(x_{j_0}) < \infty$ *for some $x_{j_0} \in S$, then for any ranks r_{nj} with some $r_{nj_0} \to \infty$,*

$$P \bigcap_j [X^*_{r_{nj}} < x_j] \to 0 . \tag{60}$$

(i′) *Similarly for end ranks s_j and s_{nj}, upon replacing X^*, r, L by $^*X$, s, $-M$, respectively.*

(ii) *For variable ranks $r_{nj} \to \infty$ with $s_{nj} = k_n + 1 - r_{nj} \to \infty$,*

$$\bar{\Delta}_n = P \bigcap_j [X^*_{r_{nj}} < x_j] - P \bigcap_j [\bar{I}^N_n(x_j) \geqq g_{r_{nj}}(x_j)] \to 0 \tag{61}$$

where $\bar{I}^N_n$ on S are normal r.f.'s with covariance γ_n.

If lim sup $|g_{r_{nj_0}}(x_{j_0})| < \infty$ *for some $x_{j_0} \in S$, then for any ranks r_{nj} with some r_{nj_0} fixed,*

$$P \bigcap_j [X^*_{r_{nj}} < x_j] \to 0 . \tag{62}$$

If lim sup $|g_{r_{nj}}(x_j)| < \infty$ *for all $x_j \in S$, then for fixed end ranks s_j,*

$$P \bigcap_j [^*X_{ns_j} < x_j] \to 1 . \tag{63}$$

PROOF. (a) We have

$$P \bigcap_j [X^*_{nr_j} < x_j] = P \bigcap_j [I_n(x_j) \geqq r_j] . \tag{64}$$

To prove the first assertion in (i) we can assume, because of the simplifying argument, that $\Delta_n \to \Delta$, $L_n \to L$, and $F_n \to F$ on S. Then lemma 2 applies to the r.f.'s I_n so that $\mathcal{L}(I_n) \to \mathcal{L}(I)$ where I on S is an L-Poisson r.f. with independent increments, while clearly $\mathcal{L}(I_n^p) \to \mathcal{L}(I)$. Thus $\Delta = 0$, and the assertion is proved. The next assertion in (i) follows, by the corresponding assertion in part (i) of theorem 1, from

$$(65) \qquad \bigcap_j \, [X^*_{r_{nj}} < x_j] \subset [X^*_{r_{nj_0}} < x_0] \, .$$

Similarly, (i′) results from

$$(66) \qquad \bigcap_j \, [{}^*X_{ns_j} < x_j] = \bigcap_j \, [I_n^c(x_j) < s_j] \, .$$

(b) We have

$$(67) \qquad P\bigcap_j \, [X^*_{r_{nj}} < x_j] = P\bigcap_j \, [\bar{I}_n(x_j) \geqq g_{r_{nj}}(x_j)] \, .$$

To prove the first assertion in (ii) we can assume, because of the simplifying argument, that $\bar{\Delta}_n \to \bar{\Delta}$, $g_{r_{nj}} \to g_j$, $F_n \to F$ on S, and $\gamma_n \to \gamma$ on $S \times S$.

If some $g_j(x_j) = +\infty$ or $-\infty$, then the probabilities of the corresponding events $[X^*_{r_{nj}} < x_j]$ and $[\bar{I}_n(x_j) \geqq g_{r_{nj}}(x_j)]$ have the same limit 0 or 1. In the first case, both terms of Δ_n converge to 0 and hence $\bar{\Delta} = 0$. In the second case both events can be dropped at the limit. Thus, it suffices to prove the assertion for finite $g_j(x_j)$. But then, according to part (ii) of theorem 1, $\sigma^2 I_n \to \infty$ on S so that lemma 2 applies, and hence $\bar{\Delta} = 0$. The assertion is proved. The following assertions in (ii) result from the last one in part (ii) of theorem 1. This completes the proof.

THEOREM 4. (*Joint ranking limit theorem.*) *Under* (H)

I. *For fixed ranks* $r_1, \cdots, r_h$

(i) *The class of joint limit laws of ranked r.v.'s* $X^*_{nr_1}, \cdots, X^*_{nr_h}$ *is that of laws* $\mathcal{L}(X^*_{r_1}, \cdots, X^*_{r_h})$ *with d.f.'s defined by*

$$(68) \quad P\{X^*_{r_1} < x_1, \cdots, X^*_{r_h} < x_h\} = P\{I(x_1) \geqq r_1, \cdots, I(x_h) \geqq r_h\}, \qquad x_1, \cdots, x_h \in R \, ,$$

with I on R an L-Poisson r.f. with independent increments where the functions L on $\boldsymbol{R}$ *are nondecreasing, nonnegative, and not everywhere finite.*

In particular, for two fixed ranks $r < r'$,

$$(69) \quad P\{X^*_r > x, X^*_{r'} < x'\} = \int_{L(x)}^{L(x')} e^{-t} dt \left\{ \int_{L(x)}^{t} \frac{(t-t')^{r'-r-1}}{(r'-r-1)!} \cdot \frac{t'^{r-1}}{(r-1)!} dt' \right\}$$

or $=0$, *according as* $x \leqq x'$ *or* $x > x'$.

$$(70) \quad \text{(ii)} \quad L_n \xrightarrow[c]{w} L \Leftrightarrow \mathcal{L}(X^*_{nr_1}, \cdots, X^*_{nr_h}) \xrightarrow[c]{w} \mathcal{L}(X^*_{r_1}, \cdots, X^*_{r_h}) \, .$$

I′. *Similarly for fixed end ranks, upon replacing* r, X^*, I, L *by* $s, {}^*X, I^c, -M$, *replacing nonnegative by nonpositive, and interchanging* x *and* x'.

II. *For variable ranks* r_{nj} *such that* $\rho_{nj} = r_{nj}/k_{nj} \to \rho_j$ *with* $0 < \rho_j < 1$ *for every* $j = 1, \cdots, h$,

(i) *The class of joint limit laws of ranked r.v.'s* $X_{r_{n1}}, \cdots, X_{r_{nh}}$ *is that of limit laws* $\mathcal{L}(X^{*\sigma_1}_{\rho_1}, \cdots, X^{*\sigma_h}_{\rho_h})$ *with d.f.'s defined by*

$$(71) \quad P\{X^{*\sigma_1}_{\rho_1} < x_1, \cdots, X^{*\sigma_h}_{\rho_h} < x_h\} = P\{\bar{I}(x_1) \geqq g_1(x_1), \cdots, \bar{I}(x_h) \geqq g_h(x_h)\}, \qquad x_1, \cdots, x_h \in R \, ,$$

where

(a) *the functions* g_j *on R are nonincreasing, not necessarily finite, such that* $g_j \leqq g_{j'}$ *for* $\rho_j < \rho_{j'}$, *and whenever* $g_j(x)$ *and* $g_j(x')$ *are both finite then* $\rho_j = \rho_{j'}$ *and*

$$(72) \qquad g_j(x) - g_j(x') = c_{jj'} = \lim \frac{(\rho_{nj} - \rho_{nj'})\sqrt{k_n}}{\sqrt{\rho_1(1-\rho_j)}}$$

are constants independent of x.

(b) *The r.f. I on R is normal with a triangular covariance defined by*

$$(73) \qquad \gamma(x, x') = \frac{\beta(x_1)}{\beta(x_2)}, \qquad x_1 = \min(x, x'), \qquad x_2 = \max(x, x'),$$

where the function β on R is nonnegative, nondecreasing, not necessarily finite, and such that $\beta^2(x) = \rho_j/(1-\rho_j)$ whenever $g_j(x)$ is finite. When the defining ratio is undetermined, set $\gamma(x, x') = 0$.

In particular, for two limit relative ranks ρ, ρ',

$$(74) \quad P\{X_\rho^{*g} < x, X_{\rho'}^{*g'} < x'\} = \frac{1}{2\sqrt{1-\gamma^2(x, x')}} \int_{g(x)}^{+\infty}\int_{g'(x')}^{+\infty} \exp\left\{-\frac{1}{2}\cdot\frac{1}{1-\gamma^2(x, x')} \cdot [t^2 - 2\gamma(x, x')tt' + t'^2]\right\} dt\,dt'$$

if $\gamma(x, x') < 1$, and

$$(75) \qquad P\{X_\rho^{*g} < x, X_{\rho'}^{*g'} < x'\} = \frac{1}{\sqrt{2\pi}} \int_{\max[g(x), g'(x')]}^{+\infty} e^{-t^2/2}dt$$

if $\gamma(x, x') = 1$.

$$(76) \quad \text{(ii)} \quad \mathcal{L}(X_{\rho_{n1}}^*, \cdots, X_{\rho_{nh}}^*) \xrightarrow[c]{w} \mathcal{L}(X_{\rho_1}^{*g_1}, \cdots, X_{\rho_h}^{*g_h}) \Leftrightarrow g_{r_{nj}} \xrightarrow[c]{w} g_j, \qquad j = 1, \cdots, h.$$

III. *The r.f.'s $\{X_r^*, r = 1, 2, \cdots\}$, $\{{}^*X_s, s = 1, 2, \cdots\}$, $\{X_\rho^{*g}, 0 < \rho < 1\}$ are independent (when they exist).*

Proof. (a) In I, assertion (ii) follows from assertion (i). The form of the limit laws in (i) follows from theorem 3. To prove that every law of the form stated in (i) is a limit law, it suffices to observe that the construction for a given L in the proof of the individual theorem applies. Finally, the particular case in (i) for $x > x'$ results from $P\{X_{nr}^* \geqq x, X_{nr'}^* < x'\} = 0$ while for $x \leqq x'$ it results from

$$\begin{aligned}(77) \quad P\{X_r^* \geqq x, X_{r'}^* < x'\} &= P\{I(x) < r, I(x') \geqq r'\} \\ &= \sum_{j=0}^{r-1} P\{I(x) = j\} \cdot P\{I(x') - I(x) \geqq r' - j\} \\ &= \sum_{j=0}^{r-1} e^{-L(x)} \frac{L^j(x)}{j!} \left\{\sum_{h=r'-j}^{+\infty} e^{-L(x')+L(x)} \frac{[L(x') - L(x)]^h}{h!}\right\} \\ &= \sum_{j=0}^{r-1} \frac{L^j(x)}{j!} \int_0^{L(x')-L(x)} \frac{y^{r'-j-1}}{(r'-j-1)!} e^{-y}dy \\ &= \int_{L(x)}^{L(x')} \left\{\sum_{j=0}^{r-1} \frac{L^j(x)}{j!} \cdot \frac{[t - L(x)]^{r'-j-1}}{(r'-j-1)!} e^{-t}\right\} dt \\ &= \int_{L(x)}^{L(x')} e^{-t}dt \left[\int_{L(x)}^{t} \frac{(t-t')^{r'-r-1}}{(r'-r-1)!} \cdot \frac{t'^{r-1}}{(r-1)!} dt'\right].\end{aligned}$$

This completes the proof of I. Similarly for I'.

(b) In II, assertion (ii) follows from assertion (i). As for assertion (i), let $\mathcal{L}(X^*_{r_{n1}}, \cdots, X^*_{r_{nh}}) \xrightarrow[c]{w} \mathcal{L}$. Then the individual laws converge and, by the individual theorem, $g_{nj} \to g_j$ for every $j = 1, \cdots, h$.

If $\rho_j < \rho_{j'}$ then, from some n on, $r_{nj} < r_{nj'}$ so that

$$(78) \qquad g_{r_{nj}} = \frac{r_{nj} - L_n}{\sigma I_n} \leqq \frac{r_{nj'} - L_n}{\sigma I_n} = g_{r_{nj'}},$$

and hence, by passing to the limit, $g_j \leqq g_{j'}$. If $g_j(x)$ and $g_{j'}(x)$ are both finite, then $F_n(x) \to \rho_j$ and $F_n(x) \to \rho_{j'}$ so that $\rho_j = \rho_{j'}$ and

$$(79) \qquad g_j(x) - g_{j'}(x) = \lim_n \frac{(\rho_{nj} - \rho_{nj'})\sqrt{k_n}}{\sqrt{\rho_j(1-\rho_j)}} = c_{jj'}$$

is independent of x. Thus, the g_j must fulfill the conditions asserted in (a) of part (i).

Furthermore, according to theorem 3, convergence to $\mathcal{L}(X)$ is equivalent to convergence (weak or complete) of the d.f.'s with values $a_n = P\{I_n^N(x_1) \geqq g_{r_{n1}}(x_1), \cdots, I_n^N(x_h) \geqq g_{r_{nh}}(x_h)\}$, where every r.f. I_n^N is normal with covariances γ_n of the r.f. I_n. Select a subsequence $F_m \to F$ as $m \to \infty$. Then elementary computations show that $\gamma_m \to \gamma$ with $\gamma(x, x') = \beta(x)/\beta(x')$ for $x \leqq x'$ and $\beta = \{F/(1-F)\}^{1/2}$, unless the ratio is undetermined [that is, unless either $F(x) = 1$ and hence $F(x') = 1$, or $F(x') = 0$ and hence $F(x) = 0$]. Therefore, lim a_n, assumed to exist (except perhaps on a countable set in R^h), is of the form stated in II. Since finiteness of $g_j(x)$ or of $g_j(x')$ implies that $F_n(x) \to \rho_j$ or $F_n(x') \to \rho_{j'}$ and since $0 < \rho_j, \rho_{j'} < 1$, it follows that in such cases $\gamma(x, x')$ is determined. Thus indeterminacy of the ratio can happen only if, say, $g_j(x_j)$ is infinite and x_j is x or x'. If $g_j(x_j) = +\infty$, then $P\{I_n^N(x_j) \geqq g_{r_{nj}}(x_j)\} \to 0$ and $P\{\bar{I}(x_j) \geqq g_{r_j}(x_j)\} = 0$ so that the limit value of the probability of the whole compound event under consideration as well as of its stated limit value is 0. If $g_j(x_j) = -\infty$, then $P\{I^N(x_j) \geqq g_{r_{nj}}(x_j)\} \to 1$ and $P\{\bar{I}(x_j) \geqq g_{r_j}(x_j)\} = 1$ so that the corresponding events may be left out of the whole compound events. Therefore, when the ratio is undetermined, the value of $\gamma(x, x')$ is immaterial and, to fix the ideas, we may take it to be 0. This completes the proof of the form of the limit laws $\mathcal{L}$. Conversely, given $\mathcal{L}$ with r_{nj}, g_j, and $\bar{I}$ fulfilling the required conditions, it follows easily that the set of equations

$$(80) \qquad \frac{\rho_{nj} - F'_n}{\sqrt{F'_n(1-F'_n)}}\sqrt{k_n} = g_j, \qquad i = 1, \cdots, h,$$

determines uniquely the functions F'_n on R which in turn determines d.f.'s $F_{nk} = F_n$ of r.v.'s by the construction in the proof of the individual theorem. The particular case results at once from the properties of nonsingular and singular two-dimensional normal d.f.'s and similar expressions can be written for h limit relative ranks.

Finally, III follows from independence of the r.f.'s I, I^c, $\bar{I}$ (when they exist), which results from the last assertion in lemma 2. This terminates the proof.

Remarks. 1. It would be of interest to give multi-integral forms, similar to the one given in the particular case in I, for any finite number of fixed ranks. Also to examine the case of variable ranks such that the limit relative ranks may be 0 or 1.

2. Since X_r^* and $^*X_s$ are independent, it follows that $\mathcal{L}(^*X_{ns} - X^*_{nr}) \to \mathcal{L}(^*X_s -$

$X_r^*) = \mathcal{L}({}^*X_s) * \mathcal{L}(-X_r^*)$. In particular, the limit law of the "range" is obtained upon taking $s = 1, r = 1$.

4. Ranking and summation

We can now return to the ranking problem within the setup of the Central Limit Problem [4]. To begin with, the r.v.'s X_{nk} under consideration are to be uniformly asymptotically negligible (u.a.n.). This leads to the following complements to the previous results.

I. *Under the u.a.n. condition*

(i) *For relative ranks* $\rho_n = r_n/k_n$ *such that* $0 < \liminf \rho_n < \limsup \rho_n < 1, X_{r_n}^* \xrightarrow{P} 0$ *and if, moreover,* $\sum 1/k_n^2 < \infty$ *then* $X_{r_n}^* \xrightarrow{a.s.} 0$.

(ii) *Every limit function L equals* $+\infty$ *on* $(0, +\infty)$ *and every limit function M equals* $-\infty$ *on* $(-\infty, 0)$ *or, equivalently, every* X_r^* *is a.s. nonpositive and every* ${}^*X_s$ *is a.s. nonnegative.*

The first assertion results, by u.a.n., from the last two propositions in section 2. The second assertion results, by u.a.n., from the definition of L and M.

Furthermore, the Central Limit Theorem and the limit theorems for normed sums yield at once the following precisions about the limit functions L and M.

II. *Let* $\mathcal{L}\left(\sum_k X_{nk}\right) \to \mathcal{L}(X)$ *where the r.v.'s* X_{nk} *are u.a.n. and X is a r.v., necessarily infinitely decomposable, with P. Lévy function* (L_X, M_X).

(i) *The limit functions L and M of the ranking theorems exist and* $L(x) = L_X(x)$ *or* $+\infty$, $M(x) = -\infty$ *or* $M_X(x)$, *according as* $x < 0$ *or* $x > 0$.

(ii) *In the case of normed sums, L and* $-M$ *are convex functions of* $\log |x|$ *on* $(-\infty, 0)$ *and* $(0, +\infty)$, *respectively.*

(iii) *In the identically distributed normed case,* $L(x) = \beta|x|^{-\gamma}$ *on* $(-\infty, 0)$ *and* $M(x) = -\beta'^{-\gamma}$ *on* $(0, +\infty)$, *where* $\beta, \beta' \geqq 0$ *and* $0 < \gamma < 2$, *with* $\beta = \beta' = 0$ *if, and only if,* $\mathcal{L}(X)$ *is normal.*

The hypothesis of II implies, by I, II and the ranking limit theorems, that the limit laws $\mathcal{L}(X_r^*, r = 1, 2, \cdots)$, $\mathcal{L}({}^*X_s, s = 1, 2, \cdots)$ exist. The question arises as to the convergence of joint laws of $\sum_k X_{nk}$ and of ranked r.v.'s, and as to the form of limit laws (if they exist). In the case of relative ranks $\rho_n = r_n/k_n \to \rho$ with $0 < \rho < 1$, the u.a.n. hypothesis alone implies limit degeneracy at 0 of $\mathcal{L}(X_{r_{nj}}^*, j = 1, \cdots, h)$ and the problem is trivial. Thus we consider only fixed ranks and fixed end ranks.

Let (C) denote the following assumption: The u.a.n. independent summands X_{nk} are such that $\mathcal{L}\left(\sum_k X_{nk}\right) \to \mathcal{L}$ infinitely decomposable with ch. f. f defined by

$$\log f(u) = iau - \frac{b^2}{2} u^2 + \int_{-\infty}^{0} \left(e^{iuy} - 1 - \frac{iuy}{1+y^2}\right) dL(y) + \int_{+0}^{+\infty} \left(e^{iuy} - 1 - \frac{iuy}{1+y^2}\right) dM(y) \tag{81}$$

with a, b real and (L, M) a P. Lévy function or, equivalently, by the Central Limit Theorem,

$$(C_1)\quad L_n \xrightarrow{w} L\,,\qquad M_n \xrightarrow{w} M\,,$$

$$(C_2)\quad \sum_k \sigma^2 X^{\epsilon}_{nk} \to b^2 \text{ as we let } n\to\infty \text{ and then } \epsilon\to 0\,,$$

$$(82)\quad (C_3)\quad \sum_k EX^{\tau}_{nk} \to a(\tau) = a + \int_{-\tau}^{-0} \frac{y^3}{1+y^2}\,dL(y) - \int_{-\infty}^{-\tau} \frac{y}{1+y^2}\,dL(y) + \int_{+0}^{+\tau} \frac{y^3}{1+y^2}\,dM(y) - \int_{+\tau}^{+\infty} \frac{y}{1+y^2}\,dL(y)\,,$$

where $-\tau$ and $+\tau$ are continuity points of L and M, respectively. (The right superscripts for X denote truncation at ϵ and at τ.)

Let ${}^xF$ and ${}^xf$ be the d.f. and the ch. f. of ${}^xX$, that is, of a r.v. X given that $X < x$ (with same subscripts for F and X, if any). If the conditioning event is of positive probability, that is, $F(x) > 0$, then ${}^xF(y) = F(y)/F(x)$ or 1 according as $y < x$ or $y \geqq x$ and hence

$$(83)\qquad {}^xf(u) = \frac{1}{F(x)} \int_0^x e^{iuy}\,dF(y)\,.$$

We require the following lemma.

LEMMA 3. *Under* (C), *for every* $x > 0$,

$$(84)\qquad \prod_k {}^xf_{nk} \to {}^xf$$

where ${}^xf$ *is an infinitely decomposable ch. f. defined by*

$$(85)\qquad \log {}^xf(u) = i\,{}^xa u - \frac{{}^xb^2}{2} u^2 + \int_{-\infty}^{-0} \left(e^{iuy} - 1 - \frac{iuy}{1+y^2}\right) d\,{}^xL(y) + \int_{+0}^{+\infty} \left(e^{iuy} - 1 - \frac{iuy}{1+y^2}\right) d\,{}^xM(y)$$

with

$$(86)\qquad {}^xa = a - \int_x^{+\infty} \frac{y}{1+y^2}\,dM(y)\,,\qquad {}^xb = b\,,\qquad {}^xL = L\,,$$

and ${}^xM(y) = M(x)$ *or* 0 *according as* $y < x$ *or* $y \geqq x$.

PROOF. Let $x > 0$. Since the X_{nk} are u.a.n. independent summands, the same is true of the ${}^xX_{nk}$. Since $F_{nk}(x) \to 1$ uniformly in k, all $F_{nk}(x) > 0$ from some n on. Then, by (C_1),

$$(87)\qquad {}^xL_n = \sum_k \frac{F_{nk}(y)}{F_{nk}(x)} \xrightarrow{w} L$$

and

$$(88)\qquad {}^xM_n(y) = \sum_k \frac{F_{nk}(y) - 1}{F_{nk}(x)} \to M(y)$$

for continuity points $y < x$ of M, while ${}^xM_n(y) = 0$ for $y \geqq x$. Similarly, by (C_2), upon taking $\epsilon < x$ (which we may)

$$(89)\qquad {}^xb_n^2 = \sum_k \sigma^2({}^xX^{\epsilon}_{nk}) = \sum_k \frac{\sigma^2(X^{\epsilon}_{nk})}{F_{nk}(x)} \to b^2$$

as we let $n \to \infty$ and then $\epsilon \to 0$. Finally, for $\tau \leqq x$,

$$(90) \quad {}^x a_n(\tau) = \sum_k E({}^x X^\tau_{nk}) = \sum_k \frac{E(X^\tau_{nk})}{F_{nk}(x)} \to a(\tau) = {}^x a(\tau) - \int_x^{+\infty} \frac{y}{1+y^2}\, dM(y)$$

[the same limit is obtained also for $\epsilon > x$ since then ${}^x a(\tau) = {}^x a(x)$].

Thus, for every fixed $x > 0$, the Central Limit Theorem applies to $\prod_k {}^x f_{nk}$, and the lemma is proved.

In what follows g is a Borel function on R.

THEOREM 5. *Let the F_{nk} be continuous from some n on. Under (C) with $M(+0) = +\infty$,*

$$(91) \quad \mathcal{L}\Big(\sum_k X_{nk},\ {}^*X_{n1}\Big) \to \mathcal{L}(X,\ {}^*X_1),$$

with ch. f. f defined by

$$(92) \quad f(u, v) = \exp\Big[\int_{+0}^{+\infty} {}^x f(u)\, e^{iux+ivg(x)} d\, {}^*F_1(x)\Big]$$

where the d.f. ${}^*F_1$ of ${}^*X_1$ is defined by ${}^*F_1 = \exp(M)$ on $(0, +\infty)$.

PROOF. Let us observe that, under (C), $M(+0) = +\infty$ is equivalent to the continuity of ${}^*F_1$ at 0. Because of the continuity assumptions, we have (from some n on)

$$(93) \quad f_n(u, v) = E \exp\Big(iu\sum_k X_{nk} + iv\, {}^*X_{n1}\Big)$$
$$= \sum_k E\Big\{\exp\Big(iu\sum_k X_{nk} + iv\, {}^*X_{n1}\Big) I_{[X_{nk} = {}^*X_{n1}]}\Big\}$$
$$= \sum_k \int_{-\infty}^{+\infty} \Big[\prod_{j\neq k} \int_{-\infty}^{x} e^{iuy} dF_{nj}(y)\Big] e^{iux+ivg(x)} dF_{nk}(y)$$
$$= \sum_k \int_{+0}^{+\infty} \Big[\prod_{j\neq k} {}^x f_{nj}(u)\Big] e^{iux+ivg(x)} \Big[\prod_{j\neq k} F_{nj}(x)\Big] dF_{nk}(x) + J_n(u, v),$$

where, by the individual ranking theorem and II of this section,

$$(94) \quad |J_n(u, v)| \leqq \int_{-\infty}^{-0} \sum_k \Big\{\prod_{j\neq k} F_{nk}(x)\, dF_{nk}(x)\Big\} = P\{{}^*X_{n1} < 0\} \to 0.$$

As for the remaining sum, observe that, by lemma 3, $\prod_j {}^x f_{nj} \to {}^x f$ for every $x > 0$ while

$$(95) \quad \Big|\prod_j {}^x f_{nj} - \prod_{j\neq k} {}^x f_{nj}\Big| \leqq |f_{nk} - 1| \to 0$$

uniformly in k, and

$$(96) \quad \int_{+0}^{x} \Big[\sum_{j\neq k} F_{nk}(y)\Big] dF_{nk}(y) = {}^*F_{n1}(x)$$

with ${}^*F_{n1} \xrightarrow{c} {}^*F_1$. It follows, by elementary computations, that the remaining sum and hence

$$(97) \quad f_n(u, v) \to f(u, v) = \exp\Big[\int_{+0}^{+\infty} {}^x f(u)\, e^{iux+ivg(x)} d\, {}^*F_{n1}(x)\Big].$$

This proves the theorem.

Remarks. 1. The same method yields the joint limit law of $\left(\sum_k X_{nk}, {}^*X_{ns}\right)$ for any fixed end ranks and of $\left(\sum_k X_{nk}\right) g\,({}^*X_{ns})$ [replace u by $ug(x)$ and v by 0]. Lengthier computations yield the joint limit laws of $\sum_k X_{nk}$ and any number of ranked r.v.'s of fixed end ranks. Similarly one obtains limit laws of $\sum_k X_{nk}$ with ranked r.v.'s of fixed ranks.

2. In the normed identically distributed case, the expressions of ${}^x f$ and M (or L) are sufficiently simple to enable us to find simpler forms for limit ch. f.'s. In particular, we can obtain the forms of the limit law of $\left(\sum_x X_{nk}\right) \Big/ {}^*X_{n1}$ due to Darling [1].

3. It would be of interest to investigate the problem of this section without the continuity assumptions.

REFERENCES

[1] D. A. Darling, "The influence of the maximum term in the addition of independent random variables," *Trans. Amer. Math. Soc.*, Vol. 73 (1952), pp. 95–107.

[2] B. V. Gnedenko, "Sur la distribution limite du terme maximum d'une série aléatoire," *Annals of Math.*, Vol. 44 (1943), pp. 423–453.

[3] Michel Loève, *Probability Theory*, New York, D. Van Nostrand, 1955.

[4] ———, "Termes variationnels dans le modèle limite central," *C. R. Acad. Sci., Paris*, Vol. 240 (1955), pp. 722–724.

[5] I. R. Savage, "Bibliography of nonparametric statistics and related topics," *Jour. Amer. Stat. Assoc.*, Vol. 48 (1953), pp. 844–906.

[6] N. V. Smirnov, "Limit distributions for the terms of a variational series," *Amer. Math. Soc., Translation No. 67*, 1952, pp. 1–64.